THE TEACHING OF MATHEMATICS
2014, Vol. XVII, 1, pp. 21–33

EULER-POINCARÉ CHARACTERISTIC—A CASE OF TOPOLOGICAL SELF-CONVINCING

Milosav M. Marjanović

Abstract. In this paper we establish a topological property of geometric objects (lines, surfaces and solids) called Euler-Poincaré characteristic. Since the paper is intended for a large profile of mathematics teachers, our approach is entirely intuitive and majority of readers can omit two addenda whose understanding requires a solid knowledge of topology.

E-P characteristic is an integer which we calculate here decomposing lines into fibers being finite sets of points, surfaces into fibers being lines and solids into fibers being surfaces.

When an object is subjected to a "plastic" deformation its shape and size changes as well as the alternating sums resulting from the method of calculation, but the value of these sums stays unchanged. This fact serves to convince the reader that E-P characteristic is a stable topological property. The same fact gives this approach an advantage over usually practiced ones which require a triangulation of geometric objects.

MathEduc Subject Classification: G95

MSC Subject Classification: 97G99

Key words and phrases: Euler-Poincaré characteristic; decomposition of lines into finite sets of points, surfaces into lines, solids into surfaces.

1. Introduction

This paper is intended for those readers who have no formal knowledge of topology. All objects that we consider are planar or solid shapes known to the reader from his/her school geometry course and, here in this paper, referred to as geometric objects. An elementary approach to ideas and methods of topology does not mean that some of its contents are presented using the techniques of introductory university courses of analysis, but it means that such an approach must be completely intuitive, based on imagination and aimed at conceptual understanding. In that way, this paper (excluding two addenda) is an approachable piece of mathematics which may be easily read by a large profile of teachers (including primary school ones).

The topic treated in this paper is the Euler-Poincaré characteristic, what is an integer being a very stable (topological) property of geometric objects. When the size and the shape of an object changes as an effect of a "plastic" deformation, its topological properties stay unchanged. Our method is the formation of alternating sums which also change under plastic deformations, but their value stays unchanged, what makes a reader be convinced that such a number is really a topological property. Possibility of this verification gives an advantage to this approach when it is compared to those ones depending on polyhedral shaping of objects (see, for example, [C-R]). This is particularly true in the case of surfaces when they are represented as the objects in $\mathbb{R}^3$ (including possibly a command for gluing points of their edges).

Two addenda are intended for those readers having a basic knowledge of topology. The second of them contains all arguments which show that our approach is legitimate (and, in addition, a condition from [M_2] is somewhat relaxed).

2. Euler's polyhedron formula

Let a convex polyhedron has v vertices, e edges and f faces, then

$$v - e + f = 2.$$

Verifying this formula in the case of some special polyhedra a) tetrahedron, b) cube, c) hexagonal pyramid, we have: a) $4-6+4=2$, b) $8-12+6=2$, c) $7-12+7=2$. Holding true for all convex polyhedra, being hidden for a quick understanding, this fascinating relation is known as the *Euler's polyhedron formula* and it appeared first in a note submitted by L. Euler to the Proceedings of the Petersburg Academy of 1752/53. But a variant of this formula was found in the Descartes' manuscript "*Progymnasmata*". Namely, for a convex polyhedron, if α is the number of its solid angles (i.e. its vertices), φ the number of its faces and ρ the number of its plane angles (i.e. angles at the vertices of the faces), then Descartes proves that

$$\rho = 2\varphi + 2\alpha - 4.$$

As each angle has two edges as its legs and each edge is the leg of four angles, $\rho = 2e$. Then, from $2e = 2\varphi + 2a - 4$, the Euler's formula $\alpha - e + \varphi = 2$ follows. Thus, it is evident that this formula of Descartes is a precursor of Euler's formula ([Z-B]).

There exist several interesting applications of the Euler's formula to the study of polyhedra and planar graphs. One of them is the so called topological proof of existence of five Platonic solids (polyhedra with congruent faces of regular polygons and the same number of faces meeting at each vertex).

Let p be the number of edges of each face and q the number of edges meeting at each vertex. Counting each edge twice their number is pf or qv. Thus, $pf = 2e$ and $qv = 2e$. Substituting in Euler's formula, we have $2e/q - e + 2e/p = 2$, what implies

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{r} > \frac{1}{2}.$$

The only solutions in (p,q) of this inequality are: $(3,3)$, $(4,3)$, $(3,4)$, $(5,3)$ and $(3,5)$, corresponding to tetrahedron, hexahedron (cube), octahedron, dodecahedron and icosahedron, respectively.

3. Topological equivalence

Number 2 in Euler's formula is a deep, stable characteristic (property) of a whole class of geometric objects called topological spheres. Our first aim here is an intuitive description of topologically equivalent geometric objects and of their topological properties.

When a geometric object is imagined to be made of the ideally plastic substance and when it is allowed to make some parts of it longer, wider stretching them in all directions or to make some other its parts smaller contracting them in all direction, then that object is said to be under a plastic deformation. Let us add that under plastic deformations two different points of an object are not allowed to coincide. But the object is allowed to be cut into pieces at some of its points (case of lines) or along some lines (case of surfaces) or along some surfaces (case of solids) and after being plastically deformed these pieces have to be glued together in the places of cutting. When an object is subjected to these allowed transformations another object of different shape and size is obtained, but such two objects are said to be *topologically equivalent* or *to have the same topological type.* A property shared by all topologically equivalent objects is called a *topological property* or *topological invariant.*

The figures that follow illustrate some of the basic topological types.

1. *Topological arc.* Let $[0,1] = \{x : 0 \leqslant x \leqslant 1\}$ be the unit interval. Deforming it, a sequence of lines is obtained each representing one and the same topological type called *topological arc.*

Fig. 1. Lines representing topological arc

2. *Topological circle.* Let $S^1 = \{(x,y) : x^2 + y^2 = 1\}$ be the unite circle. Deforming it, a sequence of lines is obtained, each representing one and the same topological type called topological circle.

Fig. 2. Lines representing topological circle

When a circle is cut at a point A, the ends can be tied together making knots in different ways.

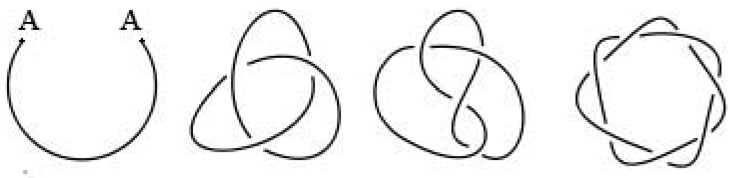

Fig. 3. Knots represent topological circle

3. *Topological disk.* Let $D^2 = \{(x,y) : x^2 + y^2 \leqslant 1\}$ be the unit disk. A sequence of topologically equivalent objects to the unit disk is represented in Fig. 4.

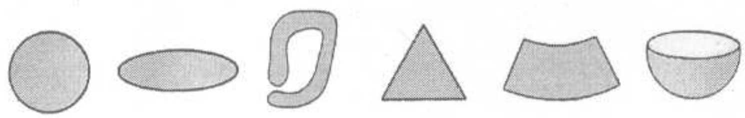

Fig. 4. Surfaces representing topological disk

4. *Topological ring (annulus).* Let $R = \{(x,y) : \frac{1}{4} \leqslant x^2 + y^2 \leqslant 1\}$ be the circular ring. Surfaces topologically equivalent to it are represented in Fig. 5.

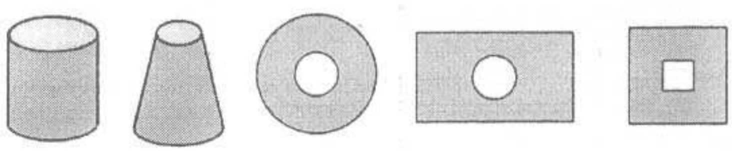

Fig. 5. Surfaces that represent topological ring

5. *Topological sphere.* Let $S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$ be the unit sphere. Surfaces topologically equivalent to it are represented in Fig. 6.

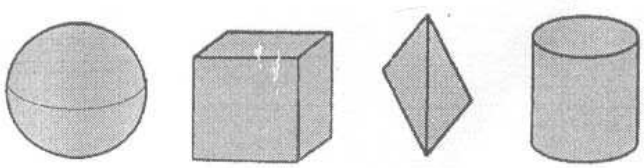

Fig. 6. Surfaces that are topological sphere

The requirement for the validity of Euler formula that a polyhedron is convex only ensures that its surface is topological sphere.

4. Addendum 1

Let us remind that a formal definition of topological equivalence of two objects (spaces) X and Y mean that there exists a mapping $f\colon X \to Y$ which is 1–1 and onto and both mappings f and f^{-1} are continuous. A property that an object has is topological whenever that property has each topologically equivalent object to it.

The generalization of Euler's formula to n-dimensional convex polyhedra (polytopes) is due to Swiss mathematician Ludwig Schläfli (1814–1895), who states that

$$f_0 - f_1 + \cdots + (-1)^{n-1} f_{n-1} = 1 - (-1)^n,$$

where $f_i,\ i = 0, 1, \ldots, n-1$ is the number of i-dimensional faces of a convex n-dimensional polyhedron.

The most significant generalization of this Euler's result is due to the French mathematician Henri Poincaré, who is also considered to be the founder of topology. Namely, for each n-dimensional polyhedron P, Poincaré considers the number

$$\chi(P) = \beta_0 - \beta_1 + \cdots + (-1)^n \beta_n,$$

where $\beta_i,\ i = 0, 1, \ldots, n$ are Betti numbers (ranks of homology groups $H_i(P)$, $i = 0, 1, \ldots, n$). Thus, $\chi(P)$ is evidently a topological property of the polyhedron P which is called *Euler-Poincaré characteristic*. In addition, Poincaré proves that

$$f_0 - f_1 + \cdots + (-1)^n f_n = \chi(P),$$

where f_i is the number of i-dimensional faces of P. The last relation also serves for the calculation of the Euler-Poincaré (E-P) characteristic of the polyhedron P.

A disadvantage of this way of calculation is the fact that for a geometric object X, to calculate $\chi(X)$ a polyhedron P which is topologically equivalent to X has to be found, what becomes more and more complex when the dimension of X increases.

5. Euler-Poincaré characteristic

This section is intended for those readers who omit Addendum 1. The requirement that a polyhedron is convex ensures only that its surface is a topological sphere. Thus, if the surface of a polyhedron is topological sphere, then Euler's formula applies to such a polyhedron or, better to say, to its surface. The number 2 in that formula is a deep property of topological spheres and it was the great classical mathematician Henri Poincaré (1854–1912) who defined an integer to be a topological property of all other geometric objects. That integer is called Euler-Poincaré (E-P) characteristic and its precise definition is beyond each elementary approach. Instead of defining it, we will described a method of calculation which produces E-P characteristics of a variety of geometric objects.

For a geometric object X, its E-P characteristic will be denoted by $\chi(X)$. When X is one point then $\chi(X)$ is taken to be 1. When X consists of n points, then $\chi(X)$ is taken to be n. More generally, when the objects X_i, $i = 1, 2, \ldots, n$ do not intersect and are at positive distance each from the other, then $\chi(\bigcup X_i) = \chi(X_1) + \cdots + \chi(X_n)$.

Now we proceed inductively and we calculate E-P characteristic of a line decomposing it into fibers having a finite number of points, of a surface decomposing it into fibers that are lines, etc. In Fig. 7 a topological arc and a topological circle are seen decomposed into fibers.

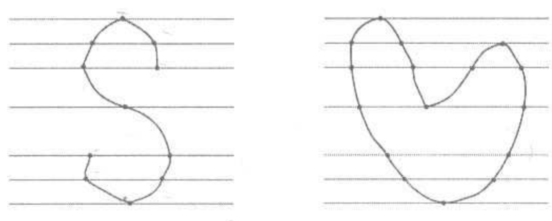

Fig. 7. a) Decomposition into fibers of an arc, b) decomposition into fibers of a circle

In the case a) (looking upwards), the fibers are: initial fiber – one point, running fibers – two points, transitional fiber – two points, running fibers – one point, transitional fiber – two points, running fibers – two points, final fiber – one point.

For the sake of simplicity, let us call initial and final fibers also transitional. Forming an alternating sum whose members are characteristics of transitional and running fibers taken in order, we get $\chi(X) = 1 - 2 + 2 - 1 + 2 - 2 + 1 = 1$. Hence, when X is a topological arc, its E-P characteristic is 1.

In the case b) the fibers are: one point, two points, three points, four points, three points, two points, one point. Calculating, we get $\chi(X) = 1 - 2 + 3 - 4 + 3 - 2 + 1 = 0$. Hence, E-P characteristic of a topological circle is 0. In Fig. 7 two poor shapes of a topological arc and a topological circle are given. In Fig. 8 two more regular shapes represent them.

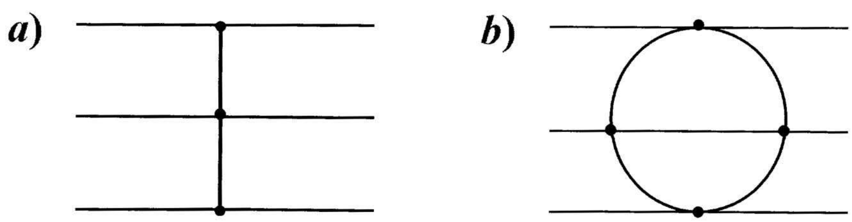

Fig. 8. "Good" shapes of a) a topological arc, b) a topological circle

Now the calculation of E-P characteristic is simpler: a) $1 - 1 + 1 = 1$, b) $1 - 2 + 1 = 0$. We see that alternating sum changes dependently on the shape of an object but its value stays unchanged being a topological property of these objects.

I still can remember my students, even those preparing to be elementary school teachers, how they liked to distort the shape of an object and verify that its E-P characteristic stayed unchanged ([M_1]). I also practiced to assign them exercises as the following one:

Which topological type is represented by the lines in the following figure?

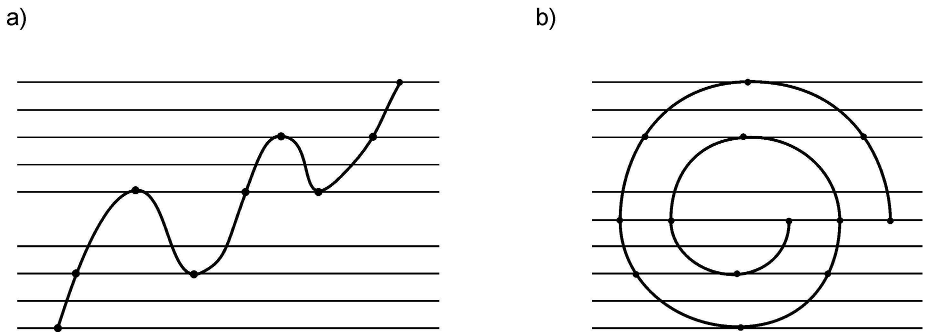

Fig. 9.

Calculate E-P characteristic in each of these cases.

(a) $1-1+2-3+3-3+2-1+1=1$, b) $1-2+3-4+5-4+3-2+1=1$) The students really enjoyed doing this type of exercises.

In the case of two touching circles

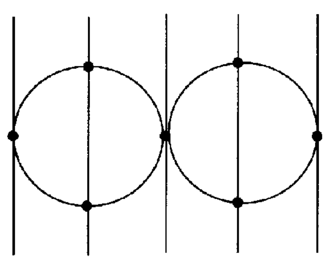

Fig. 10. Decomposition into fibers of two touching circles

we have $\chi = 1-2+1-2+1 = -1$.

Next we consider the examples of surfaces decomposed into fibers being lines. In Fig. 11 such a decomposition of a sphere and of a torus is seen.

a) Fibers are: one point, running circles, one point. Hence, $\chi = 1-0+1 = 2$.

b) Fibers are: one point, running circles, two touching circles, running pair of disjoint circles, two touching circles, running circles, one point. Hence, $\chi = 1-0+(-1)-0+(-1)-0+1 = 0$.

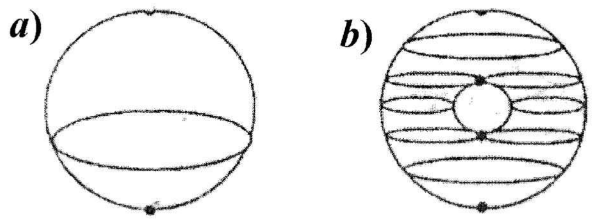

Fig. 11.

6. Surfaces

The most familiar surfaces are those that arise as the boundaries of solids: a sphere is the boundary of a ball, a torus is the boundary of a solid ring, etc. We can say for the torus that it is a sphere with one hole and *each surface that is boundary of a solid is a sphere with g holes*, $g = 0, 1, \ldots$, which is usually denoted by Mg and whose decomposition into fibers is represented in Fig. 12.

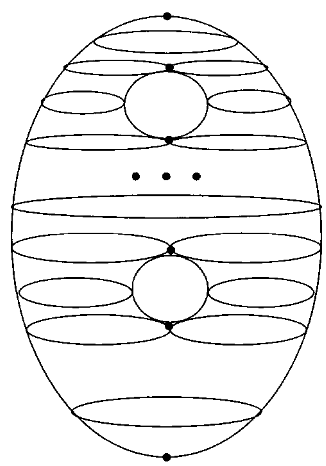

Fig. 12. Decomposition of sphere with g holes into fibers

It is easy to verify that $\chi(Mg) = 1 - 2g + 1 = 2 - 2g$. (Let us notice that as soon as the decomposition passes through a hole, -2 is added as a summand).

When a long narrow rectangular paper strip is twisted 180 degrees and then, two opposite sides of such "rectangle" are glued together, the model of a surface is obtained which is called the *Möbius strip* (after its discoverer, German mathematician A. P. Möbius, 1790–1869), Fig. 13.

This surface is remarkable for being one-sided. (A lady-bird can walk on this surface, starting from a point and arriving at that the same point being on its opposite face without crossing its edge). Let the reader checks that the Möbius strip M has a decomposition into fibers: one arc, two running disjoint arcs, one

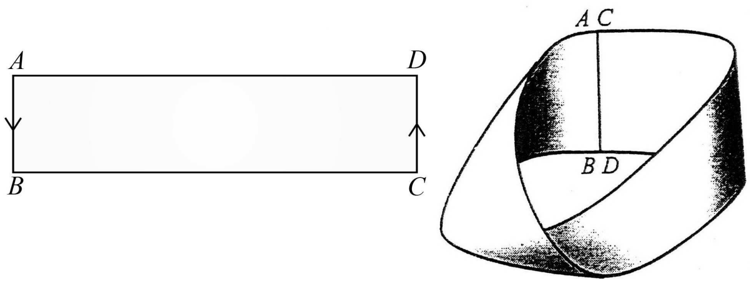

Fig. 13. The Möbius strip

arc and that $\chi(M) = 1 - 2 + 1 = 0$. It is somewhat more challenging to see the decomposition: one circle, one running circle, one circle and accordingly $\chi(M) = 0 - 0 + 0 = 0$.

When diametrically opposite points of a circle are glued together, a half-circle is obtained with two its end points left to be glued together.

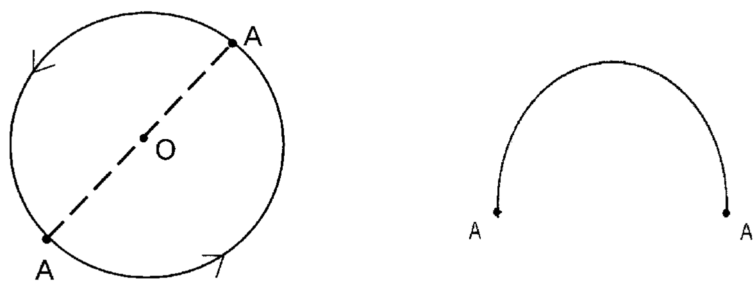

Fig. 14.

Thus, the line that arises after this gluing of points is a circle again.

When diametrically opposite points of a boundary circle of a ring are glued together, a Möbius strip is obtained again. To see it, the ring is cut along two arcs α and β (see Fig. 15), two its halves are straightened out, one of them is turned upside down and, they are joined to form a rectangle whose two opposite sides have to be glued together, producing so a Möbius strip in an already described way.

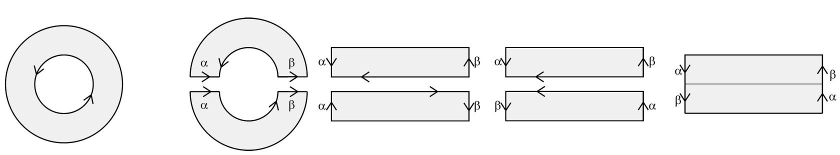

Fig. 15. Another model for the Möbius strip

In general, a surface (without boundaries) is a geometric object each point of which has a neighborhood topologically equivalent to a open disk in the Euclidean plane. (Let you look at a torus and see how small neighborhoods of its points look just like somewhat distorted plane disks).

In the next figure, two one-sided (non-orientable) surfaces are represented.

Fig. 16. a) Projective plane, b) Klein bottle

When a cap is cut from the sphere and diametrically opposite points of the boundary circle are glue together, a one-sided surface is obtained, called *projective plane* (Fig. 16a)). When two such caps are cut off and the points of boundary circles are glued as in the former case, a one-sided surface is obtained, called *Klein bottle*, (F. Klein, German mathematician, 1849–1925) (Fig. 16b)).

In the case a), a decomposition into fibers is: one point, running circles, a circle and therefore, $\chi = 1 - 0 + 0 = 1$. In the case b), a decomposition is: a circle, running circles, a circle and $\chi = 0 - 0 + 0 = 0$.

Let us notice that a parallel divides projective plane into a topological disc and a Möbius strip (Fig. 15) and similarly a Klein bottle is divided by such a parallel into two Möbius strips. Of course, as soon as a Möbius strip is a part of a surface, such a surface is one-sided. When a cap is cut off a sphere with k holes, $k = 0, 1, 2, \dots$ and diametrically opposite points of the boundary circle are glue together, a one-sided surface is obtained (Fig. 17a)). When two caps are cut off and points of boundary circles are glued together as before, another one-sided surface is obtained (Fig. 17b)).

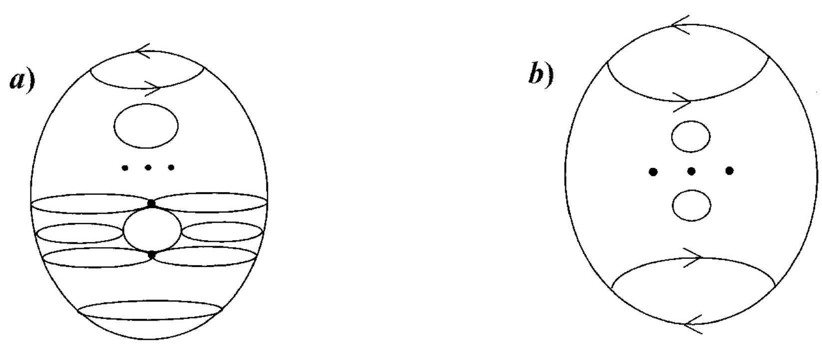

Fig. 17.

In the former case (under a)) a decomposition has for its initial fiber: one point, and the final one: a circle. Such two fibers in the latter case (under b)) are both a circle while the summand -2 is added whenever the decomposition passes through a hole. Hence, we have

$$a)\quad \chi = 1 - 2k + 0 = 1 - 2k, \qquad b)\quad \chi = 0 - 2k + 0 = -2k.$$

Let us add at the end that surfaces represented in Fig. 17 are, up to topological equivalence, all one-sided surfaces without boundary.

7. Addendum 2

All spaces that we consider are supposed to be Hausdorff. Given the spaces X and Y, let $I = [0,1]$ be the unit interval and $Y \times I$ the topological product of the spaces Y and I. Suppose that the spaces X and $Y \times I$ are disjoint and let $W = X \oplus (Y \times I)$ be their disjoint topological sum.

Given a continuous mapping $f\colon Y \to X$, then when the points $(y,0)$ and $f(y)$ are identified for each $y \in Y$, a quotient space $\overline{W}$ is obtained (called in homotopy theory the mapping cylinder of f). The quotient mapping $x \mapsto [x]$ which maps each point $x \in X$ onto its equivalence class $[x]$ in $\overline{W}$ is an embedding and the homeomorphic copy of X in $\overline{W}$ will be denoted by $\overline{X}$.

1. *The space $\overline{X}$ is a strong deformation retract of the space $\overline{W}$.*

Proof. Let $\alpha\colon W \times I \to W$ be given by $\alpha(x,u) = x$ for each $x \in X$ and each $u \in I$ and let $\alpha((y,t),u) = (y, t(1-u))$, for $y \in Y$ and $t \in I$ and $u \in I$. Let $p\colon W \to \overline{W}$ be the natural projection and $p \times i\colon W \times I \to \overline{W} \times I$ be given by $(p \times i)(w,u) = ([w],u)$. Let $H\colon \overline{W} \times I \to \overline{W}$ be given by $H([x],u) = [x]$, for each $x \in X$ and $u \in I$ and $H([(y,t)],u) = [(y,t(1-u))]$, for each $y \in Y$, $t \in I$ and $u \in I$. Since $p \circ \alpha = H \circ (p \times i)$ and being $p \circ \alpha$ continuous and $p \times i$ quotient, it follows that H is continuous. Thus H is a strong deformation retraction and $\overline{X}$ is a strong deformation retract of $\overline{W}$.

Now we describe a quotient model which will be the basis for the calculation of E-P characteristic. Let (X_i), $i = 0,1,\ldots,n$ and (Y_j), $j = 1,\ldots,n$ be two sequences of disjoint spaces. Further, we suppose that all spaces X_i, $i = 0,1,\ldots,n$ and $Y_j \times [j-1,j]$, $j = 1,\ldots,n$ are mutually disjoint and let Y be their disjoint topological sum. Let the mappings

$$f_0\colon Y_1 \times \{0\} \to X_0, \quad \underline{f}_i\colon Y_i \times \{i\} \to X_i,$$
$$\overline{f}_i\colon Y_{i+1} \times \{i\} \to X_i, \ \ i = 1,\ldots,n-1, \quad f_n\colon Y_n \times \{n\} \to X_n$$

be continuous.

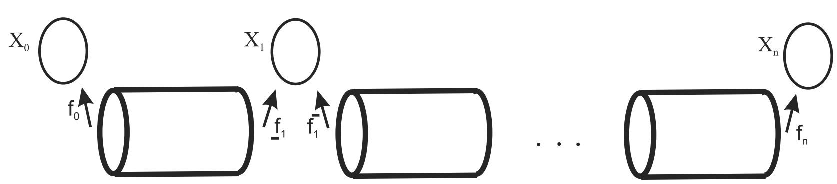

Fig. 18.

Let us write formally $\underline{f}_0 = \overline{f}_0 = f_0$ and $\underline{f}_n = \overline{f}_n = f_n$. Now we suppose that for each $i = 0, 1, \dots, n$ the points (a, i) and $\underline{f}_i(a, i)$ are identified for each $a \in Y_i$, as well as the points (b, i) and $\overline{f}_i(b, i)$ for each $b \in Y_{i+1}$. Each other point of Y is identified with itself. The quotient space which is obtained by this identification will be denoted by X.

The spaces X_i and $Y_i \times \{t\}$, $t \in (i-1, i)$ are homeomorphic to their embedded copies $\overline{X}_i$ and $\overline{Y}_i(t)$, called fibers of X and these fibers make an order decomposition, called here the fibrous decomposition of X and denoted by $X_0(Y_1) \dots (Y_n)X_n$. The number n will be called the *length* of this decomposition and when $n = 0$, the decomposition reduces to X_0. Mapping each fiber $\overline{X}_i$ onto i and each $\overline{Y}_i(t)$ onto t, a function $\varphi\colon X \to [0, n]$ is defined and $\varphi^{-1}([0, k])$, $(k < n)$ is a subspace of X whose fibrous decomposition is $X_0(Y_1) \dots (Y_k)X_k$. We call φ the *function associated with the given fibrous decomposition.* (As it is seen, we use a terminology to avoid confusion with the existing one based on the morpheme "fiber").

A finite space X will be called 0-*fibrous* and the space X itself will be considered to be its own fibrous decomposition. Proceeding inductively we call a topological space m-*fibrous* when it has a fibrous decomposition each fiber of which is k-fibrous for some $k \leqslant m-1$. Now we are ready to prove the statement that follows.

2. *Let X be an m-fibrous space having its fibrous decomposition given by the sequences X_0, X_1, ... , X_n and Y_1, ... , Y_n and the set of mappings $\underline{f}_i$, $\overline{f}_i$, $i = 0, 1, \dots, n$. Then, the Euler-Poincaré characteristic is defined for all spaces X_0, X_1, ... , X_n and Y_1, ... , Y_n as well as for X and*

$$\chi(X) = \chi(X_0) - \chi(Y_1) + \chi(X_1) - \dots + \chi(X_{n-1}) - \chi(Y_n) + \chi(X_n).$$

Proof. This statement is trivially true when $m = 0$. Let us suppose that it is true for all spaces which are k-fibrous for some $k \leqslant m-1$. Let X be an m-fibrous space. When X has a fibrous decomposition of the length 0, then X is k-fibrous for some $k \leqslant m-1$ and the statement is true. Let us suppose it is true for all m-fibrous spaces having a fibrous decomposition of the length less than n. Let X be an m-fibrous space and let X_0, X_1, ... , X_n and Y_1, ... , Y_n be sequences which together with the set of mappings $\underline{f}_i$, $\overline{f}_i$, $i = 0, 1, \dots, n$ determine a fibrous decomposition of X. According to the definition of an m-fibrous space, the spaces X_0, X_1, ... , X_n and Y_1, ... , Y_n are k-fibrous for some $k \leqslant m-1$ and according to the inductive hypothesis on m, the E-P characteristic is defined for all of them.

Let φ be the function associated with the fibrous decomposition of X. Modifying slightly the statement **1**, $\varphi^{-1}([0, n-1])$ is a strong deformation retract of $\varphi^{-1}([0, n-1/3))$, as it is also $\varphi^{-1}(n)$ of $\varphi^{-1}((n-2/3, n])$. As we have already noticed it, the E-P characteristic is defined for $\varphi^{-1}(n) \approx X_n$ and, according to the induction hypothesis on n, it is also defined for $\varphi^{-1}([0, n-1])$. From the following homotopy equivalences

$$\varphi^{-1}([0, n-1]) \simeq \varphi^{-1}([0, n-1/3)), \qquad \varphi^{-1}(n) \simeq \varphi^{-1}((n-2/3, n]),$$

it follows that E-P characteristic is also defined for the spaces $\varphi^{-1}([0, n-1/3))$ and $\varphi^{-1}((n-2/3, n])$. Since these spaces are open in X, we find that

$$(X, \varphi^{-1}([0, n-1/3)), \varphi^{-1}((n-2/3, n]))$$

is an excisive triad. Using now a very well known property of E-P characteristic (see for example [D]), we can write

$$\chi(X) = \chi(\varphi^{-1}([0, n-1/3)) + \chi(\varphi^{-1}(n-2/3, n])) - \chi(\varphi^{-1}((n-2/3, n-1/3))).$$

Since $\varphi^{-1}(n-2/3, n-1/3)) \simeq Y_n$ and using already established homotopy equivalences, we have

$$\chi(X) = \chi(\varphi^{-1}([0, n-1])) + \chi(\varphi^{-1}(n)) - \chi(Y_n).$$

Using the induction hypothesis on n, we replace $\chi(\varphi^{-1}([0, n-1]))$ by the corresponding alternating sum, obtaining so the following equality

$$\chi(X) = (\chi(X_0) - \chi(Y_1) + \cdots + \chi(X_{n-1})) + \chi(X_n) - \chi(Y_n).$$

Thus, we have proved all conclusions of the statement **2**.

REFERENCES

[C-R] R. Courant and H. Robbins, *What is Mathematics?*, Oxford University Press, London - New York -Toronto.

[D] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, 1972.

[M_1] M. M. Marjanović, *Metric Euclidean Projective and Topological Properties*, The Teaching of Mathematics, IV, (1), 2001, 41–70.

[M_2] M. M. Marjanović, *Calculating Euler-Poincaré Characteristic Inductively*, arXiv, December 1, 2012.

[Z-B] G. M. Ziegler and Ch. Blatter, *Euler's polyhedron formula—a starting point of today's polytope theory*, Write-up of a lecture given by GMZ at the International Euler Symposium in Basel, May 31/June 1, 2007.

Serbian Academy of Science and Arts, Kneza Mihaila 35, 11000 Beograd, Serbia

E-mail: milomar@beotel.net