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NEW METHODS FOR CALCULATIONS OF SOME LIMITS

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Abstract. The purpose of the present paper is to establish some properties of certain classes of sequences.

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1. Introduction

Teaching is a dynamic process involving many aspects such as learning, questioning/responding and interpersonal skills. The most inspiring teachers are those who can transmit enthusiasm for their subject to students. If the teacher cannot get excited about the subject, then why should the students?

We take it as a personal responsibility to pass on to others the techniques and concepts that have been acquired. We attempt to do so in a cheerful way by injecting humor whenever possible. The adopted teaching philosophy can best be summed up by the phrase: teach by examples, and that is what we do here.

We present a method for finding limits of sequences which appeared in problem solving math journals.

2. Main results

Throughout the paper we denote: $\mathbb{R}_+^* = (0, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbf{N} = \{0, 1, 2, \dots\}$, $\mathbf{N}^* = \{1, 2, 3, \dots\}$. Our aim is to give a general method to calculate the limits of sequences from some special classes. In what follows:

- $(a_n)_{n\geqslant 1}$ is a positive real sequence with $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}_+^*$ and $\lim\limits_{n\to\infty} n(a_{n+1} - a_n) = b \in \mathbb{R}$;
- $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a function having a continuous derivative $f'\colon \mathbb{R}_+^* \to \mathbb{R}$;
- $(x_n)_{n\geqslant 1}$ is a positive real sequence such that for some $t \in \mathbb{R}$, $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{n^{t+1} x_n} = x \in \mathbb{R}_+^*$.

Theorem.

1. $\lim\limits_{n\to\infty} n(f(a_{n+1}) - f(a_n)) = bf'(a)$.

2. $\lim\limits_{n\to\infty}\left(\dfrac{f(a_{n+1})}{f(a_n)}\right)^n = e^{\frac{bf'(a)}{f(a)}}$.

3. $\lim\limits_{n\to\infty}\dfrac{n^{t+1}}{\sqrt[n]{x_n}} = \dfrac{e^{t+1}}{x}$.

4. If $u_n = \dfrac{\sqrt[n+1]{x_{n+1}}}{(n+1)^t}\cdot\dfrac{n^t}{\sqrt[n]{x_n}}$, $n \geqslant 2$, then $\lim\limits_{n\to\infty} u_n = 1$ and $\lim\limits_{n\to\infty} u_n^n = e$.

5. $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt[n+1]{x_{n+1}}}{(n+1)^t} - \dfrac{\sqrt[n]{x_n}}{n^t}\right) = xe^{-(t+1)}$.

6. If $B_n = f(a_{n+1})\cdot\dfrac{\sqrt[n+1]{x_{n+1}}}{(n+1)^t} - f(a_n)\cdot\dfrac{\sqrt[n]{x_n}}{n^t}$, $n \geqslant 2$, then

$$\lim_{n\to\infty} B_n = \frac{x}{e^{t+1}}(f(a) + bf'(a)).$$

Proof of 1. On each interval $[a_n, a_{n+1}]$, $n \in \mathbf{N}^*$ the function f verifies the conditions of Lagrange Theorem, hence there exists ξ_n situated between a_n and a_{n+1}, such that $f(a_{n+1}) - f(a_n) = (a_{n+1} - a_n)f'(\xi_n)$, which is equivalent to $n(f(a_{n+1} - f(a_n)) = n(a_{n+1} - a_n)f'(\xi_n)$, $n \in \mathbf{N}^*$. Passing to the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} n(f(a_{n+1} - f(a_n)) = \lim_{n\to\infty} n(a_{n+1} - a_n)f'(\xi_n) = bf'(\lim_{n\to\infty}\xi_n).$$

Since ξ_n is situated between a_n and a_{n+1}, $n \in \mathbf{N}^*$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = a$, by the continuity of function f' we have that $\lim\limits_{n\to\infty} f'(\xi_n) = f'(\lim\limits_{n\to\infty}\xi_n) = f'(\lim\limits_{n\to\infty} a_n) = f'(\lim\limits_{n\to\infty} a_{n+1}) = f'(a)$, which yield that $\lim\limits_{n\to\infty} n(f(a_{n+1}) - f(a_n)) = bf'(a)$. ■

Proof of 2. Since $\lim\limits_{n\to\infty}\dfrac{f(a_{n+1})}{f(a_n)} = \dfrac{f(a)}{f(a)} = 1$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{f(a_{n+1})}{f(a_n)}\right)^n \\
&\quad = \lim_{n\to\infty}\left(\left(1 + \frac{f(a_{n+1}) - f(a_n)}{f(a_n)}\right)^{\frac{f(a_n)}{f(a_{n+1}) - f(a_n)}}\right)^{\frac{n(f(a_{n+1}) - f(a_n))}{f(a_n)}} \\
&\quad = e^{\lim\limits_{n\to\infty}\frac{n(f(a_{n+1}) - f(a_n))}{f(a_n)}} = e^{\frac{bf'(a)}{f(a)}}. \quad ■
\end{aligned}$$

Proof of 3. By the equality of Cauchy-D'Alembert limits, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{n^{t+1}}{\sqrt[n]{x_n}} &= \lim_{n\to\infty}\sqrt[n]{\frac{n^{n(t+1)}}{x_n}} = \lim_{n\to\infty}\frac{(n+1)^{(n+1)(t+1)}}{x_{n+1}}\cdot\frac{x_n}{n^{n(t+1)}} \\
&= \lim_{n\to\infty}\left(\frac{x_n\cdot n^{t+1}}{x_{n+1}}\cdot\left(\frac{n+1}{n}\right)^{(n+1)(t+1)}\right) = \frac{1}{x}\cdot e^{t+1} = \frac{e^{t+1}}{x}. \quad ■
\end{aligned}$$

Proof of 4. We have

$$u_n = \frac{\sqrt[n+1]{x_{n+1}}}{(n+1)^{t+1}} \cdot \frac{n^{t+1}}{\sqrt[n]{x_n}} \cdot \frac{n+1}{n}, \quad \forall n \in \mathbf{N}^* \setminus \{1\},$$

so $\lim_{n\to\infty} u_n = \frac{x}{e^{t+1}} \cdot \frac{e^{t+1}}{x} \cdot 1 = 1$ and then $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$.

Also we have

$$\begin{aligned}\lim_{n\to\infty} u_n^n &= \lim_{n\to\infty} \left(\frac{x_{n+1}}{x_n} \left(\frac{n}{n+1} \right)^{n(t+1)} \frac{1}{\sqrt[n+1]{x_{n+1}}} \right) \\ &= \lim_{n\to\infty} \left(\frac{x_{n+1}}{n^{t+1} x_n} \left(\frac{n}{n+1} \right)^{(n+1)(t+1)} \frac{(n+1)^{t+1}}{\sqrt[n+1]{x_{n+1}}} \right) = x \cdot e^{-t} \cdot \frac{e^{t+1}}{x} = e. \quad \blacksquare\end{aligned}$$

Proof of 5. We have

$$C_n = \frac{\sqrt[n+1]{x_{n+1}}}{(n+1)^t} - \frac{\sqrt[n]{x_n}}{n^t} = \frac{\sqrt[n]{x_n}}{n^t}(u_n - 1) = \frac{\sqrt[n]{x_n}}{n^{t+1}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n,$$

for all $n \in \mathbf{N}^* \setminus \{1\}$, so we deduce that

$$\begin{aligned}\lim_{n\to\infty} C_n &= \lim_{n\to\infty} \frac{\sqrt[n]{x_n}}{n^t} \cdot \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} \cdot \lim_{n\to\infty} \ln u_n^n \\ &= xe^{-(t+1)} \cdot 1 \cdot \ln(\lim_{n\to\infty} u_n^n) = xe^{-(t+1)} \ln e = xe^{-(t+1)}. \quad \blacksquare\end{aligned}$$

Proof 1 of 6. We have

$$B_n = f(a_n) \cdot \frac{\sqrt[n]{x_n}}{n^t} \left(\frac{f(a_{n+1})}{f(a_n)} \cdot u_n - 1 \right) = f(a_n) \cdot \frac{\sqrt[n]{x_n}}{n^t}(v_n - 1),$$

for all $n \geqslant 2$, where $v_n = \frac{f(a_{n+1})}{f(a_n)} \cdot u_n$. We obtain $\lim_{n\to\infty} v_n = \frac{f(a)}{f(a)} \cdot 1 = 1$, so we get $\lim_{n\to\infty} \frac{v_n - 1}{\ln v_n} = 1$. Taking this into account, we deduce that

$$\lim_{n\to\infty} v_n^n = \lim_{n\to\infty} \left(\frac{f(a_{n+1})}{f(a_n)} \right)^n \cdot \lim_{n\to\infty} u_n^n = e^{\frac{bf'(a)}{f(a)}} \cdot e = e^{\frac{f(a)+bf'(a)}{f(a)}}.$$

Since $B_n = f(a_n) \cdot \frac{\sqrt[n]{x_n}}{n^{t+1}} \cdot \frac{v_n - 1}{\ln v_n} \cdot \ln v_n^n$, for all $n \geqslant 2$, we obtain

$$\begin{aligned}\lim_{n\to\infty} B_n &= f(a) \cdot \frac{x}{e^{t+1}} \cdot 1 \cdot \ln e^{\frac{f(a)+bf'(a)}{f(a)}} = \frac{f(a) \cdot x(f(a) + bf'(a))}{f(a) \cdot e^{t+1}} \\ &= \frac{x(f(a) + bf'(a))}{e^{t+1}}. \quad \blacksquare\end{aligned}$$

Proof 2 of 6. We have

$$\begin{aligned}B_n &= f(a_{n+1}) \cdot \frac{\sqrt[n+1]{x_{n+1}}}{(n+1)^t} - f(a_n) \cdot \frac{\sqrt[n]{x_n}}{n^t} \\ &= f(a_{n+1}) \cdot \frac{\sqrt[n+1]{x_{n+1}}}{(n+1)^t} - f(a_{n+1}) \cdot \frac{\sqrt[n]{x_n}}{n^t} + f(a_{n+1}) \cdot \frac{\sqrt[n]{x_n}}{n^t} - f(a_n) \cdot \frac{\sqrt[n]{x_n}}{n^t} \\ &= f(a_{n+1}) \left(\frac{\sqrt[n+1]{x_{n+1}}}{(n+1)^t} - \frac{\sqrt[n]{x_n}}{n^t} \right) + \frac{\sqrt[n]{x_n}}{n^{t+1}} (f(a_{n+1}) - f(a_n)) \cdot n,\end{aligned}$$

for all $n \in \mathbf{N}^* \setminus \{1\}$, and passing to the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} B_n = f(a)\cdot\frac{x}{e^{t+1}} + \frac{x}{e^{t+1}}\cdot\lim_{n\to\infty}((f(a_{n+1}) - f(a_n))\cdot n)$$
$$= x(f(a) + bf'(a))e^{-(t+1)}. \quad \blacksquare$$

3. Applications

We will present some examples of applications of the previous results.

A1. If $x_n = n!$, $a_n = n^{1/n}$, $n \in \mathbf{N}^*$; $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(x) = x$, then:

$$\lim_{n\to\infty}\frac{x_{n+1}}{nx_n} = \lim_{n\to\infty}\frac{(n+1)!}{n\cdot n!} = 1 = x, \quad t = 0, \quad a = 1,$$
$$b = \lim_{n\to\infty} n(a_{n+1} - a_n) = \lim_{n\to\infty} n(\sqrt[n+1]{n+1} - \sqrt[n]{n})$$
$$= \lim_{n\to\infty}(((n+1)\sqrt[n+1]{n+1} - n\sqrt[n]{n}) - \sqrt[n+1]{n+1}) = 1 - 1 = 0.$$

Therefor in this case,

$$B_n = \sqrt[n+1]{n+1}\,\sqrt[n+1]{(n+1)!} - \sqrt[n]{n}\,\sqrt[n]{n!}, \ \ \forall n \in \mathbf{N}^* \setminus \{1\}, \ \text{and} \ \lim_{n\to\infty} B_n = e^{-1}.$$

A2. If $x_n = n!$; $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(x) = 1$ and $(a_n)_{n\geqslant 1}$ is a positive real sequence, then:

$$B_n = f(a_{n+1})\cdot\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^t} - f(a_n)\cdot\frac{\sqrt[n]{n!}}{n^t} = \frac{\sqrt[n+1]{(n+1)!}}{(n+1)^t} - \frac{\sqrt[n]{n!}}{n^t},$$

$t = 0$, $a = \lim_{n\to\infty} a_n$, $x = 1$, so we obtain

$$B_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}, \quad \forall n \in \mathbf{N}^* \setminus \{1\},$$

i.e., we obtain the sequences considered by T. Lalescu in [3], with $\lim_{n\to\infty} B_n = e^{-1}$.

A3. Let $(u_n)_{n\geqslant 1}$, $(v_n)_{n\geqslant 1}$ be real positive sequences and let there exist $t \in \mathbb{R}^*$ such that $\lim_{n\to\infty}\frac{u_{n+1}}{n^{t+1}u_n} = u \in \mathbb{R}_+^*$, $\lim_{n\to\infty}\frac{v_{n+1}}{n^{t+1}v_n} = v \in \mathbb{R}_+^*$. Then we can calculate (as a generalization of [4])

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{u_{n+1}}{v_{n+1}}} - \sqrt[n]{\frac{u_n}{v_n}}\right).$$

Solution. We have $\lim_{n\to\infty}\frac{u_{n+1}v_n}{nv_{n+1}u_n} = \lim_{n\to\infty}\frac{u_{n+1}}{n^{t+1}}\cdot\lim_{n\to\infty}\frac{n^t v_n}{v_{n+1}} = \frac{u}{v} \in \mathbb{R}_+^*$, and taking $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(x) = 1$; $x_n = \frac{u_n}{v_n}$, $n \in \mathbf{N}^*$, we get that $\lim_{n\to\infty}\frac{x_{n+1}}{nx_n} = x = \frac{u}{v} \in \mathbb{R}_+^*$, $t = 0$ and

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{u_{n+1}}{v_{n+1}}} - \sqrt[n]{\frac{u_n}{v_n}}\right) = \frac{x}{e}(f(a) + bf'(a)) = \frac{x}{e} = \frac{u}{ve},$$

where $(a_n)_{n\geqslant 1}$ is a positive sequence with $\lim_{n\to\infty} a_n = a \in \mathbb{R}_+^*$, $\lim_{n\to\infty} n(a_{n+1} - a_n) = b$.

If $u_n = b_n$, $v_n = a_n$, $a_n, b_n \in \mathbb{R}_+^*$, $n \in \mathbf{N}^*$, $\lim\limits_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = \lim\limits_{n\to\infty} \frac{b_{n+1}}{n^3 b_n} = a \in \mathbb{R}_+^*$, then

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{b_{n+1}}{a_{n+1}}} - \sqrt[n]{\frac{b_n}{a_n}}\right) = \frac{a}{ae} = e^{-1},$$

i.e., we have solved Problem 24 from the journal MathProblems [4].

A4. Let $f, g\colon \mathbb{R} \to \mathbb{R}$, $f(x) + g(x) = 1$ for all $x \in \mathbb{R}$, and

$$B_n(f,g) = n^{f(x)}\big((\sqrt[n+1]{(n+1)!})^{g(x)} + (\sqrt[n]{n!})^{g(x)}\big).$$

Then we can calculate $\lim\limits_{n\to\infty} B_n(f,g)$.

Solution. We note that

$$B_n(f,g) = n^{f(x)}(\sqrt[n]{n!})^{g(x)}(w_n(x) - 1) = n^{f(x)+g(x)}\left(\frac{\sqrt[n]{n!}}{n}\right)^{g(x)}(w_n(x)-1)$$

$$= n\left(\frac{\sqrt[n]{n!}}{n}\right)^{g(x)}(w_n(x)-1) = \left(\frac{\sqrt[n]{n!}}{n}\right)^{g(x)}\frac{w_n(x)-1}{\ln w_n(x)}\cdot \ln(w_n(x))^n,$$

for all $n \in \mathbf{N}^* \setminus \{1\}$, where $w_n(x) = \left(\dfrac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^{g(x)}$. Hence,

$$\lim_{n\to\infty} w_n(x) = \left(\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n}{\sqrt[n]{n!}}\cdot\frac{n+1}{n}\right)\right)^{g(x)} = \left(\frac{1}{e}\cdot e\cdot 1\right)^{g(x)} = 1,$$

and $\lim\limits_{n\to\infty}\dfrac{w_n(x)-1}{\ln w_n(x)} = 1$.

Also, we have

$$\lim_{n\to\infty}(w_n(x))^n = \left(\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right)^n\right)^{g(x)} = \left(\lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{1}{\sqrt[n+1]{(n+1)!}}\right)^{g(x)}$$

$$= \left(\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^{g(x)} = e^{g(x)}.$$

Therefore,

$$\lim_{n\to\infty} B_n(f,g) = \left(\frac{1}{e}\right)^{g(x)}\cdot 1\cdot \ln(e^{g(x)}) = \frac{g(x)}{e^{g(x)}} = g(x)e^{-g(x)}.$$

If $f(x) = \cos^2 x$, $g(x) = \sin^2 x$, $x \in \mathbb{R}$, then

$$B_n(f,g) = L_n(x) = n^{\cos^2 x}\left((\sqrt[n+1]{(n+1)!})^{\sin^2 x} - (\sqrt[n]{n!})^{\sin^2 x}\right),$$

and $\lim\limits_{n\to\infty} B_n(f,g) = \lim\limits_{n\to\infty} L_n(x) = \sin^2 x\cdot e^{-\sin^2 x}$, i.e., we have solved Problem 67 from MathProblems [5].

Remark. By the methods used in the proofs given above, one can also solve the following problems (as well as many others from various math problem solving journals):

Problema 579, Gazeta Matematică, Vol. VI, 1900–1901, pp. 33-38;

Problem 692, The Pentagon, Vol. 71, No. 1, 2011, p. 54;

Problem 5208, School Science and Mathematics Journal, April 2012;

Problem 24, MathProblems, Vol. 1, No. 4, 2011, p. 33;

Problem 43, Math Problems, Vol. 2, No. 3, 2012, p. 91;

Problem 67, MathProblems, Vol. 3, No. 2, 2013, p. 140;

Problem 704, The Pentagon, Vol. 71, No. 2, 2012, p. 42;

Problem 11676, The American Mathematical Monthly, Vol. 119, No. 9, November 2012, p. 801;

Problem 3713, Crux Mathematicorum, Vol. 38, No. 2, February 2012, p. 63;

Problem 715, The Pentagon, Vol. 72, No. 1, p. 44;

Problem 234, La Gaseta de la RSME, Vol. 16, No. 3, 2013, p. 502;

Problem 241, Revista Escolar de la Olimpiada Iberoamericana de Matematica, No. 49, 2013;

Problem 3764, Crux Mathematicorum, Vol. 38, No. 7, September 2012, p. 285;

Problem W3, József Wildt International Mathematical Competition, The Edition XXIII, 2013, Octogon Mathematical Magazine, Vol. 21, No. 1, April, 2013, p. 229;

Problem 75, MathProblems, Vol. 3, No. 3, 2013, p. 171.

4. Conclusion

Success in problem solving requires effort. There are three important aspects of learning mathematics.

First, getting the idea or the concept.

Second, you must practice the skills that you hope to develop and need for the homework problems. Without this skill development, the understanding of the concepts will not get you very far.

The third aspect of learning mathematics is the assimilation process which enables you to recognize ideas you have encountered in other contexts and gives you the confidence to make the leap to solving problems, the likes of which you have not seen before. This is where the problems of the “problem solving” journals come in. These are not routine exercises. They are problems whose solutions depend of trying something new.

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[4] D. M. Bătineţu-Giurgiu, N. Stanciu, *Problem 24*, MathProblems, **1** (4) (2011), 33.

[5] D. M. Bătineţu-Giurgiu, N. Stanciu, *Problem 67*, MathProblems, **3** (2) (2013), 140.

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