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GEOMETRY AND MATHEMATICAL SYMBOLISM OF THE 16TH CENTURY VIEWED THROUGH A CONSTRUCTION PROBLEM

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Abstract. This paper represents a construction problem from *Problematum geometricorum IV* written by Simon Stevin from Bruges in 1583. The problem is used for illustrating the geometry practice and mathematical language in the 16th century. The large impact of Euclid and Archimedes can be noted. In one part of the construction, Stevin expressed the need for using numbers for greater clarity. Hence, the link with the work of Descartes and the further geometry development is pointed out.

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1. Introduction

From the times of Ancient Greece until the 16th century there was no significant geometrical contribution that would expand the existing geometrical knowledge. Only in the second half of the 16th century we can find geometry works of Simon Stevin from Bruges (1548–1620). It is believed that his influence was neglected in the history of mathematics and that his name should be mentioned together with the name of his contemporary Galileo Galilei, from whom Stevin was a whole generation older (16 years) [8].

It is considered that Stevin, as a predecessor of Descartes, prepared a path for introducing correspondence between numbers and points on the line by studying the 10th book of Euclid's Elements and translating it to numbers [9]. Besides his work on the *Problematum geometricorum libri V*, he wrote *Tomus secundus de geometriae praxi* (in 1605) which is: ... *different from the Problemata geometrica and inferior to it; it is also a collection of geometrical problems but it is not arranged as logically as the former; it was chiefly made to complete the Prince's geometrical training* [6, p. 261].

Problematum geometricorum libri V is a collection published in 1583, which consists of five books with different geometry problems. The fourth book deals with the construction problem which will be presented in this paper.

2. The terminology used in Problematum geometricorum, book IV

The fourth book of *Problematum* presents the construction of a solid, similar to the one given solid and equal to the other given solid. It is a problem analogous to the problem in plane proved as Proposition 25 in the Sixth Book of Euclid's Elements (here and after referred to as Proposition VI.25). Proposition VI.25 states: *To construct one and the same figure similar to a given rectilinear figure and equal to another given rectilineal figure*. We can see that the terminology used to formulate VI.25 is the same as in formulation of a problem in *Problematum*. Stevin uses terms equal and similar in the same way as Euclid does in formulation of VI.25.

Common notion 5 (or 7) in the First Book of Elements states that: *Things which coincide with one another are equal to one another*. But that is not the equality that Euclid uses in formulation of VI.25. It is the equality used in formulation, for example, of Pythagoras' Theorem. It allows decomposition of a figure and rearrangement of its pieces. In this way, the resulting figure will not be equal to the starting one in the sense of Common notion 5, but its pieces will be. The equality of the starting and the resulting figure is stated by Common notion 2: *If equals be added to equals, the wholes are equal* [7], [1].

Similar rectilinear figures are defined in the Sixth Book of Elements in Definition 1 as *figures that have their angles equal and the sides about the equal angles proportional*. Stevin's problem stated above deals with similar solids, and its definition is already given in the XI Book as Definitions 9 and 24. The definitions respectively define *similar solid figures* as those contained by similar planes equal in multitude, and *similar cones and cylinders* as those in which the axes and the diameters of the bases are proportional. Despite of the existence of these definitions, and the definition of a solid, given as *that which has length, breadth, and depth* (Definition 1 in Book XI) Stevin gives a description of geometrical solid in the following note:

> *We call Geometrical solid a solid which is constructed by a Geometrical law, such as Sphere, a Segment of a sphere, a Sector of a sphere, Spheroids, a Segment of a spheroid, a Conoid, a conoidal Segment, a Column, of which there are two kinds, viz. the Cylinder and the Prism, a Pyramid, the regular Solids, the augmented regular solids, the truncated regular solids; the construction of all of which will be dealt with fully in our Geometry. Indeed, we call these solids and others which are constructed Geometrically Geometrical solids to distinguish them from bodies such as, generally, stones, fragments of stones, and the like* [9, 313].

This expansion of Euclid's definitions is due to the fact that Stevin knew the work of Archimedes and his predecessors, so Stevin includes spheroids, conoids, etc., which are not mentioned in the Elements. His construction is presented through examples of cone and cylinder. Pyramid is used for proof through numbers. Most of Stevin's work on this problem is based on the work of Archimedes *On the Sphere* *and Cylinder I*, and if all propositions needed for the construction and its proof should be stated here, almost whole book would be presented.

"Constructional tool" for Problematum IV

Beside the "constructional tool" from the postulates and proposition in Elements which is in use nowadays (e.g. to bisect a given finite straight line, to draw a straight line at right angles to a given straight line from a given point on it, etc.), in *Problematum IV* Stevin uses the construction of the third proportional (Euclid's Elements VI.11), the fourth proportional (Euclid's Elements VI.12) and the mean proportional (Euclid's Elements VI.13). The construction of two mean proportionals Stevin exposes in *Problematum*, as the first problem. Since the above mentioned constructions are well known, we will just give a short reminder in modern symbolism for the third, the fourth, mean and two mean proportional(s). The third proportional for the given line segments a and b is a line segment c for which is $a : b = b : c$. The fourth proportional for three given line segments a, b and c is a line segment d for which is $a : b = c : d$. Mean proportional for two line segments a and b is the line segment c for which $a : c = c : b$. The construction of these proportionals is sufficient for the analogue problem in plane resolved by Eucild in VI.25, but for Stevin's problem in space, the construction of two mean proportioanls is needed. For two given line segments a and b two mean proportionals are line segments c and d for which $a : c = c : d = d : b$.

3. The first three problems of Problematum IV

Before formulating the construction problem, Stevin first deals with three geometry problems necessary for the construction. The problems are represented in the way they are exposed in Elements, only with the explicit indication what is given, what is required, what construction, proof and conclusion are.

The First problem

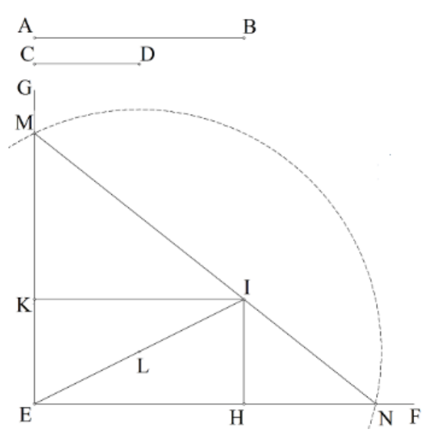

Figure 1. Construction of two mean proportionals

The first problem is the problem of construction of two mean proportionals for two given line segments. The construction is conducted in the manner of Hero. Stevin refers that the proof should be found in Eutocius comments to the second book *On the sphere and cylinder* of Archimedes.

The Hero's construction of two mean proportionals is well known, so it is illustrated here only by Figure 1. It is sufficient to know that AB and CD are two given line segments, and KM and HN are two mean proportionals.

The Second problem

The second problem is the construction of a cone equal to a given cone, with a given altitude. Here we represent the construction in the same way as Stevin did. Given is the cone ABC, with altitude AD and the base diameter BC, and the altitude EF. Required is to construct another cone, equal to the cone ABC and with the given altitude EF.

Construction. First, mean proportional G for the AD and EF is found. Then, the fourth proportional HI for G, AD, and BC respectively is found. FHI is the required cone (HI is the diameter and EF is the altitude) (Figure 2).

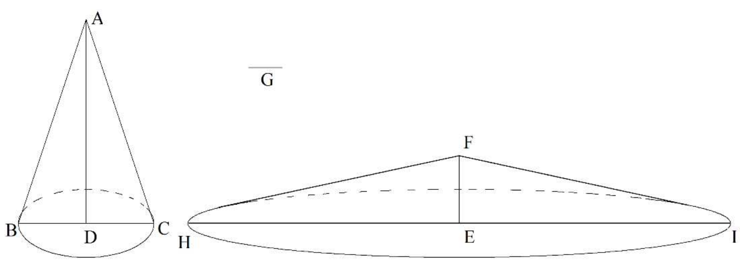

Figure 2. Construction of a cone equal to the given cone ABC with the given altitude EF

Proof. Here, we will first cite the proof from Stevin's *Problematum*, and then expose the proof using the help of modern symbolism.

„*Section 1. AD has to EF the duplicate ratio of that of AD to G, for G is their mean proportional by the construction.*

Section 2. The circle HI is to the circle BC in the duplicate ratio of that of the homologous line HI to the homologous line BC, as is inferred from the 20th proposition of Euclid's 6th book. But as the line HI is to the line BC, so is AD to G, by the inverted ratio of the construction. Consequently, the circle HI is to the circle BC in the duplicate ratio of that of the line AD to G. But it has been proved in Section 1 that AD is to EF in the same duplicate ratio of that of AD to G. Consequently, as the segment AD is to the segment EF, so is the circle HI to the circle BC. Therefore, the solids are cones whose bases and altitudes are inversely proportional, so that, by the 15th proposition of Euclid's 12th book, the

> *cones ABC and EHI are equal to one another. Moreover, it is evident from the construction itself that the cone EHI has been constructed with the given altitude HF."* [9, 307]

Using modern symbolism, per construction, G is mean proportional for AD and EF, so

$$AD : G = G : EF,$$

wherefrom it follows

$$AD : EF = AD^2 : G^2. \tag{1}$$

Circle with diameter HI to the circle with diameter BC is $HI^2 : BC^2$. For this conclusion, Stevin refers to Elements VI.20 [9, p. 299], which states: *Similar polygons are divided into similar triangles, and into triangles equal in multitude and in the same ratio as the wholes, and the polygon has to the polygon a ratio duplicate of that which the corresponding side has to the corresponding side.* Per construction, HI is the fourth proportional for G, AD and BC, so

$$G : AD = BC : HI, \text{ i.e. } HI : BC = AD : G,$$

wherefrom we conclude that the circle with diameter HI to the circle with diameter BC is $AD^2 : G^2$. From equality (1) and from the previous, the circle with diameter HI to the circle with diameter BC is $AD : EF$. By the Elements XII.15, "*In equal cones and cylinders the bases are reciprocally proportional to the heights; and those cones and cylinders in which the bases are reciprocally proportional to the heights are equal*", cones ABC and FHI are equal.

In a note, Stevin concludes that similar construction and proof can be conducted to cylinders, for Proposition XII.15 includes cylinders too.

The third problem

The third problem is the construction of a cone equal to the segment of a sphere, and having the same base as the segment of a sphere. The construction and the proof will not be exposed here. Stevin refers to the proof in *On the sphere and cylinder II* by Archimedes, for which numerous propositions from *On the sphere and cylinder I* are needed (only some of them are Propositions I.35–I.44).

4. The fourth problem – main construction

The fourth problem, and as Stevin says "what is sought in this Fourth book" [9, p. 313] is "*for given any two Geometrical solids, to construct a third solid, similar to one of the given solids and equal to the other*". This problem Stevin exposes through four examples.

Example 1. Let there be any two solids, the two cones ABC and DEF, and let the altitude of the cone DEF be the line DG, and the diameter of the base EF. Required is to construct the third cone, similar to the cone DEF and equal to the cone ABC (Figure 3).

Construction. Let us construct the cone HIK equal to the cone ABC with altitude equal to DG, using the construction presented in the previous, second problem. Let us denote the diameter of the base by IK. Let L be the third proportional for EF and IK, and let MN be the first of two mean proportionals for EF and L respectively. If we denote by OP the fourth proportional for EF, DG and MN, respectively, then the cone OMN, with base diameter MN and altitude OP is the required cone.

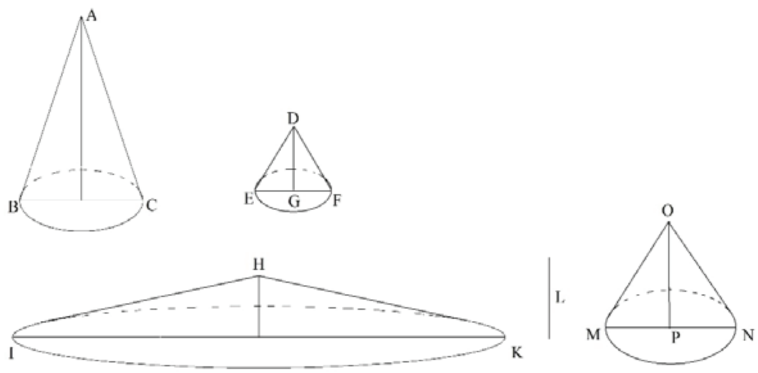

Figure 3. Construction of a cone equal to the cone ABC and similar to the cone DEF

Proof. Regarding the construction, OP is the fourth proportional for EF, DG and MN, so $EF : DG = MN : OP$, hence, cones DEF and OMN are similar.

L is the third proportional for EF and IK, so "*the segment EF is to the segment L in the duplicate ratio of that of the line EF to IK*" [9, p. 315]. In the original work, there is no further explanation for this statement, but this statement was frequently used in geometry problems. Using modern notation, it states that from $EF : IK = IK : L$ it can be concluded that $EF : L = EF^2 : IK^2$. But, EF and IK are diameters of the circles, so the circle with diameter EF to the circle with diameter IK is as EF to L. The circle with diameter EF to the circle with diameter IK is as the cone DEF to the cone HIK because cones have equal altitude per construction (this is regarding Elements XII.11). Hence, the cone DEF is to the cone HIK as the line EF to the line L.

Per construction, MN is the first of the two mean proportionals, so "*the segment EF is to L in the triplicate ratio of that of EF to the segment MN*" [9, p. 317]. Similar to the previous, it is frequently used that from ratio used in defining two mean proportionals one can conclude that $EF : L = EF^3 : MN^3$. It was earlier proved that the cones DEF and OMN are similar, hence using the previous equality (and considering EF and MN are base diameters of the cones), the cone DEF to the cone OMN is as EF to L (this is stated in Elements XII.12). It was also proved that the cone DEF is to the cone HIK as EF to L, so the cones OMN and HIK are equal. Per construction, the cone ABC is equal to the cone HIK, so the cone OMN is equal to the cone ABC.

Example 2. In Example 2, it was only stated that the previous construction could be conducted on cylinder.

Example 3 is the construction of the segment of a sphere similar to the first and equal to the second given segment of a sphere. For this construction, the constructions exposed as Problems 2 and 3 are needed. The construction will not be given in this paper.

Example 4. In this example, Stevin states: "*This problem of ours can also be demonstrated by means of numbers, which may be effected, for greater clarity, in the following way.*" In this example the construction of a pyramid equal to the pyramid ABC and similar to the pyramid EFG is given. The pyramid ABC has a square base with 2 feet side, 12 feet altitude and 16 feet volume and EFG has a square base with 8 feet side and 3 feet altitude (Figure 4).

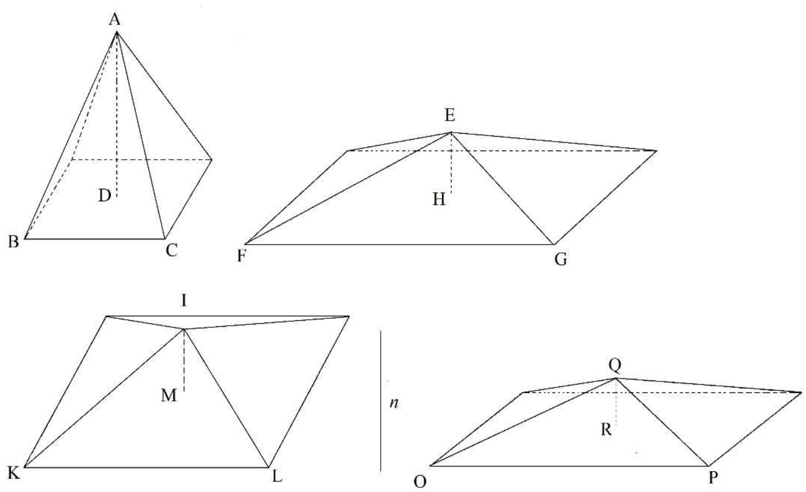

Figure 4. Construction of a pyramid equal to the pyramid ABC and similar to the pyramid EFG by the mean of numbers

Representing the construction in terms of numbers was the novelty at the time, so we expose the way Stevin conducted that construction, with some comments for easier understanding.

> *"Construct a pyramid IKL equal to the pyramid ABC, with the altitude IM equal to the altitude EH, viz. 3 feet, so that its base (in order to make a pyramid whose volume be 16 feet) will be a square whose side KL will be 4 feet. Then find the third proportional, the first term being $FG = 8$, the second $KL = 4$; then the third will be $N = 2$ feet."*

In modern symbolism, $N = 2$ from equality $8 : 4 = 4 : N$.

> *"Subsequently find the two mean proportional between $FG = 8$ and $N = 2$, the one of these mean proportionals which follows FG being OP,*

the cube root of 128*; this is proved because* 8*, and the cube root of* 128*, and the cube root of* 32*, and* 2 *are four numbers in continuous proportion."*

This statement does not have further explanation, it is stated here as general knowledge. Nowadays it can be easily concluded that from the equality $8 : OP = OP : x = x : 2$, OP is equal to $\sqrt[3]{128} = 4\sqrt[3]{2}$.

"Then find the fourth proportional, the first term being $FG = 8$, the second $EH = 3$, the third $OP =$ the cube root of 128*; then the fourth, viz. the altitude QR, will be the cube root of $\frac{3456}{512}$".*

From $8 : 3 = \sqrt[3]{128} : QR$, one can conclude that $QR = \sqrt[3]{\frac{3456}{512}} = \frac{3\sqrt[3]{2}}{2}$.

"Subsequently, on the square, whose side is OP and with the altitude QR construct a pyramid QOP; its volume will be 16 *feet. The reason is that the square whose side is $OP =$ the cube root of* 128 *will be the cube root of* 16384*, which, when multiplied by the altitude $QR =$ the cube root of $\frac{3456}{512}$ gives the product $=$ the cube root of $\frac{56623104}{512}$ the third part of which, viz. the volume of the pyramid QOP, is the cube root of $\frac{56623104}{13824}$, i.e. the cube root of* 4096*, which makes, as said above,* 16 *feet".*

This part of construction is calculation that attests that the volume indeed is 16 feet, i.e. $\frac{1}{3}(4\sqrt[3]{2})^2 \cdot \frac{3\sqrt[3]{2}}{2} = 16$.

"I say that a third pyramid QOP has been constructed, by means of numbers, in the same order as has been done above by segments, similar to the pyramid EFG and equal to the pyramid ABC; as was required. The proof is evident from the fact that the pyramid QOP is similar to the pyramid EFG and equal to the pyramid ABC, by the numerical construction itself". [9, p. 327, 329].

In the end, it is noted that it is possible to construct a cone equal to any Geometrical solid.

5. Conclusion

Representing this construction, one can see the great impact that Euclid and Archimedes still had on geometry in the 16th century. The Elements have all the constructional tools that Stevin uses, and Archimedes proved Problem 3 as Proposition 5 in *On the sphere and cylinder II.* Stevin just generalized this problem to other solids and gave a different proof.

Mathematical language and symbolism Stevin uses in this book is similar to the language used in Elements. But, the need for exposing the problem by the means of numbers is a novelty. After number construction, Stevin says: "*This example can also be dealt with by means of the rule called Algebra, but since this is common knowledge, we have not thought it necessary to set it forth here.*" [9, p. 331]

Hence, in the 16th century there was a need for linking geometry to numbers and algebra. Our opinion is that this work illustrates the mathematical circumstances in which Descartes made the correspondence between geometry and numbers. Further development of geometry is well known.

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