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ON CATALAN NUMBERS

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Abstract. In this article we introduce one of the most frequently encountered sequences in mathematics, the sequence of Catalan numbers. The numerous and various occurrences and applications in combinatorial problems as well as their relations with other famous numerical sequences make Catalan numbers very suitable for creative teaching of mathematics at all levels.

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1. Introduction

We start with a brief historical overview of the exciting mathematical background of Catalan numbers. An interested reader may refer to the article [5]. The recent discovery by Luo Jianjin in 1988 addressed the first appearance of Catalan numbers to Chinese mathematician Ming Antu (c. 1692–c. 1763) who wrote a book in 1731 which included some trigonometric expansions involving Catalan numbers [4]. Leonard Euler (1707–1783) in his letter from 1751 to Christian Goldbach (1690–1764) defines Catalan numbers C_n as the numbers of triangulations of $(n+2)$-gon and finds the generating function for these numbers. Another correspondent of Euler was Johann Segner (1704–1777) who found the recurrence relation for Catalan numbers. In 1838 Eugene Catalan (1814–1894) studied the problem of different parenthesizations of n factors. Arthur Cayley (1821–1895) counted planar trees in 1859 by using the generating function method. Despite their omnipresence, the Catalan numbers have been named recently and the wide adoption of the name begins in early 1970's.

Richard Stanley, the leading modern combinatorialist, in his famous book Enumerative Combinatorics [6] provided a list of 66 combinatorial interpretations of Catalan numbers. This collection has been published recently in a separate monograph [7]. In a 2008 interview [1], he confessed that Catalan numbers are his favorite number sequence. Another extensive monograph on Catalan numbers is [3].

The Catalan numbers are determined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \ n = 0, 1, 2, \dots . \tag{1.1}$$

The first members of the sequence $\{C_n\}_{n\geq 0}$ are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4869, 16796, 58786, 208012, \dots$$

The Catalan sequence is numbered by A000108 in the Neil Sloan's On-Line Encyclopedia of Integer Sequences (OEIS) as probably its longest entry.

In this paper we present the major combinatorial interpretations of Catalan numbers. The choice is arbitrary and depends on space and preference of the author. The paper is only an invitation to further study this subject and may serve as a starting point for own research. In Section 2 we count binary trees and derive the major number theoretic properties. We show the relation of Catalan numbers to Fibonacci sequence and Chebyshev polynomials. In Section 3 we count binary terms and introduce Tamari order on the set of binary trees. We introduce the special convex polytope called associahedron whose vertices are Catalan objects. Section 4 includes more examples on combinatorial counting: triangulations of convex polygon, upper-diagonal walking and ballot problem and pattern avoiding problems for permutations.

2. Binary trees and number theoretic properties

In Graph Theory, a tree is a simple graph with no cycles. The nodes of a tree which are incident to a unique edge are called leaves, the remaining nodes are called internal nodes. The rooted tree is a tree with a distinguished node called the root. Any rooted tree gives a partial order on the set of nodes by $u \leq v$ if u lies on the unique path from v to the root. A tree is called a full binary tree if all internal nodes including root have exactly two successors. A binary tree is a planar tree if it is embedded in the plane, so the ordering of the successors of the internal nodes are given from left to right. Let $\mathcal{T}$ be the class of full planar rooted binary trees graded by the number of internal nodes $\mathcal{T} = \bigsqcup_{n\in\mathbb{N}} \mathcal{T}_n$. For the sake of shortness, we will simply call the elements $T \in \mathcal{T}$ the binary trees. There is a binary operation $\bullet$ on the set $\mathcal{T}$ that associates to any trees T_1 and T_2 the rooted tree $\bullet(T_1, T_2)$ obtained by grafting the roots of T_1 and T_2 on a common new root. The grafting operation $\bullet$ is not commutative since it depends on the order of factors. The class $\mathcal{T}$ may be defined recursively by

(i) $\mathcal{T}_0 = \{\bullet\}$

(ii) If $T_1 \in \mathcal{T}_{n_1}$ and $T_2 \in \mathcal{T}_{n_2}$ then $\bullet(T_1, T_2) \in \mathcal{T}_{n_1+n_2+1}$.

Define the generating function

$$C(x) = \sum_{T\in\mathcal{T}} x^{|T|}. \tag{2.1}$$

Since for $n \geq 1$ any binary tree $T \in \mathcal{T}_n$ is of the form $T = \bullet(T_1, T_2)$, we have

$$\sum_{T\in\mathcal{T}} x^{|T|} = 1 + \sum_{T_1,T_2\in\mathcal{T}} x^{|\bullet(T_1,T_2)|} = 1 + x \sum_{T_1\in\mathcal{T}} x^{|T_1|} \sum_{T_2\in\mathcal{T}} x^{|T_2|}.$$

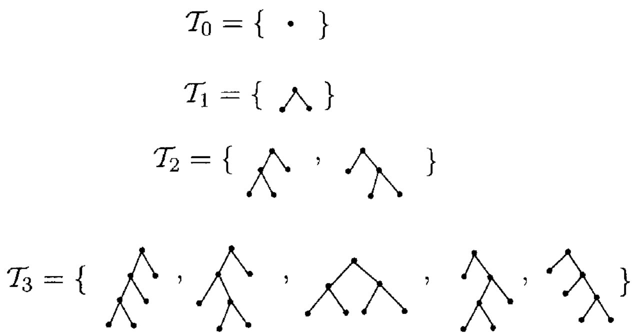


Fig. 1. Binary trees with $n \leq 3$ internal nodes

Therefore, the function $C(x)$ satisfies functional relation

$$C(x) = 1 + xC(x)^2. \tag{2.2}$$

Consequently, since $C(0) = 0$, we obtain

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

The expansion of the square root term into a power series gives

$$C(x) = \frac{1}{2x}\left[1 - \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n\right],$$

which by a simple calculation leads to

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.$$

According to (2.1), we obtain that the numbers $C_n = |\mathcal{T}_n|$ of binary trees $T \in \mathcal{T}_n$ are Catalan numbers (1.1).

From the functional relation (2.2) and Cauchy product formula for power series, we obtain

$$\sum_{n=0}^{\infty} C_n x^n = 1 + \sum_{n=0}^{\infty} (\sum_{i+j=n} C_i C_j) x^{n+1},$$

which implies the recurrence relation for Catalan numbers

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1},\ C_0 = C_1 = 1. \tag{2.3}$$

Let L_n be a path graph on n nodes, see Figure 2. Define the adjacency matrix $A_n = (a_{i,j})_{n \times n}$ by $a_{i,j} = 1$ if (i,j) is the edge of L_n, and $a_{i,j} = 0$ otherwise. Let $q_n(u)$ be the characteristic polynomial

$$q_n(u) = \det(uE - A).$$

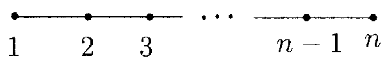


Fig. 2. Path graph L_n

By expanding the determinant on elements of the first row we obtain

$$q_{n+1}(u) = uq_n(u) - q_{n-1}(u),\ q_0(u) = 1,\ q_1(u) = u.$$

The substitution $U_n(u) = q_n(2u)$ leads to

$$U_{n+1}(u) = 2uU_n(u) - U_{n-1}(u),\ U_0(u) = 1,\ U_1(u) = 2u,$$

which is exactly the recurrence relation satisfied by Chebyshev polynomials of the second kind, which are defined as

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

Define the generating function $G(u,v) = \sum_{n=0}^{\infty} q_n(u)v^n$. By using the recurrence relation for $q_n(u)$ and summing up, we obtain

$$G(u,v) = \frac{1}{1 - uv + v^2}.$$

On the other hand, by rewriting the equation (2.2) in the form

$$C(x) = \frac{1}{1 - xC(x)},$$

and iterating this identity, we get the continued fraction expansion

$$C(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \frac{x}{1-\cdots}}}}.$$

The convergents to this continued fraction are defined recursively

$$P_1(x) = 1,\ P_n(x) = \frac{1}{1 - xP_{n-1}(x)},\quad n \geq 1.$$

Suppose we have $P_n(x) = \frac{p_{n-1}(x)}{p_n(x)}$ for some sequence of polynomials $p_n(x)$, which therefore satisfies

$$p_{n+1}(x) = p_n(x) - xp_{n-1}(x),\ p_0(x) = p_1(x) = 1.$$

In particular, for $x = -1$, we obtain that $F_n = p_n(-1)$ is the Fibonacci sequence and that $\frac{F_{n-1}}{F_n}$ are convergents to continued fraction expansion of the golden ratio

$\phi = C(-1) = \frac{1+\sqrt{5}}{2}$. Summing up the generating function $F(x,y) = \sum_{n=0}^{\infty} p_n(x)y^n$ by using the recurrence relation for $p_n(x)$ we get

$$F(x,y) = \frac{1}{1-y+xy^2}.$$

The substitution of variables $x = 1/u^2, y = uv$ gives $F(1/u^2, uv) = G(u,v)$, which implies

$$q_n(u) = u^n p_n\big(\frac{1}{u^2}\big).$$

This identity relates Catalan numbers and Chebyshev polynomials.

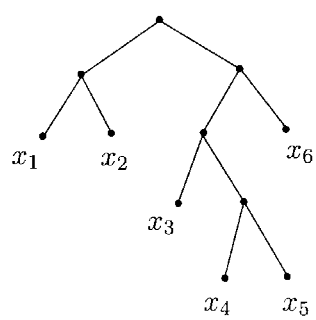
3. Binary terms and associahedron

The set of binary terms $\mathbf{T}$ on a set of variables $X = \{x_1, x_2, x_3, \dots\}$ and a binary function symbol $\cdot$, is recursively defined:

(i) every variable is a term $X \subset \mathbf{T}$

(ii) if $t_1, t_2 \in \mathbf{T}$ then $(t_1 \cdot t_2) \in \mathbf{T}$.

The length of a term $t \in \mathbf{T}_n$ is the number n of appearances of the functional symbol in the expression of t. There is a well known bijection between binary trees with n internal nodes and binary terms of the length n obtained by parenthesizing the string $x_1 \cdot x_2 \cdots x_n \cdot x_{n+1}$. To each internal node associate the binary function symbol. The $n+1$ leaves of T are ordered rightward, so to the i^{th} leaf associate the variable x_i for any $i = 1, 2, \dots, n+1$, see Figure 3.



$$(x_1 \cdot x_2) \cdot ((x_3 \cdot (x_4 \cdot x_5)) \cdot x_6)$$

Fig. 3. Binary tree and the corresponding binary term

A grupoid is a set G with a binary operation $*\colon G \times G \to G$. Assigning values to variables $e\colon X \to G$ gives rise to a valuation map on terms $\widetilde{e}\colon \mathbf{T} \to G$. If the operation $*$ is associative, $(a * b) * c = a * (b * c)$, for any $a, b, c \in G$, then all terms corresponding to the trees of the same size have the same value.

There is a natural partial order on the set $\mathbf{T}_n$ induced by the relation $(x\cdot y)\cdot z \le x\cdot(y\cdot z)$. We define $t_1 \le t_2$ if and only if term t_2 may be obtained from term t_1 by only rightward application of associativity law. Note that this is also an ordering of binary trees by correspondence to binary terms. In Figure 4 is presented the partially ordered set of terms on four variables.

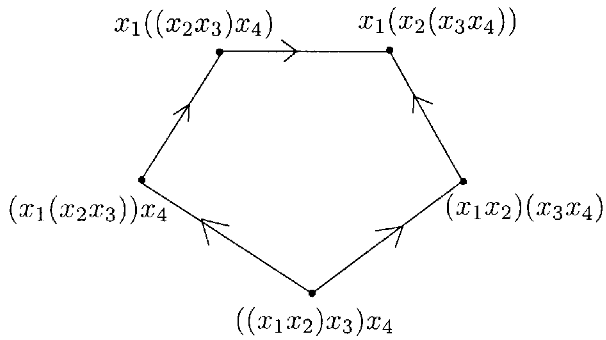


Fig. 4. Tamari lattice $\mathbf{T}_3$

Polytopes are high dimensional analogues of polyhedra. A *convex polytope* P in $\mathbb{R}^n$ is a convex hull $P = \mathrm{Conv}\{a_1, \dots, a_m\}$ of a finite set of points $a_1, \dots, a_m \in \mathbb{R}^n$. A dimension of P is its affine dimension. A supporting hyperplane of the polytope P is an affine hyperplane H such that $H \cap P$ is nonempty and P is contained in one of the half-spaces determined by H. The intersections of P with supporting hyperplanes are called faces of P. The set of faces is ordered by inclusion. Each face of P, in itself, is a polytope of lower dimension. Vertices are zero-dimensional faces, edges are one-dimensional faces and faces of codimension one are called facets. The graph $G(P)$ of polytope is its one-dimensional skeleton, i.e. the union of vertices and edges of P. The simplest polytopes are the simplex Δ^{n-1} and the cube I^{n-1}. The simplex is realized as the convex hull $\Delta^{n-1} = \mathrm{Conv}\{e_1, e_2, \dots, e_n\}$, where e_i is the i^{th} coordinate vector in $\mathbb{R}^n$ and the cube I^{n-1} is the convex hull of the indicator vectors $e_S \in \{0,1\}^{n-1}$ of subsets $S \subset \{1, 2, \dots, n-1\}$. Hence, the numbers of vertices are $f_0(\Delta^{n-1}) = n$ and $f_0(I^{n-1}) = 2^{n-1}$. Any face of the simplex $F_S \subset \Delta^{n-1}$ is encoded by the subset $S \subset \{e_1, \dots, e_n\}$. Thus the face lattice of the simplex Δ^{n-1} is a Boolean algebra B_n on the n-element set.

The overview of Catalan numbers would not be complete without the associahedron, which is a convex polytope whose vertices enumerate Catalan objects. The first appearance of the associahedron was in 1963, when James Stasheff, while he was studying the homotopy of loop spaces, constructed a cell complex whose vertices correspond to the binary terms of the length $n-1$. This cell complex turns out to be a boundary complex of a convex polytope As^{n-1} which is called Stasheff's polytope or the associahedron. Since the vertices of As^{n-1} correspond to the binary terms, i.e. to the binary trees, it follows that the number of vertices $f_0(As^{n-1}) = C_n$ is the Catalan number.

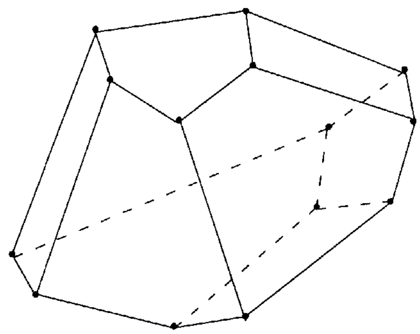


Fig. 5. Associahedron As^3

The easiest way to realize the associahedron As^{n-1} is by truncations of faces of the simplex Δ^{n-1}. We start with the path graph L_n on n nodes and write sets S of vertices such that the induced subgraphs are connected. Then we perform truncations in a direct order on faces of Δ^{n-1} which correspond to complements S^c. For example, if $n = 4$ and L_4 is the path with edges $\{12, 23, 34\}$, we perform truncations on faces $\{1, 4, 12, 14, 34, 123, 124, 134, 234\}$, see Figure 5.

4. More combinatorial counting

In this section we present the three most known enumeration problems related to Catalan numbers.

4.1. Triangulations of a convex polygon

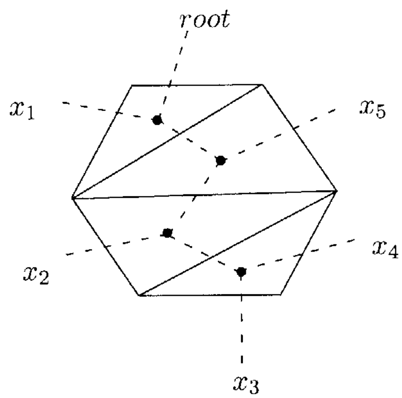
Catalan numbers first occurred in Euler's problem from 1751: given a convex $(n + 2)$-gon, find all different ways to divide it into triangles by nonintersecting diagonals.

To any such triangulation we can associate a planar binary tree in a unique way, such that triangles correspond to the internal vertices and edges of triangulations correspond to the edges of the three. It remains to choose the distinguished edge of polygon which determines the root. This also uniquely associates the binary term on $(n + 1)$ variables to the triangulation of $(n + 2)$-gon.

4.2. Upper-diagonal walking or Ballot problem or Dyck words

The ballot problem is introduced in 1887 by Joseph Bertrand (1822–1900): in an election where two candidates receive the same number of votes, what is the number of different possible voting so that the first candidate never has fewer votes than the second candidate.

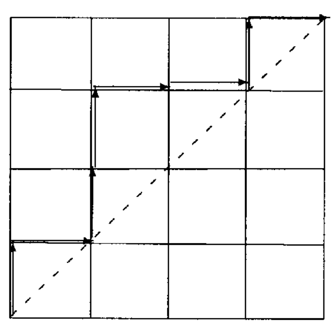
This problem is also known as a upper-diagonal walking: given a $n \times n$-tableau on which we can walk only rightwards and upwards, what is the number of allowed paths from the initial position $(0, 0)$ to the final position (n, n). A path is allowed if it is above the main diagonal. It is obvious that a unique allowed path could be associated to each voting order and that it is equivalent to making a word consisting



$$x_1((x_2(x_3x_4))x_5)$$

Fig. 6. Triangulation of a hexagon and the corresponding tree

of n letters A and n letters B such that no initial segment of the word has more B's than A's. These words are called Dyck words and may also be interpreted as possible ways of correctly matching pairs of parentheses if $A =$ (and $B =$), see Figure 7.



$$ABAABBAB = ()(())()$$

Fig. 7. The allowed path and the corresponding Dyck word

Let us construct the set $S = \cup_{n\geq 0} S_n$ of Dyck words recursively, where S_n contains words of length n:

- ⋄ $S_0 = \{\}$
- ⋄ If $x \in S_{n-1}$, then $(x) \in S_n$
- ⋄ If $x \in S_i$ and $y \in S_j$, then $(xy) \in S_{i+j+1}$.

Let $|S_n|$ be the number of words of length n. By definition it is obvious that $|S_n|$ satisfies the recurrence relation (2.3) for Catalan numbers, so $|S_n| = C_n$.

4.3. Avoiding patterns of permutations

Percy MacMahon (1854–1929) first proved a result in pattern avoidance problems for permutations. His two volume *Combinatorial Analysis* (1915/6) was one of the first monographs in combinatorics.

A permutation ω on n letters is said to be (123)-avoiding if there are no $i < j < k$ such that $\omega(i) < \omega(j) < \omega(k)$. Let $A_n(123)$ be the set of such permutations. Count the number $|A_n(123)|$ of its elements. For example $A_3(123) = \{132, 213, 231, 312, 321\}$ and $A_4(123) = \{1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312, 4321\}$. It can be immediately seen that $\omega \in A_n(123)$ if and only if ω can be divided into two decreasing subsequences.

We construct a bijection between (123)-avoiding permutations and Dyck words. Given $\omega \in A_n(123)$, we say that $\omega(i)$ is a right-to-left maximum if $\omega(i) > \omega(j)$ for any $j > i$. For example, the right-to-left maxima of 58327641 are $1, 4, 6, 7, 8$. Then for the right-to-left maxima $m_1, \ldots, m_s$ we have the presentation

$$\omega = w_s m_s w_{s-1} m_{s-1} \cdots w_1 m_1,$$

where w_i is a subword (possible empty) of ω between m_{i+1} and m_i. The sequence $w_s w_{s-1} \ldots w_1$ is decreasing since ω is (123)-avoiding. Reading the decomposition of ω from right to left, to each m_i we associate subword $A^{m_i - m_{i-1}}$ (with $m_0 = 0$) and to each w_i we associate $B^{|w_i|+1}$. The obtained word is a Dyck word. For example

$$58327641 \mapsto ABA^3BA^2BAB^3AB^2.$$

To the upper-diagonal walk on Figure 7 is associated permutation 4231.

The reader can try to prove that $|A_n(\omega)| = C_n$ regardless to $\omega \in S_3$. The proof may be found in the famous Donald Knuth's monograph [2].

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