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BORROMEAN RINGS AND MATHEMATICAL STORYTELLING

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Abstract. Diverse mathematical concepts like Gray codes, linking number, Borromean and Brunnian rings, mapping degree (and others), are incorporated in mathematical stories which should make the subject more attractive and mathematics more accessible and easier to comprehend.

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Key words and phrases: mathematical storytelling; Borromean rings; Brunnian rings; Gray codes; linking number.

1. Introduction

With the advent of *Wikipedia, MathOverflow*, and other searchable resources, the need for comprehensive reference texts has perhaps diminished and may continue to do so. To some extent, the demand for the classical definition-theorem-prof text is also somewhat lessened, since one can look up a standard proof on-demand. What has not been eliminated is the need for a story with drama and characters. Overarching narratives are not easily modularized; connections and applications between areas require a global view.

Robert Ghrist,
Elementary Applied Topology
`https://www.math.upenn.edu/~ghrist/notes.html`

If what Robert Ghrist says about writing mathematics is true, then something similar, possibly to a much higher degree, should be true for teaching of mathematics. It is plausible that in the near future the teaching of mathematics will undergo a major change with the standard student-teacher relationship modified or given a different meaning.

Many teachers of mathematics are well aware that mathematical storytelling is often a valuable method, both for the popularization of mathematics and for motivating students to study the subject. Here we collect a few examples from our own practise, developed and used in our "Živa matematika" (Math alive) project, supported by the Serbian Ministry of Science, Technology and Education.

2. TV plays and Gray codes

Figure 1 exhibits the essentially unique balanced, 4-bit Gray code. Recall that a Gray code is an ordering of the binary numeral system (all 0-1 words of given length) such that two successive words differ in only one bit (binary digit). Gray codes have been widely used to facilitate error correction in digital communications. The original motivation was to minimize the effect of error in the conversion of analog signals to digital and these codes are still used for this purpose today.

The code shown in Figure 1 is balanced in the sense that each digit (associated to one of the 4 tracks) changes equal number of times.

A purely mathematical application of this code can be found in [3] where this code describes an equipartition of the curve

$$\Gamma = \{(\cos(t), \sin(t), \cos(2t), \sin(2t)) \mid 0 \leq t \leq 2\pi\} \subset \mathbb{R}^4$$

into 16 arcs of the same 'length' by 4 hyperplanes in $\mathbb{R}^4$.

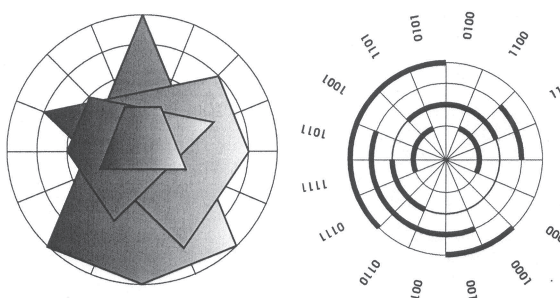

Fig. 1. The unique balanced, 4-bit Gray code

There is one more attractive way to describe this code. Recall the TV play "Quad" by S. Beckett [1]

https://www.youtube.com/watch?v=LPJBIvv13Bc&t=45s

https://www.youtube.com/watch?v=Q7DZgHA6798&t=67s

https://www.youtube.com/watch?v=gXyIkFd4z5M&t=215s

where "*. . . Four actors, whose colored hoods make them identifiable yet anonymous, accomplish a relentless closed-circuit drama . . .* " (R. Frieling).

As a variation on the theme of "Quad" one can design a scheme for a play based on the balanced Gray code. In this scheme the stage begins and ends empty, 4 actors enter and exit one at a time, running through all 16 possible subsets, and each actor is supposed to enter (leave) the stage precisely 2 times.

I learned about the link between Gray codes and play scenarios from Donald Knuth [2]. In Exercise 65 (on page 314) he attributes this idea to Brett Stevens and discusses other Gray codes that appear naturally in this context.

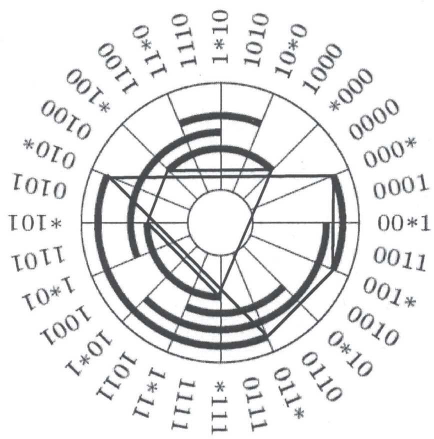

Fig. 2. Knuth's picture of the balanced, 4-bit Gray code

2.1. Additional symmetry of the balanced Gray code

An image of the balanced 4-digit Gary code (reproduced here in Figure 2) appears on page 293 in [2, Section 7.2.1.1] (see Fig. 33. (b)). By comparing this image with our Figure 1 one observes that Knuth's image is rotated counterclockwise through the angle of $\frac{3\pi}{8}$. (Images of a deltoid (kite) and an isosceles trapezoid are added in Figure 2 to make the comparison easier.)

Our Figure 1, taken from [3], reveals an additional symmetry of this code which was apparently not well known or emphasized before. This symmetry was one of the key observation in [3], leading to a new 4-equipartition result for a class of measures with symmetries.

3. Topology of washing machines

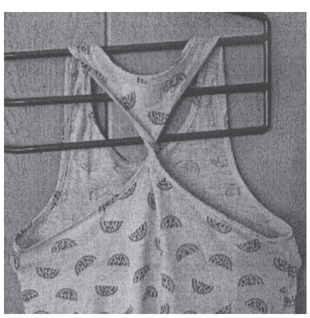

Fig. 3. Mysterious T-shirt

Fig. 4. Topological T-shirt

Žarko M. is a friend of mine and a fellow mathematician. One day he came with a T-shirt (Figure 3) claiming that it was a perfectly normal piece of garment before it was put in a washing machine.

If you take a closer look you will notice that one of the bretelles is twisted by 360°.

A perfectly legitimate topological model of the T-shirt shown in Figure 3 is a closed twisted ribbon with two holes, resembling the one shown in Figure 4. More precisely, the closed ribbon from Figure 4, after some stretching, can be turned into the T-shirt from Figure 3.

Now we observe that the boundary of the twisted ribbon consists of two circles which are linked with the linking number ± 1.

Recall that if $e_i : S^1 \to \mathbb{R}^3$ are two embeddings such that $L_1 \cap L_2 = \emptyset$ and $L_i = \mathrm{Image}(e_i)$, than the linking number $lk(L_1, L_2)$ is defined as the degree of the map,

$$\phi : S^1 \times S^1 \to S^2, \qquad \phi(s,t) = \frac{e_1(s) - e_2(t)}{\|e_1(s) - e_2(t)\|}. \tag{1}$$

Conclusion. In a (topological) washing machine a T-shirt T_1 which enters the machine and the T-shirt T_2 that leaves the machine (after the end of the washing cycle) are isotopic! Therefore if two boundary circles of T_2 are linked with a positive linking number, the same must be true for the T-shirt T_1. Therefore, the T-shirt was already twisted when it was put in the washing machine.

Remark: As visible from the formula (1) the degree (= the linking number) is not changed even if the T-shirt is allowed to self-intersect, during the washing cycle, provided the self-intersection is in the interior of the ribbon. What happens if regular homotopies (homotopies through immersions) are permitted!?

4. Borromean and Brunnian rings

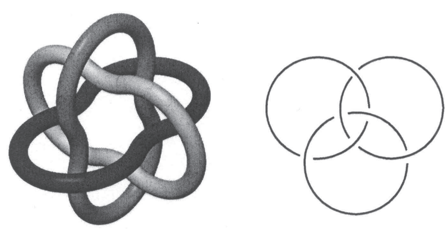

Fig. 5. The emblem (logo) of the International Mathematical Union

Borromean rings. Figures 5 and 6, as a symbol of unity, may be found in different cultures and in different contexts. They appear on the coat of arms of the aristocratic Borromeo family from Northern Italy, symbolize the Christian trinity in early religious texts, appear as the *"Valknut"*, the knot of slain warriors in Scandinavian mythology. They also appear in more mundane contexts, from girls pony tails and braids, to a commercial where the rings represent the purity, body, and flavor of a beer,

https://en.wikipedia.org/wiki/Borromean_rings `http://mathworld.wolfram.com/BorromeanRings.html`

`https://www.youtube.com/watch?v=qxJY4lna5q0`

Borromean rings symbolize a fragile unity where if one of the rings is broken then the remaining two rings are no longer held together. This is a very interesting phenomenon which opens many possibilities for interpretation and storytelling.

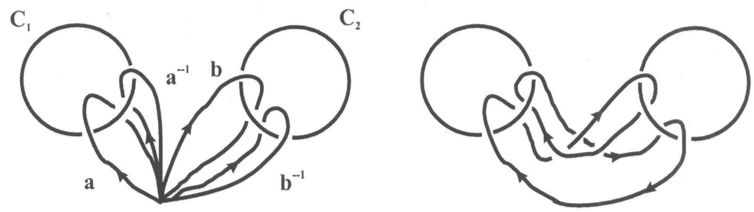

Fig. 6. The commutator $[a, b] = aba^{-1}b^{-1}$ as the Borromean "genetic code"

A natural question is whether there exist 'fragile links' with four, five or more rings. In other words is it possible to create a system of n rings (for each $n \geq 3$) which is unstable in the sense that after the removal of any of the rings the whole system falls apart. The answer is affirmative and such systems of rings (links) are called *Brunnian*.

The mathematics of Borromean (Brunnian) links is very interesting, involving an attractive mixture of Topology and Group theory. Some of the relevant key words and phrases are *fundamental group, braiding, commutator, links and knots*, etc.

The reader will have a lot of fun browsing the *Google*, either for study or for pure pleasure. Here we focus on a 'practical problem' where these objects naturally appear.

4.1. Paradoxical hanging of pictures

Retired professor Simeon (Sima) Čvorović was very proud of his collection of paintings and other artwork, which occupied all the walls in his house. Our story begins when professor Sima started preparation for a family gathering. He liked these meetings, with grandchildren and their friends running around, touching everything, including his precious paintings. It is important to say that Sima was not one of those grumpy types. Quite the contrary, he and his wife enjoyed these meetings very much and the family unity was highly valued and cherished in their home.

However, he wanted to make some precautions (this time) after the last year's experience, when one of the youngsters was caught trying to climb the wall and remove his precious "White square hyperbolic tessellation".

With a flesh of inspiration, he decided to hang all his paintings anew. This time he decided to use two or more nails for each of the paintings, especially for those pictures with abstract mathematical content (which somehow attracted the most

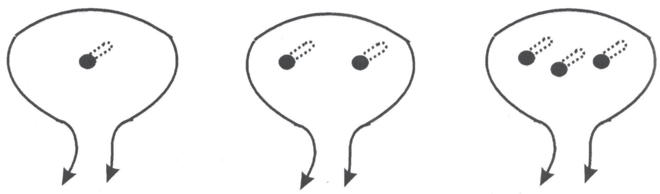

Fig. 7

and puzzled his little guests). The commutator $[a, b] = aba^{-1}b^{-1}$ as the Borromean "genetic code"

Sima was very pleased with his idea. Instead to use a single nail, as in Figure 7 on the left, professor Čvorović opted for more reliable variants, as in Figure 7 in the middle and on the right. Now he was sure that even if some of the nails gets loose and falls out (by accident or by some external force), then the painting will still stay safely in its place, hanging on the remaining nails.

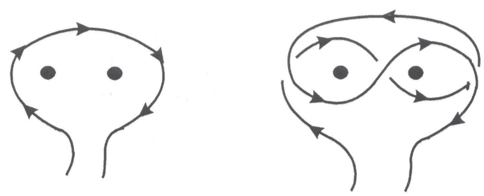

Fig. 8

Happily experimenting with this idea Sima realized that, rather then simply using two nails (as in Figure 8 on the left) a much better and apparently safer way of hanging a picture with two nails is shown in Figure 8 on the right.

Very pleased with his ingenuity he hanged the picture like that and proudly invited his wife to make an experiment by removing one of the nails from the wall.

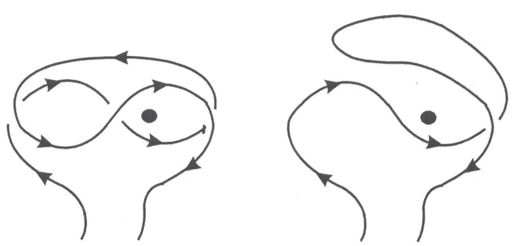

Fig. 9

However he fainted and almost fell unconscious after realizing that the right nail is perfectly useless, and that the picture would fall down upon the removal of the left nail. Indeed, this is confirmed by a simple 'visual experiment', as shown if Figure 9.

"It is good after all that the left nail does the job", thought Sima. However, to his amazement, something similar happens if one removes the right nail. Indeed, this can be also checked by a simple 'visual experiment', or even better, by looking to the Figure 9 in the mirror.

Totally dismayed and disappointed, professor Čvorović exclaimed: "What on Earth is keeping the picture on the wall if none of the two nails is doing the job!?"

Somehow it appears that the whole system (two nails together with the carefully arranged rope) is stable (keeps the painting on the wall) in spite of the fact that neither of the nails alone is sufficient.

4.2. Paradoxical words

An *alphabet with n symbols* is an arbitrary collections of n symbols (for example letters, numbers, etc.) where each symbol has two versions, the *upper case* (capital) and *lower case* (small letters). For example $\mathcal{A} = \{A, a, B, b, C, c\}$ is an alphabet with three symbols.

A *word* in an alphabet is an arbitrary sequence of symbols in the chosen alphabet. Two words w_1 and w_2 can be written one after another (concatenation of words), creating a new word $w = w_1w_2$.[1]

Concatenation is an operation on words. We alow an *empty word* $\emptyset$ which has the property $w = \emptyset w = w\emptyset$.

We say that a word is *shrinkable* if the upper case and lower case of the same symbol appear side by side (adjacent) in the word. For example $abCccDaA$ is shrinkable. A shrinking of a shrinkable word $w = \dots xX \dots = w_1xXw_2$ (where x and X are the capital and the small case instance of the same symbol) is a new word $w' = w_1w_2$ obtained by removing adjacent symbols x and X.

Example: For example, the word $w = abCccDaA$ can be shrunk in two different ways to either $w' = abcDaA$ or $w'' = abCccD$, depending on whether we remove the pair Cc or the pair aA.

Shrinking operation can be repeated by a successive removal of adjacent instances (of the upper and lower case) of the same symbol. (Note that the operation of shrinking words is nothing but the usual operation of cancelation where we interpret upper case letters as inverses, i.e. if we instead of A, B, C we write a^{-1}, b^{-1}, c^{-1}.)

Here is an example of the shrinking operation on a (multiply) shrinkable word.

$$aCb(aA)BcC \longrightarrow aCbB(cC) \longrightarrow aC(bB) \longrightarrow aC$$

A word is *completely shrinkable* if it can be shrunk to the empty word $\emptyset$. For example the word $aABb$ is completely shrinkable.

We say that a word is *nonshrinkable* if it does not alow any shrinking at all. A word is *paradoxical* if,

[1] Students can be reminded that our words resemble very much usual monomials with an important difference that our multiplication (concatenation of words) is not commutative!

1. it is nonshrinkable and,
2. if we choose a symbol that appears in the word and remove all (upper case and lower case) instances (appearances) of this symbol in the word, we obtain a shrinkable word.

For example, the word $AbaB$ is paradoxical since both Aa and bB are shrinkable words.

Exercise. We already observed that the word $AbaB$ is paradoxical. Construct a paradoxical word in the alphabet $\{a, A, b, B, c, C\}$. More generally, show that such words exist in any alphabet with $n \geq 2$ symbols. In other words show that for any collection of n capital and small letters $\mathcal{A} = \{a_1, A_1, a_2, A_2, \dots, a_n, A_n\}$ there exists an unshrinkable word $w = x_1x_2\dots x_k$, written in this alphabet, ($x_i \in \mathcal{A}$ for each $i = 1, \dots, k$), such that if we choose an arbitrary index $j = 1, \dots, n$ and remove all letters a_j and A_j from the word w, we obtain a completely shrinkable word.

4.3. Words are pictures, and pictures are words!

Let us show how paradoxical words can be associated with the paradoxical hanging of the pictures. More importantly, we show how paradoxical words can be used as mnemonic rules for memorizing paradoxical hanging of paintings (which after very little practise allows us to do it without any hesitation.)

(1) Each nail is associated a symbol in some alphabet. For example we associated with Figure 7 (on the right) the alphabet $\mathcal{A}_3 = \{a, A, b, B, c, C\}$ while the Figure 8 (on the right) is associated the alphabet $\mathcal{A}_2 = \{a, A, b, B\}$.

(2) Let us suppose that somebody has already hanged the picture on the wall and now we want to take a closer look. We carefully traverse the rope (starting at the (left) point where the rope is attached to the painting). Each time we move above a nail (associated to a symbol x) we write x if we move from *left to the right*, otherwise (if while traversing the rope we move *from right to the left*) we write X (the upper case of the symbol x).

Let us look at some examples. Figure 7 (on the right) is associated the word abc. Indeed, by moving slowly along the rope, we moved above the leftmost nail (from left to the right) and consequently wrote a. After that we moved above the second nail (again from left to the right), which was recorded (so far) as ab. Finally we obtained abc, by taking into account that in the final stage of the moving along the rope we traversed the third nail from left to the right.

Figure 8 is by the same general rule associated the word $aBAb$. Let us emphasize that a new letter (symbol) is added only if we move exactly above one of the nails. All other cases, including the part of the rope that is strictly bellow one of the nails (!), are ignored.[2] Recall that the word $aBAb$ is paradoxical.

The following observation is intuitively very convincing. However, we do not offer a formal proof. Rather, we use it to enhance our understanding of paradoxical hanging of pictures and as a guide form experiments.

[2] This makes sense since, after all, we focus to those parts of the rope which are (potentially) useful in "fighting the gravitation".

Observation. Each paradoxical word produces a paradoxical hanging of a picture. (Here we tacitly assume that we avoid self-knotting of the rope, say by following a self-avoiding procedure where at each crossing the rope moves upwards.) Conversely, if we have a paradoxical hanging of a picture, we can "read off" the associated paradoxical word.

As an exercise, the reader is invited to examine Figure 10 with a more complex paradoxical hanging of a picture with 4 "nails" (cylinders). Check that the associated paradoxical word is,

$$(aBAb)(cDCd)(BabA)(DcdC).$$

(The parentheses are added for some extra clarity.)

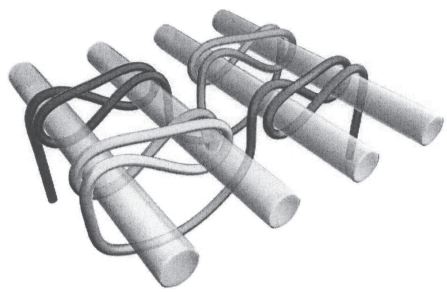

Fig. 10

The reader who is not yet convinced is cordially invited to look at the animation,

nhttps://www.youtube.com/watch?v=rSdMvkY3NAc

https://vimeo.com/9802595

created by Dušan Živaljević. For this and other Dušan's animations and models the reader is referred to,

http://www.rade-zivaljevic.appspot.com/

4.4. Exercises

1. If w is a word in some alphabet then its "inverse word" w^{-1} has the property that ww^{-1} is completely shrinkable (reducible to the empty word). Convince yourself that each word has a unique inverse word, for example if $w = abC$ then $w^{-1} = cBA$, if $w = a$ then $w^{-1} = A$, if $w = B$ then $w^{-1} = b$, etc.

2. Let $w = x_1x_2\dots x_n$ be a word of length n in some alphabet

$$\mathcal{A} = \{a, A, b, B, c, C, \dots\}.$$

Show that the inverse word w^{-1} of w is obtained if we read the word w backwards and each letter x_i is replaced by its inverse x_i^{-1},

$$w^{-1} = x_n^{-1}x_{n-1}^{-1}\dots x_1^{-1}.$$

3. *The commutator* $[X, Y]$ of words $X = x_1 \dots x_n$ and $Y = y_1 \dots y_m$ is defined as the word,

$$[X, Y] = XYX^{-1}Y^{-1} = x_1 \dots x_n y_1 \dots y_m x_n^{-1} \dots x_1^{-1} y_m^{-1} \dots y_1^{-1}.$$

Show that if X and Y are paradoxical words and the words XY i YX^{-1} are nonshrinkable, then the commutator $[X, Y]$ is also a paradoxical word.

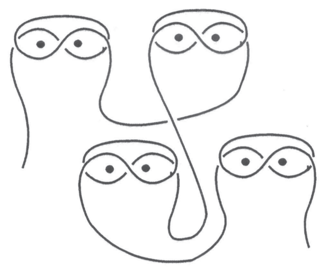

Fig. 11. "Four Owls"

4. Analyze the hanging of a picture on a wall with 8 nails exhibited in Figure 11 (Four Owls configuration). Associate variables x_1, x_2, x_3, x_4 (from left to the right) to the nails in the upper row and similarly, the variables y_1, y_2, y_3, y_4 to the nails in the lower row. Convince yourself that the word w associated to the "Four Owls configuration" is the following,

$$w = [x_1, x_2^{-1}][x_4^{-1}, x_3][y_2^{-1}, y_1][y_3, y_4^{-1}].$$

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