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TRANSFORMING THE DISK MODEL OF HYPERBOLIC GEOMETRY TO THE UPPER HALF-PLANE MODEL

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Abstract. The isomorphism between the two Poincaré models of Hyperbolic Geometry is usually proved through a formula using the Möbius transformation. In this article, we present a geometrical procedure which transforms one model onto the other and leads to these transformations. This can be utilized in the educational process.

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1. Introduction

The fact that the disk model and the upper half-plane model of Hyperbolic Geometry are isomorphic, is usually proved through a formula using the Möbius transformation [1, p. 322], [2, p. 288], [3, p. 209]. In what follows, we explain how someone can end up with this formula. We present in a more 'geometrical sense', utilizing a suitable inversion, how we can get the upper half-plane model from the disk model (or conversely). This may be helpful to math teachers and especially to students of mathematics who study related subjects. We present first the properties of inversions in a circle and all the required elements of Hyperbolic Geometry in the Poincaré disk model, so that the article can easily be read also by readers who have little familiarity with this kind of geometry. Subsequently, we explain, presenting an explanatory figure, a transformation of the disk model to the upper half-plane model and how the geometric features of the latter arise from the corresponding features of the first one. In the end, using the basic algebra of complex numbers, we derive algebraic formulas mapping the disk model onto the upper half-plane model.

2. Inversions in circles

Definition. Let $\gamma(O, r)$ be a circle with center O and radius r. A *circle inversion* is the transformation of the plane which maps a point A onto the point A' of the ray OA satisfying the condition $(OA) \cdot (OA') = r^2$.

The main properties of an inversion are summarized in the following proposition [2, pp. 194–216]:

Proposition.

1. *An inversion with respect to the circle* $\gamma(O, r)$ *transforms*:
   a. *A circle that does not pass through* O *onto a circle that does not pass through* O*. The image of the center* K *of a circle is not the center of its image.*
   b. *A line that passes through* O *onto a line that passes through* O*. Only the common points with the circle* γ *are invariant.*
   c. *A circle that passes through* O *onto a line not passing through* O *and conversely.*
2. *Inversions preserve angles. Angles are transformed to equal angles with opposite direction, like a reflection. The inverses of two orthogonal lines (straight lines, circles etc.) are orthogonal lines.*
3. *The cross-ratio* $(AB, CD) = \dfrac{(AC)(BD)}{(AD)(BC)}$ *of four points* A, B, C, D *is invariant under inversions.*

3. The disk model

The disk model has been analyzed in the references [1], [2] and is presented in the following Figure 1. Let $\Omega(O, 1)$ be the unit circle which defines the model. The plane of the model is the interior D of the unit circle Ω and its points are the interior points of the circle.

Its lines are either open diameters of Ω (a, b in Figure 1), or open arcs of circles orthogonal to Ω (c, d). Lines that intersect on Ω, as a and c, are limiting parallel lines.

Angles are Euclidean, defined through the tangent ray or rays in case one or two lines are arcs. (??)

Let X and Y be two points on a line AB of the model. We consider the cross-ratio $(XY, AB) = \dfrac{(XA)(YB)}{(XB)(YA)}$, where the lengths of the Euclidean segments XA, XB, YA, YB are measured with the Euclidean metric. The metric of the model is defined as

$$d_H : D \times D \to \mathbb{R}^+, \quad \text{where } d_H(X, Y) = |\ln(XY, AB)|.$$

Circles have the shape of Euclidean circles inside $\Omega(s)$. Every circle in the interior of Ω is a hyperbolic circle. Its hyperbolic center coincides with its Euclidean center only if that point is O.

Lobachevskii defined horocycles (or boundary lines) as curves such that all the perpendicular bisectors of their chords are limiting parallels. Let XA be a ray. In Euclidean Geometry the horocycle with axis XA is the straight line perpendicular to XA at the point X. On the contrary, in Hyperbolic Geometry any three points of a horocycle are never collinear. In the disk model they have the shape of Euclidean circles tangent internally to the circle of the model excluding the point of tangency (p, q). Every such circle is a horocycle.

In Hyperbolic Geometry, the locus of points equidistant from a straight line is not a line. For every given line in the model and a distance d_H, the equidistant curve has the shape either of an open arc (and it is symmetric with respect to the hyperbolic line) not orthogonal to the circle of the model (m, n), or of an Euclidean open chord (k), excluding diameter lines, which is symmetric to the equidistant arc-curve passing through O. All such arcs and chords are equidistant curves of hyperbolic lines. In Figure 1, k and m are equidistant curves of the line d and n is a equidistant curve of the line a.

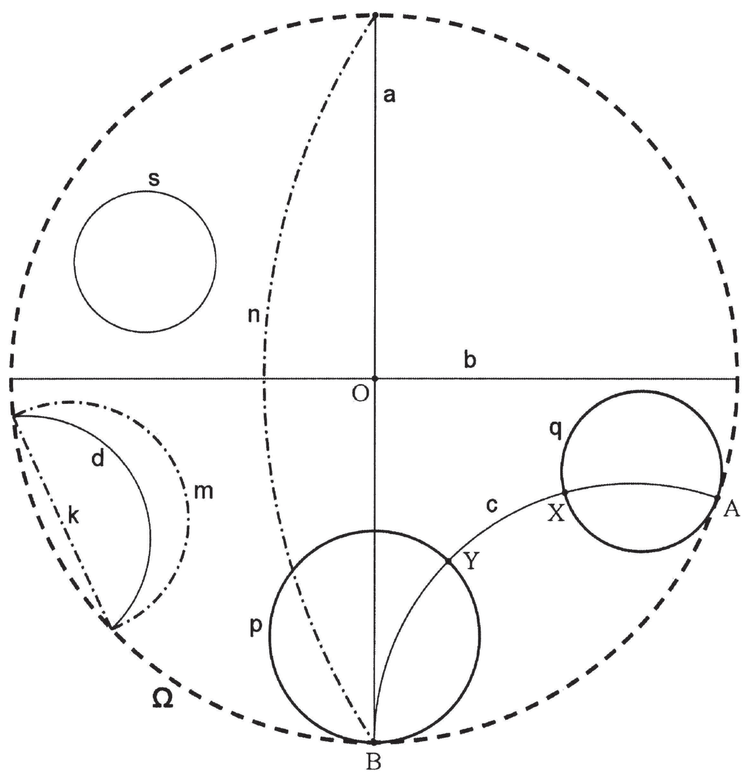

Figure 1. The disk model

4. Transformation of the disk model into the upper half-plane model

We consider the circle δ with center at the point B and radius equal to the diameter of Ω. The inversion with respect to δ transforms the disk model D onto the upper half-plane model, which is analyzed in the reference [3].

The inverse of the boundary circle Ω is the boundary line Ω' (Proposition 1c). The inverses of the points of plane D are the points of one of the two half-planes into which Ω divides the Euclidean plane. We call it the upper half-plane.

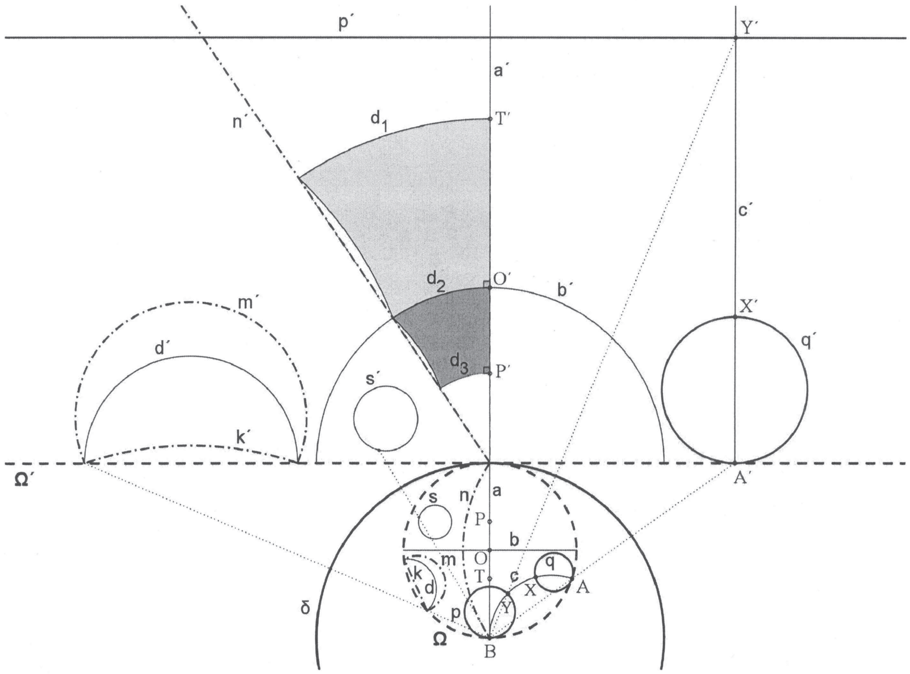

Figure 2. The transformation

Line a of the disk model is transformed into the open ray a' perpendicular to Ω' (Proposition 1b-2), line b into the open semicircle b' orthogonal to Ω' (Proposition 1c-2), line c into the open ray c' perpendicular to Ω' (Proposition 1c-2) and line d into the open semicircle d' orthogonal to Ω' (Proposition 1a-2). Such semicircles orthogonal to Ω' and open rays perpendicular to Ω' are lines in the upper half-plane model. The limiting parallels are two open semicircles tangent to Ω', or a semicircle and an open ray tangent on Ω', or two open rays. Therefore, a trebly asymptotic triangle (a triangle whose sides are limiting parallels) as the one in Figure 3a, has in the upper half-plane model one of the two shapes shown in Figure 3b.

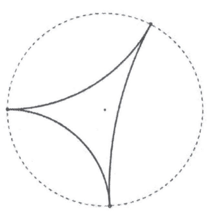

Figure 3a

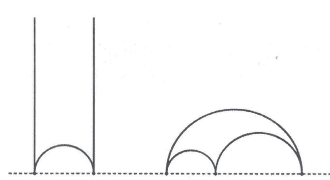

Figure 3b

In the same way we get the following.

Angles are Euclidean, defined through the tangent ray or rays in the case one or two lines are semicircles (Proposition 2). Segments can be measured using the metric d_H (Proposition 3). In the case a segment lies on an open ray where one of the two boundary points is at infinity (the image of B in Figure 2), we get

$$\begin{aligned} d_H(X', \Psi') &= \big|\lim_{B'\to\infty}(X'Y', A'B')\big| = \lim_{B'\to\infty}\Big|\ln\frac{(X'A')(Y'B')}{(X'B')(Y'A')}\Big| \\ &= \Big|\lim_{B'\to\infty}\Big(\ln\frac{(X'A')}{(Y'A')} + \ln\frac{(Y'B')}{(X'B')}\Big)\Big| \\ &= \Big|\ln\frac{(X'A')}{(Y'A')} + \ln 1\Big| = \Big|\ln\frac{(X'A')}{(Y'A')}\Big|. \end{aligned}$$

Horocycles have the shape either of Euclidean circles tangent to the boundary line Ω' excluding the point of tangency (q'), or of lines parallel to Ω' (p').

Equidistant curves have the shape either of an open arc not orthogonal to Ω' (k', m'), or of an open ray not perpendicular to Ω' (n'). The open ray n' is an equidistant curve of the line a' and the open arcs k' and m' of the line d'.

Similarly to the disk model, in the upper half-plane model, as we approach the boundary line Ω', short Euclidean lengths correspond to long hyperbolic lengths. In Figure 2, the points P and T in the disk model are symmetrical with respect to the center O of the disk, hence $d_H(P, O) = d_H(O, T)$. The same occurs for the corresponding lengths of the segments $P'O'$ and $O'T'$ in the upper half-plane. Also, since the open ray n' is an equidistant curve of the line a', the three distances d_1, d_2 and d_3 are equal. In the same figure, the two shadowed Saccheri quadrilaterals are equal, as well. Recall that a quadrilateral $ABCD$ is called a Saccheri quadrilateral with basis AB, if $\angle A = \angle B = 90°$ and $AD = BC$. In Hyperbolic Geometry, $\angle C = \angle D < 90°$ applies.

5. The algebraic formula for transformation

The points of the disk model are the points of the set $D = \{z : |z| < 1\}$, i.e., the interior points of the unit circle Ω. The points of the upper half-plane model are the points of the set $H = \{z : \mathrm{Im}(z) > 0\}$.

Proposition. *The anti-Möbius transformation $f\colon D \to H$ given by the formula*

$$w = f(z) = \frac{-2i\bar{z} + 2}{\bar{z} - i}$$

maps D one-one onto H.

Proof. We consider a coordinate system with the center O of the boundary circle Ω as the origin, and the axis of the abscissas parallel to the boundary line Ω'. The circle δ of inversion has the center at the point $B(-i)$ and the radius $r = 2$. The inversion with respect to δ maps a point $X(z)$ onto the point $X'(u)$ so that

$$(BX')\cdot(BX) = 2^2 \iff |u + i|\cdot|z + i| = 4.$$

The point X' lies on the ray OX, so

$$u + i = k \cdot (z+i), \quad k > 0 \implies k = \frac{4}{|z+i|^2}.$$

Thus we get

$$u = \frac{4 \cdot (z+i)}{|z+i|^2} - i = \frac{4 \cdot (z+i)}{(z+i)\cdot(\bar{z}-i)} - i = \frac{-i\bar{z}+3}{\bar{z}-i}.$$

Under the inversion, a point $X(z)$ on the circle Ω is mapped onto the point $X'(u)$ on the parallel line to the axis of the abscissas with $\mathrm{Im}(u) = 1$. A point $X(z)$ of D is mapped onto a point $X'(u)$ of the half-plane with $\mathrm{Im}(u) > 1$. The translation $w = u - i$ maps these points onto the points of the upper half-plane H. Hence, we conclude that

$$w = u - i = \frac{-i\bar{z}+3}{\bar{z}-i} - i = \frac{-2i\bar{z}+2}{\bar{z}-i}.$$

The geometrical construction ensures that the transformation is one-one onto H, a fact that can also be easily proved algebraically. Furthermore, since its formula is equivalent to

$$w = \frac{(-2i\bar{z}+2)\cdot(z+i)}{(\bar{z}-i)\cdot(z+i)} = \frac{2(z+\bar{z})}{|z+i|^2} + \frac{2(1-z\bar{z})}{|z+i|^2}\, i,$$

we can easily observe that $|z| < 1 \iff \mathrm{Im}(w) > 0$. ■

Remark 1. Using the product of a transformation which maps H one-one onto itself with f, we get other transformations that map D one-one onto H. An example is the transformation $g(z) = -\bar{z}$ (a reflection with respect to the axis of ordinates, $z \mapsto -\bar{z}$), which maps H one-one onto H. The product

$$h(z) = g(f(z)) = 2i \cdot \frac{i-z}{i+z}$$

is the Möbius transformation mapping D one-one onto H.

Remark 2. We can also use any point of Ω as the center of the inversion circle δ. If, for example, the center is the image of the complex number $z = 1$, following the same procedure, we get that the anti-Möbius transformation

$$f(z) = \frac{2\bar{z}+2}{\bar{z}-1}$$

maps D one-one onto the 'left' half-plane. Its product with $g(z) = \frac{1}{2}(-i\bar{z})$, which consists of a reflection in the axis of abscissas ($z \mapsto \bar{z}$) and a dilative anti-clockwise rotation by $90°$ ($z \mapsto \frac{1}{2}(-iz)$), is the Möbius transformation

$$h(z) = i \cdot \frac{1+z}{1-z}$$

which maps D one-one onto H.

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