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TO BE INTEGER OR NOT TO BE RATIONAL: THAT IS THE QUESTIO$\sqrt{N}$

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Abstract. Another proof is given of the fact that the square root of a nonnegative integer is either an integer or an irrational. Bibliography on this theme is presented.

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The square root of a nonnegative integer is either an integer or an irrational. To show this, let N be a nonnegative integer.

(i) If $N = m^2$ for some nonnegative integer m, then $\sqrt{N} = m$.

(ii) But if this is not the case, then $b^2 < N < (b+1)^2$ for some positive integer b. Note that $\sqrt{N}$ being rational implies that there exist positive integers m, n, with n minimal, such that $x = \sqrt{N} + b = m/n$. In addition, the bound $\sqrt{N} < b+1$ gives us $m/n < (b+1)+b$ and so $m - 2nb < n$. Now observe that $x(x-2b) = (\sqrt{N}+b)(\sqrt{N}-b) = N - b^2 > 0$, and consequently

$$x = \frac{N-b^2}{x-2b} = \frac{N-b^2}{\frac{m}{n}-2b} = \frac{n(N-b^2)}{m-2nb},$$

which contradicts the minimality of n.

Further comments. The argument above may be considered as a generalization of the one in [36], which was designed to prove the irrationality of $\sqrt{2}$.

Should it be of interest to the readers

There are many proofs of the above result disperse in the literature. We refer the reader to [7] as a first approach (this reference is mainly concerned with the case $N = 2$, but also includes the proofs in [4, 5, 13, 17, 31, 40, 43] of the general case). As a way of classifying the different proofs, let us say that some of them are of arithmetic nature: they rely on the prime factorization of N (see, for example

This note is one of many slightly varied proofs of the irrationality of square root from any natural number not being a square itself. Attached list of references serves the interested reader to follow these variations. The Editors

[22]). Other proofs rely on the division algorithm [4, 18, 26, 28, 31], or on Bezout's Lemma [17, 24, 38]. The use of algebraic relations is exploited in [2, 5, 6, 8, 11, 13–16, 20, 21, 23, 29, 30, 39–41, 43]. Geometric constructions, as "proofs without words", can be found in [1, 3, 9, 12, 25, 32, 34, 35, 37]. Also convergence criteria are used to prove irrationality, as [10, 25, 42] show. Finally, the digit representation of N in some base is also a plausible argument used in [19, 27, 33].

We must also point out that [6, 15, 40] essentially coincide with [39]. Also, our contribution is close to [16], but it may be considered as a simpler form of exploiting a similar idea.

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