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A METHOD FOR OBTAINING ASYMPTOTES OF SOME CURVES

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Abstract. We describe a quick and a simple method for obtaining the asymptotes of the curve $F(x,y)=0$, where $F(x,y)$ is a polynomial in x and y, with emphasis on second order polynomials. Using this method we obtain a simple classification of second order curves.

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1. The usual text-book procedure for finding common points of the straight line

$$y = kx + n \tag{1}$$

and the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2 \qquad (a>0,\, b>0) \tag{2}$$

consists in solving the system (1), (2). Eliminating y between (1) and (2) we obtain the equation

$$(b^2 - a^2k^2)x^2 - 2a^2knx - (a^2n^2 + a^2b^2) = 0, \tag{3}$$

and analysing this quadratic equation (it may have two, one or no solutions in $\mathbf{R}$) we arrive at the well-known conclusions regarding the line (1) and the curve (2). However, it is often overlooked that (3) is a quadratic equation only if $b^2 - a^2k^2 \neq 0$. If $b^2 - a^2k^2 = 0$ the equation (3) reduces to

$$\pm 2abnx - (a^2n^2 + a^2b^2) = 0,$$

that is to say to two linear equations, each having exactly one solution, provided that $n \neq 0$.

If $b^2 - a^2k^2 = 0$, $n = 0$, the equation (3) becomes $a^2b^2 = 0$, which cannot be true, signifying that in this case (1) and (2) have no common points. But for $b^2 - a^2k^2 = 0$, $n = 0$, the equation (1) reduces to $y = \pm\frac{b}{a}x$, and those are, as we know, the asymptotes of the hyperbola (2). In other words, the asymptotes of (2) are obtained when the coefficients of x^2 and x in (3) are equated to 0, and the constant term is not 0.

The above remarks concerning the hyperbola (2) suggest the following general method for finding the asymptotes of the curve

$$a_1x^2 + a_2xy + a_3y^2 + b_1x + b_2y + c = 0 \qquad (a_1^2 + a_2^2 + a_3^2 > 0). \tag{4}$$

Eliminating y between (1) and (4) we arrive at the equation

$$(a_1 + a_2k + a_3k^2)x^2 + (a_2n + 2a_3kn + b_1 + b_2k)x + (a_3n^2 + b_2n + c) = 0,$$

and by analogy with the hyperbola (2), we look for the asymptotes of the form (1) from the conditions

$$(5)\quad a_1 + a_2k + a_3k^2 = 0, \qquad a_2n + 2a_3kn + b_1 + b_2k = 0, \qquad a_3n^2 + b_2n + c \neq 0.$$

If we put

$$(6)\qquad P(t) = a_1 + a_2t + a_3t^2, \qquad Q(t) = b_1 + b_2t,$$

the first two equations from (5) can be written in the form

$$(7)\qquad P(k) = 0, \qquad P'(k)n + Q(k) = 0,$$

and so the coefficients k and n in the equation of the asymptote (1) are determined by the system (7) under the condition

$$(8)\qquad a_3n^2 + b_2n + c \neq 0.$$

2. We first examine what happens if the system (7) has a solution and if the condition (8) is not fulfilled. In other words, we suppose that there exist real numbers p and q such that

$$(9)\qquad P(p) = 0, \qquad P'(p)q + Q(p) = 0, \qquad a_3q^2 + b_2q + c = 0.$$

Denote the left-hand side of (4) by $F(x,y)$. Equalities (9) imply that $F(x, px+q) \equiv 0$, and this means that the polynomial $F(x,y)$ is divisible by $y - px - q$, and since $F(x,y)$ is of second degree, we have $F(x,y) = (y - px - q)(Ax + By + C)$, and so the equation $F(x,y) = 0$, i.e. the equation (4), represents two straight lines (which may coincide).

In fact, it can be shown that the equalities (9) imply that

$$(10)\qquad D = 0 \qquad \text{where} \qquad D = \begin{vmatrix} 2a_1 & a_2 & b_1 \\ a_2 & 2a_3 & b_2 \\ b_1 & b_2 & 2c \end{vmatrix}.$$

But it is well known that (10) is a necessary and sufficient condition which ensures that the polynomial $F(x,y)$ can be factorised into linear factors; namely

$$F(x,y) = a_1x^2 + a_2xy + a_3y^2 + b_1x + b_2y + c = (Ax + By + C)(Dx + Ey + F).$$

The coefficients A, B, C, D, E, F may be all real, in which case (4) represents two straight lines (which may coincide), or some of them may be complex, in which case (4) represents the empty set (in real analytical geometry). Hence, in both cases there is no point in looking for the asymptotes of (4).

3. Return to the curve (4), the polynomials P and Q defined by (6) and the determinant D defined by (10).

Suppose that the equation $P(t)=0$ has two distinct solutions in t: α and β, which is equivalent to $a_3\neq 0$, $a_2^2-4a_1a_3>0$. Then $P(\alpha)=P(\beta)=0$, $P'(\alpha)\neq 0$, $P'(\beta)\neq 0$, and we have two cases.

(i) If $D\neq 0$, the curve (4) has two asymptotes

$$y=\alpha x-\frac{Q(\alpha)}{P'(\alpha)},\qquad y=\beta x-\frac{Q(\beta)}{P'(\beta)}; \tag{11}$$

in other words, it is a hyperbola.

(ii) If $D=0$, the equation (4) represents two straight lines which intersect and whose equations are (11).

Example 1. For the curves

$$x^2+2xy-3y^2+2x+y+3=0 \tag{12}$$

and

$$x^2+2xy-3y^2+2x+y+\frac{15}{16}=0 \tag{13}$$

we have: $P(t)=1+2t-3t^2$, $P'(t)=2-6t$, $Q(t)=2+t$. The equation $P(t)=0$ has two solutions: 1 and $-1/3$.

For the equation (12) we have $D=-66$, and so it represents a hyperbola whose asymptotes are given by

$$y=x+\frac{3}{4}\quad\text{and}\quad y=-\frac{1}{3}x-\frac{5}{12}. \tag{14}$$

For the equation (13) we have $D=0$, and so it represents two straight lines whose equations are (14). △

So far, we searched for the asymptotes of (4) of the form $y=kx+n$. But the curve (4) may also have a vertical asymptote whose equation $x=\lambda$ cannot be obtained from $y=kx+n$. From (4) and $x=\lambda$ we get

$$a_3y^2+(a_2\lambda+b_2)y+(a_1\lambda^2+b_1\lambda+c)=0$$

and a vertical asymptote may exist only if $a_3=0$. However, the supposition of this section that the equation $P(t)=0$ has two distinct solutions implies $a_3\neq 0$ and so, in this case, the curve (4) has no vertical asymptotes.

4. Suppose that the equation $P(t)=0$ has exactly one solution α which is equivalent to

$$a_3\neq 0,\quad a_2^2-4a_1a_3=0\quad\text{or}\quad a_3=0,\quad a_2\neq 0.$$

Those two cases will be considered separately.

First, let $a_3\neq 0$, $a_2^2-4a_1a_3=0$. Then $P(\alpha)=P'(\alpha)=0$. If $Q(\alpha)\neq 0$, the curve (4) has no asymptotes. Let $Q(\alpha)=0$. From the equalities

$$a_2+2a_3\alpha=0,\quad b_1+b_2\alpha=0\quad(a_3\neq 0)$$

we get $b_2 \neq 0$, and $a_2b_2 = 2a_3b_1$. Also, from $a_2^2 - 4a_1a_3 = 0$, $a_3 \neq 0$ follows $a_1 = a_2^2/4a_3$, and (4) can be transformed into

$$a_2^2x^2 + 4a_2a_3xy + 4a_3^2y^2 + 2a_2b_2x + 4a_3b_2y + 4a_3c = 0, \tag{15}$$

which can be written in the form

$$(a_2x + 2a_3y)^2 + 2b_2(a_2x + 2a_3y) + 4a_3c = 0.$$

Hence (15) represents two parallel straight lines, one straight line or the empty set, depending on the value of $b_2^2 - 4a_3c$.

As in the previous section, the condition $a_3 \neq 0$ implies that (15) has no vertical asymptotes.

Consider now the second case when $P(t) = 0$ has exactly one solution, that is to say the case $a_3 = 0$, $a_2 \neq 0$. The system (7) has unique solution

$$k = -\frac{a_1}{a_2}, \quad n = \frac{a_1b_2 - b_1a_2}{a_2^2},$$

and the curve (4), under the condition $D \neq 0$, has the asymptote

$$y = -\frac{a_1}{a_2}\,x + \frac{a_1b_2 - b_1a_2}{a_2^2}. \tag{16}$$

But in this case the curve (4) also has a vertical asymptote. Indeed, since $a_3 = 0$, $a_2 \neq 0$, from (4) and $x = \lambda$ we get

$$(a_2\lambda + b_2)y + (a_1\lambda^2 + b_1\lambda + c) = 0,$$

and, under the condition $D \neq 0$, there is a vertical asymptote whose equation is

$$x = -\frac{b_2}{a_2}. \tag{17}$$

Hence, if $a_3 = 0$, $a_2 \neq 0$, $D \neq 0$, the curve (4) is a hyperbola with asymptotes (16) and (17).

As before, if $D = 0$, we do not look for the asymptotes of (4), but note that in this case (4) represents the straight lines (16) and (17).

Example 2. For $c = 9$, the equation $3x^2 - xy + 7x - 5y + c = 0$ represents a hyperbola with asymptotes $y = 3x - 8$ and $x = -5$; for $c = -40$ it represents the straight lines $y = 3x - 8$ and $x = -5$. △

Remark 1. If $a_3 = 0$, the equation (4) can be written as

$$y = f(x), \quad \text{where} \quad f(x) = -\frac{a_1x^2 + b_1x + c}{a_2x + b_2}.$$

Determine, by standard methods, all the asymptotes of this function f, and compare the obtained results with the above. Consider, in particular, the case when $D = 0$, i.e. $c = (a_2b_1b_2 - a_1b_2^2)/a_2^2$.

5. Suppose that the equation $P(t) = 0$ has no solutions, which is equivalent to: $a_3 \neq 0$, $a_2^2 - 4a_1a_3 < 0$ or $a_3 = a_2 = 0$, $a_1 \neq 0$. In this case the curve (4) has no asymptotes.

6. Having in mind the above considerations, it is not difficult to determine which curve is represented by the equation (4), depending on the polynomials P and Q, defined by (6) and the determinant D, defined by (10).

We distinguish between the following cases.

1. The equation $P(t) = 0$ has two solutions α and β.

1.1. If $D \neq 0$, the equation (4) represents a hyperbola with asymptotes (11).

1.2. If $D = 0$, the equation (4) represents two straight lines which intersect and whose equations are (11).

2. The equation $P(t) = 0$ has exactly one solution α.

2.1. If $a_3 \neq 0$, $Q(\alpha) \neq 0$, the equation (4) represents a parabola.

2.2. If $a_3 \neq 0$, $Q(\alpha) = 0$, the equation (4) represents two parallel straight lines, one straight line or the empty set of points.

2.3. If $a_3 = 0$, $D \neq 0$, the equation (4) represents a hyperbola with asymptotes (16) and (17).

2.4. If $a_3 = 0$, $D = 0$, the equation (4) represents two straight lines which intersect and whose equations are (16) and (17).

3. The equation $P(t) = 0$ has no solutions.

3.1. If $a_3 \neq 0$, the equation (4) represents an ellipse (special cases: circle, point) or the empty set.

3.2. If $a_3 = 0$, $b_2 \neq 0$, the equation (4) represents a parabola.

3.3. If $a_3 = b_2 = 0$, the equation (4) represents two vertical straight lines, one vertical straight line or the empty set.

7. We can use the same idea to find the asymptotes of the curve $F(x, y) = 0$, where $F(x, y)$ is a polynomial in x and y of degree higher than 2. Namely, in order to find the asymptotes of the form $y = kx + n$, we write the equation $F(x, kx + n) = 0$ as a polynomial in x, and then we equate to zero the leading coefficient, then the next one, etc. and as we go on, we analyse the possible cases. We proceed similarly when we look for vertical asymptotes of the form $x = \lambda$.

Without going into details (which become more and more involved as the degree of $F(x, y)$ increases) we only work an example.

Example 3. We are looking for the asymptotes of the so-called Folium of Descartes, i.e. the curve

$$x^3 + y^3 - 3xy = 0. \tag{18}$$

Substituting $y = kx + n$ into (18) we get

$$(1 + k^3)x^3 + (3k^2n - 3k)x^2 + (3kn^2 - 3n)x + n^3 = 0, \tag{19}$$

and equating the coefficients of x^3 and x^2 to zero we obtain the system

$$1 + k^3 = 0, \qquad 3k^2n - 3k = 0,$$

with unique solution: $k = -1$, $n = -1$. Since the coefficient of x and the constant term of (19) do not both vanish for $k = n = -1$, we may conclude that $y = -x-1$ is the asymptote of (18). It is also easily verified that (18) has no vertical asymptotes.

REMARK 2. Determining the asymptotes of the curve $F(x, y) = 0$, where $F(x, y)$ is a polynomial in x and y, is a topic considered in many text-books on analysis; we mention, for instance, [1] and [2]. However, the methods used in those books are more complicated than the methods suggested here. In particular, the reader might find it interesting to compare the method applied in [1] to the curve (18) with the method of the above Example 3.

8. The suggested method for finding the asymptotes is efficient, in the sense that it enables us to arrive quickly and simply at the correct result. Hence, the method "works", but the question is why? Excepting the analogy with the hyperbola (2) and its asymptotes, we have given no other justification for the described procedure.

We shall now give an explanation which might be said to justify the method. The explanation will be intuitive, rather than formal, and we shall only be concerned with asymptotes of the form $y = kx + n$. The case of vertical asymptotes can be explained analogously.

If $y = kx + n$ is an asymptote of the curve $F(x, y) = 0$, this means that as $|x|$ becomes greater and greater, the curve $F(x, y) = 0$ becomes nearer and nearer to the line $y = kx + n$. Further, this means that the equation $F(x, kx + n) = 0$ must not become senseless as $|x|$ increases.

In order to illustrate this, consider again the Folium of Descartes (18). As before, from (1) and (18) we get (19), and after dividing by x^3 we get

$$1 + k^3 + (3k^2n - 3k)\,\frac{1}{x} + (3kn^2 - 3n)\,\frac{1}{x^2} + n^3\,\frac{1}{x^3} = 0. \tag{20}$$

As $|x|$ increases, the values of the expressions $1/x$, $1/x^2$, $1/x^3$ become more and more close to 0, and the equation (20) becomes very close to the equation $1+k^3 = 0$, i.e. $k = -1$. Now, for $k = -1$ the equation (19) becomes $(3n + 3)x^2 + (-3n^2 - 3n)x + n^3 = 0$, and after dividing by x^2

$$3n + 3 - (3n^2 + 3n)\,\frac{1}{x} + n^3\,\frac{1}{x^2} = 0,$$

and the last equation, as $|x|$ increases, becomes very close to the equation $3n+3 = 0$, i.e. $n = -1$.

Why is it that the curve

$$x^2 - 2xy + y^2 - x = 0 \tag{21}$$

does not have an asymptote of the form $y = kx + n$? From (1) and (21) we get

$$(1 - 2k + k^2)x^2 + (2kn - 2n - 1)x + n^2 = 0, \tag{22}$$

and after dividing by x^2:

$$1 - 2k + k^2 + (2kn - 2n - 1)\,\frac{1}{x} + n^2\,\frac{1}{x^2} = 0, \tag{23}$$

and as $|x|$ increases, (23) becomes very close to the equation $1 - 2k + k^2 = 0$, i.e. $k = 1$. However, for $k = 1$, (22) becomes $-x + n^2 = 0$, and after dividing by x: $-1 + n^2\,\frac{1}{x} = 0$. We see that as $|x|$ increases, the last equation becomes very close to the senseless equation $-1 = 0$.

REMARK 3. As we said before, this was only an intuitive explanation certainly not a proof, why the asymptotes can be determined by the exposed method. A formal proof, based upon approximative methods, can be constructed, but this is not the subject of this note.

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