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ALGEBRA AS A TOOL FOR STRUCTURING NUMBER SYSTEMS

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Abstract. Numbers in their natural dependence on sets of visible things are abstract products which result from ignoring nature of elements of these sets and any kind of their organization (the way how they are arranged, grouped, etc). We call formulation of this way of cognition *Cantor principle of invariance of number* and, in this paper, we apply it to a coherent exposition of the first grade arithmetic.

Reacting to the way how the elements of a set are grouped, one expression is written, regrouping these elements, another expression is written. Then, such two expressions are equated since they denote one and the same number. Thus, this interplay between meaning and symbolic expressing is the ground upon which the range of numbers up to 20 is structured.

Two disjoint sets together with their union make an additive scheme to which an addition or a subtraction task may be attached. Dependently on such a task, sums and differences are written as expressions denoting numbers. To reach the unique decimal (in digits) denotation of a number some steps of transforming the corresponding expressions are made. Displaying these intermediate steps leads to a thorough understanding of the arithmetic procedures, which should precede their suppressing which leads to automatic performance. In particular, methods of adding and subtracting when the ten-line is crossed are treated in detail.

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1. Introduction

We consider the gradual building of number systems to be the main objective of learning and teaching mathematics in the type of schools in which children of age 6–15 are educated. Different names are used to denote this type of schools and we will call this division of obligatory education elementary school (and its lower stage primary school). The content of its mathematical curriculum does not differ much from one country to another, but what does differ is the way how that content is didactically transformed and, it is particularly true, when arithmetic is concerned. Our aim in this paper is an approach to the first grade arithmetic elaborated as a gradual building of number blocks (up to 10 and to 20) in which some of the technique of early algebra is employed as a tool. It is just the first step in studying such approaches in cases of all other number systems, concluding with the system of real numbers. We find that the Freudenthal's didactical analysis [4, 5] is the right way of such studying when a content of school mathematics is exposed to considerations based on mathematics as a science, its history and experience gathered from the long-running school practice, but also on developmental and cognitive psychology. Many concepts of school mathematics are in the state of development and their scientific formalization projects their meaning clearly and sharply. Historically these concepts are seen in the state of creation and their gradual formation in the mind of a child is reflected by psychology.

First we will be considering those instances when algebraic content is clearly present (use of letters, transformation of literal expressions, etc.) and when that content is an application of algebra to some important themes of school mathematics.

We take metric geometry as the first instance of this kind, where area of figures (planar shapes) and volume of solids are expressed in terms of lengths of their line elements which uniquely determine them (up to the congruence). Lengths of such line elements are seen denoted by letters and the corresponding areas and volumes are determined in the form of literal expressions. For example, the area of a rectangle is given by the formula $A = ab$, where a and b denote the lengths of its sides expressed as numbers depending on a unit of measure. This is the case when the letters are first in the role of variables denoting any natural (or rational) number and afterwards (8-th grade) any real number.

The second instance is the case of solving linear equations, when letters are used to denote unknowns (and coefficients). The frame within which solutions are sought usually starts as the system of natural numbers and ends as the system of reals

The third and very important instance where algebra is involved, is the case of elaboration of the theme "Real numbers". Summarizing very briefly, it goes like this:

- letters are used to denote lengths of line segments which are considered to be conveyors of the meaning of numbers,
- some of this lengths are not rational numbers as, for example, such is the diagonal of a square when measured by its side,
- in contrast to rational numbers these new numbers are called irrational and they both, taken together, constitute the system of real numbers,
- and, at the end, it is said that we operate with irrational numbers as they were rational (having so expressed the fact that algebraic properties of ordered fields are the same).

Without a systematic interpretation of real numbers and operations with them as constructions performed on segments lying on a line, this approach lacks in meaning as well as without having teachers be acquainted with the way how the idea of real number had been developing from Eudoxus to Descartes, their deeper understanding of the didactical transformation of this theme cannot be expected, either.

The fourth instance is the case when linear and quadratic functions enter the content of (usually) 9th grade curriculum. These functions are used to express various interdependencies and their graphs are constructed, what requires a thorough discussion of the properties of expressions $ax + b$ and ax^2.

2. Brief observations on historical development of number systems

Considering the sensory aspects of existence, let us point out the sharp psychological difference that exists between the ways how are mentally represented the experience of discrete appearances and that of continuous ones. At the reflective level, this difference is the ground upon which two different ways of conceptualization in mathematics stand—one leading to the creation of number systems and the other to the creation of geometric ideas. The border line between these two modes of thought has been reflected on school mathematics dividing it into two main courses—arithmetic with algebra and geometry.

Gradual building of the natural number system and the system of rational numbers as its extension comprises almost all content of school arithmetic. Taken historically, this content corresponds to the way how the whole numbers and their ratios were understood and used by Pythagoreans. But when they discovered incommensurable magnitudes (attributed to Hippasus of Metapontum), it caused a great logical crisis which was overcome by the Eudoxian theory of proportions. In that theory, the idea of number was taken in an extended sense, to be the ratio of magnitudes of the same kind and Eudoxus created a system of numbers which corresponds to the contemporary system of (positive) reals. Being embedded in the frame of geometry, this rather complicated theory with its operations which were performed on magnitudes, never left Euclid's *Elements* to enter school curricula.

Though the modern system of real numbers has its roots in Eudoxiam theory, it is much more abstract, logically better founded, incomparably more operative and ideologically, on the tracks of ancient arithmetic. This system is also the foundation stone of modern mathematics and such its rise would not be possible without some important mathematical contributions from the time of the Renaissance.

Some use of letters to denote a general algebraic type of calculation was known even in antiquity. But Vieta was the first to use letters to denote species of numbers and magnitudes which were not only unknowns but also coefficients of equations. In that way he introduced in mathematics the fundamental idea of variable, creating so his algebra (*logistica speciosa*) which still was rhetoric and with the operations performed on magnitudes, what is evidently displayed by the terms involved in his equations as, for example, they are: "*D plano in A*", "*Z solido*", etc. The addition of magnitudes was bound to those of the same kind, what was known as the requirement of homogeneity. Algebra for Vieta was a method for solving geometric problems and the interdependence of algebra and geometry was also the occupation of his disciple Marin Getaldić (Latin name Marino Ghetaldi). But certainly, the most significant of his followers was Descartes, who began where Vieta left off.

Gradually, algebraic symbolism was developed and improved by the mathematicians who appreciated Vieta's ideas (Harriot, Girard, Oughtred et al.) including Descartes who used first letters of the alphabet for known quantities and last letters for unknowns, further on instead of $a \times b$ he wrote ab and he used the symbols a, a^2, a^3, ... to denote powers of a. Thus, his algebraic symbolism was almost the same as the modern one is. Therefore, we can say that with Descartes rhetoric algebra ends and symbolic one begins.

In *La Géométrie*, one of the three famous appendices to his *Discours de la Méthode*, Descartes represented equivalent ratios of magnitudes as a unique relation between the length of a line segment and the unit segment (which corresponds to the number 1). Taking a ray emanating from a point O (its origin), all segments were taken to lie on this ray having one of their ends at O. Thus, the length of such a segment measured by the unit segment becomes the conveyor of the meaning of a real number and different segments of different numbers. Performing simple geometric constructions sums, differences, products and quotients of line segments were again line segments. What is particularly important, this geometric model of the real number system is closed with respect to these operations (excluding subtraction of longer segments from the shorter ones) and this model served as a ground upon which all further modifications of the real number system have been made. This was also a case of unification of ancient arithmetic, Eudoxian theory and the Renaissance algebra.

Denoted by letters and interpreted geometrically as points on a line, real numbers constitute an important didactical theme of school mathematics (usually first encountered in the curriculum of the eighth grade).

3. Some terminological remarks

Though we will be concerned with the way how teaching and learning of arithmetic is approached in the first grade of elementary school, we find that it is necessary to precise how some general terms and expressions are used in this paper, when those processes through which that piece of elementary school mathematics is synthesized, are described. Thus, for example, the language of set theory is used when description of some real world situations and actions are expressed to ensure, in that way, the needed precision. This, of course, means that this language will be used as the part of terminology of didactics of mathematics and it will enter school contents only if and when is deliberately planed.

Further, we use the term “idea” more freely to mean an abstraction that results from the observation of some relationships and properties. On the other hand we use the term “concept” more strictly. Namely, following and somewhat extending a scheme of R. Skemp [17], we take a concept to be a tripartite entity which consists of three components—the *class of corresponding examples* providing the meaning, *mental image* being a representation in the mind formed by experiencing the corresponding examples and *name* being a word labelling the concept [14].

Examples of some concepts are concepts themselves and a concept A is said to be *more abstract* (*general*) than a concept B if B is an example for A. In this case B is also said to be *less abstract* (*general*) than A. When all examples which correspond to a concept are observable things, then such a concept is said to be at the sensory level and, as the classical logic teaches us, “things are not concepts”.

As the theory of sets was created to be a language more general than the existing languages of algebra and geometry, its fundamental idea—the concept of set was taken to be more general than any other concept of classical mathematics. With such generality, the concept of set has its examples at all levels of abstractness and its full meaning can be acquired only after years of learning. But as the meaning of arithmetic concepts arise from observation of and the activity with some collections of real world objects and iconic signs that represent them, the term set at sensory level is used to denote such a collection.

Let us recall that *mental (cognitive) scheme* is taken to be a blend of mutually related mental images. Following Vygotskii [19] a *system* is taken to mean a set of concepts together with all linking relations between them. When such a system is formulated in terms of sets and set theoretic relations, then it is called *(mathematical) structure*. Hence, a system of concepts and a structure (as it has been just defined) have nearly the same meaning, while the mental scheme is their reflection in the mind.

Let us also point out that two kinds of iconic signs are used and that they should be distinguished. When a drawing is used to represent a real world appearance then it is called a *pictogram*. As being simplified drawings, pictograms are not only used to represent an appearance that is not immediately experienced, but they also serve to reduce the existing noise (inessentials) present in reality. In that way they display the essential features of observed objects helping the process of abstraction. An iconic sign that is used to represent a concept, projecting clearly its essential properties is called an *ideogram*. Typical examples of ideograms are geometric drawings and number pictures used in arithmetic to represent initial sets of numbers.

4. Counting or matching—an apparent dilemma

Names of numbers from an initial set (up to 10 or 20) exist in all natural languages and they had been formed in the remote prehistoric past. And, as we know it from the history of mathematics, that sequence of number names was extended in the ancient civilizations to meet the needs of those civilizations. According to the systems of signs used to represent numbers, it is evident that such extensions were done on the additive (and multiplicative) basis (bases), as it is also the case with our contemporary decimal system. Therefore, we should notice that counting is not an endless reciting of number names, but contrary to it, numbers and their names have been constructed out of those from an initial set (in case of decimal system, numbers 0, 1, 2, … , 9).

On the other hand, simple matching of sets (that was seen to overwhelm school books in the period of New Math) is not sufficient to develop the idea of finite cardinal number. On the contrary, the concepts of the individual numbers 1, 2, 3, … are formed first, and after years of learning a more general concept of natural number (finite cardinal) gains its full meaning.

Let us recall that Georg Cantor was using the symbol $\overline{\overline{A}}$ to denote the cardinal number of the set A. Two over-bars stand to denote two abstractions: ignoring the nature of elements of the set A and the way how they are arranged. This was the way how Cantor expressed the cognitive process leading to the creation of numbers in their natural dependence on sets of observable objects. Slightly modifying this way, we can formulate the *Cantor principle of invariance of number* to read:

Starting with the observation of a concrete set A and forgetting

(i) *the nature of elements of A,*

(ii) *any kind of their organization,*

an abstract idea of number is left behind.

Let us note that the term "organization" is less formal than a mathematically more precise expression "structure on the set A" and it includes all ways how the elements of A may be arranged, grouped etc. Also, instead of establishing formally equivalences of sets by using one-to-one correspondences, the Cantor principle is used as a guide for designing didactical procedures in arithmetic by means of which numbers and their systems are constructed. Thus, arithmetic is neither built on counting nor on matching but, in reality of school practice, by a gradual construction of number blocks (up to 10, to 20, to 100 etc) as systems having their own structure and having each, its own package of didactical tasks attached. This is true now as it has always been, only didactical procedures change in time.

With now obligatory schooling being prolonged, learning, at any price, of the mere arithmetic facts is no longer the main goal of teaching arithmetic. Understanding of the involved procedures and the ability to express them in words and symbolically is a contemporary task of intelligent learning in primary school. The use of symbols not only shortens expression in words but also accelerates thoughts. As such, this way of thinking requires a lot of practice as it is the case with numerous exercises which are done when arithmetic and algebra are learnt. But when symbols and the expressions that they make are devoid of meaning, then the mere practice cannot provide understanding. In traditional school, when goal-oriented arithmetic was completely separated from algebra, the transition from the former to the latter appeared to be an abrupt semantic jump. Difficulties in learning such algebra have been thoroughly studied and several adjustments suggested what led to the creation of a new branch of didactics of mathematics known as early algebra. There is a rich literature on this subject and majority of the authors consider the introduction of some bits of algebraic content in the primary grades as the way to facilitate students understanding later on. C. Kieran has been a particularly active researcher who has produced several papers [7–10] suggesting foci: on relations and not only on calculation, on operations and their properties, on transformation of expression, on letters in the role of variable, on representation and solving problems. Several papers deal with cognitive analysis of algebraic thinking, as for example, Arzarello et al. [1], Lins [12], etc. Some papers have their focus on functional thinking, as for example, Blanton & Kaput [2], Radford [16] and finally, let us also add that all papers in the proceedings Cai & Knuth [3] are devoted to early algebra.

It is a widely spread opinion that approaches how school arithmetic is elaborated vary very much from one country to another and that there is no complete survey of the ways how arithmetic should be taught and learnt in the contemporary school. Some interesting books do exist [11, 13, 18] etc., which throw light on some important aspects of learning and teaching arithmetic, but the structural analysis of the whole its content, sequenced into didactical themes which would be worked out with great care and nicety of detail is seriously lacking. On the other hand, several papers exist which treat this or that specific topic of learning and teaching arithmetic as, for example, they are [6, 15], etc.

An active use of arithmetic expressions (sums, differences, etc.), establishing of their properties and their transformation from one form to another is recognized as an instance of early algebra. In this paper we use that segment of early algebra as a tool for didactical transformation of arithmetic content of the first grade of primary school.

5. Block of numbers up to 10. A list of didactical tasks

The order in which the tasks that follow are listed indicates their performance in the sequence. Some of them will be just mentioned, some others followed with short addenda and only those where the technique of early algebra is used, will be discussed at greater length.

1. *Counting up to 10.* All children are supposed to know to recite in order the number names up to 10. They are involved in activities of counting numbers of elements of "small" sets, assigning the number names to these elements. Changing the order of assigning in different ways and obtaining each time one and the same number, thus they become aware of the fact that the number is independent of the way of counting.

2. *Display of additive schemes.* Let us recall that an additive scheme is a pair of disjoint sets A and B, together with their union $A \cup B$. An addition task is associated with such a scheme when cardinal numbers of A and B are given and cardinal number of $A \cup B$ is sought for. When cardinal numbers of $A \cup B$ and of one of the sets A or B are given and cardinal number of the other of them is sought for, then we say that a subtraction task is associated with that scheme.

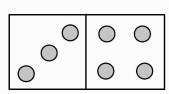
In the same way how the image of a set precedes the idea of its cardinal number, the image of an additive scheme precedes addition and subtraction, dependently on which of the two tasks is associated with it. Didactical realization of the task 2 consists of a display of pictorial representations of pairs of sets. The sets A and B should be chosen so that their elements, as well as the elements of $A \cup B$, are named differently (for example, red pencils, blue pencils and pencils; orange juices, tomato juices and juices; etc.). Then, on hearing the names of these elements, children select the sets A and B together with their union $A \cup B$. This selection is a mental operation provoked by questions: How many of these (say, orange juices) are there, How many of those (say, tomato juices) are there, How many altogether (juices) are there. Of course, all numbers are found by counting and all answers are given orally. Let us add also that, when various real world activities are described and which can be modelled as an additive scheme, the terms as "putting more", "adding", "taking away", etc. are used. Teachers should, of course, help the children to see the models of such situations by using the corresponding pictorial representations which show "what was" and "what is now".

3. *Numbers 1, 2, ... , 10 as individual concepts.* Contrary to the view that by counting, children have already formed the concepts of numbers up to 10, we hold an opinion that each of these numbers is an individual concept having its meaning, name and symbolic notation. And after the practice of counting, each of these numbers deserves a separate lesson to expand its meaning and to establish all relations which connect that number with all numbers that precede it. At an early stage of learning, number names are not separated from the names of objects and, for example, looking at a picture and after counting, children say: "3 red pencils and 4 blue pencils make 7 pencils", etc. When a lesson deals with an individual number, say with 3, it contains pictures illustrating "putting 3 objects more", "taking away 3 objects", etc. Such illustrations are iconic representations of situations to which sums with a summand 3 and differences with a subtrahend 3 are assigned. For this kind of activities we could say that they are instances of intuitive addition and subtraction that precede a more formal establishment of these operations.

4. *Writing sums and differences.* From the very start, numbers involved in operations should be taken independently of the sets which provide their meaning. To make this abstraction develop, the language expressing operations with numbers must be clearly different from that expressing operations with sets. For this purpose, the sets A and B which make an additive scheme are more convenient if they have for their elements objects of the same kind (for example, dots, pips, counters, etc.).

Examples that would be inducements for children to write sums are like this one:

Look at the domino



We see 3 *pips and* 4 *pips. To denote the total number of pips, we write:* 3 + 4 *and we read it: three plus four*.

Fig. 1. Domino as an example of an additive scheme

After having a number of similar examples done with the help of the teacher, children continue to write sums helped by place holders. For instance such an assignment is:

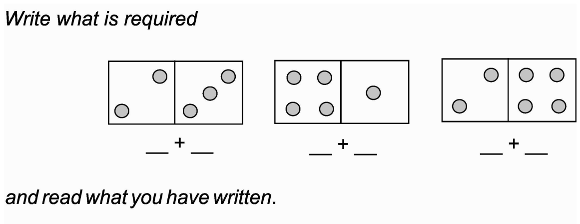


Fig. 2. Dominoes as additive schemes—place holders suggest addition tasks

Similarly, differences are attached to the additive schemes followed with a subtraction task. For the start, examples as the following one can be used:

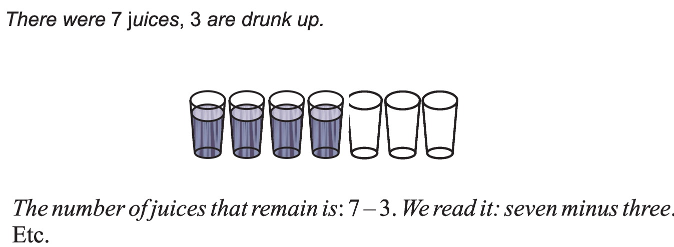


Fig. 3. Illustration of an additive scheme suggesting subtraction task

Afterwards, children do the exercises like this one:

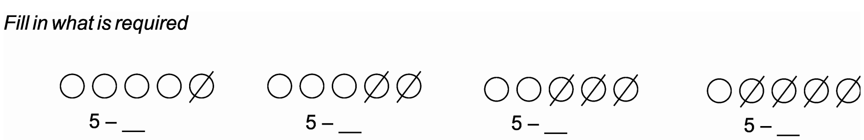


Fig. 4. Additive schemes suggesting subtraction task

As a result of these activities sums and differences that are assigned to the schemes denote the total number of elements or the number of those which remain. Thus, this is an instance when numbers are denoted differently than only by writing digits.

It is worth noting that children accept quickly and spontaneously the technique of space holders. When these holders are exclusively used to be indicators of blank spaces, then they make a mighty didactical technique which allows access to the elaboration of several arithmetic topics (avoiding a fatal rigmarole of words).

5. *The use of the equal sign.* Children should not be hurried over the task of writing sums and products. But after they have enough practice, they are taught to use the equal sign to equate two different denotations of the same number. For example, if on seeing a domino, $4+5$ is written and after counting the total of pips, 9 is written, then these two denotations are equated and $4 + 5 = 9$ is written and read: four plus five is equal to nine. Following this procedure of equating, children do similar exercises supplied by space holders:

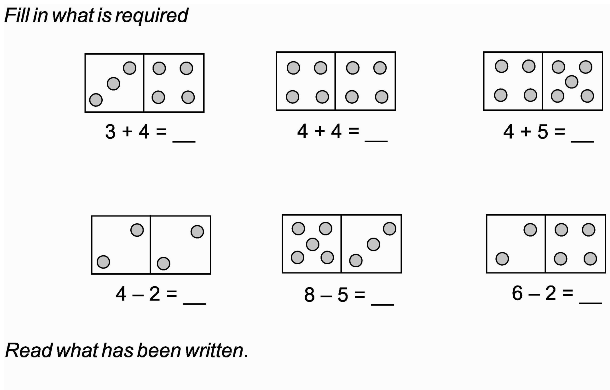


Fig. 5. Dominoes as additive schemes followed by addition tasks and subtraction tasks. Numbers of pips are recognized at the first glance

This is the right place for some comments to do. It is wrong to think (and to say) that calculating $4 + 5$ the number 9 is found. Both expressions are obtained on seeing the same set of pips, once as two groups of 4 and 5 pips and the other time as a single group of 9 pips. The numbers of elements of the involved sets are found by counting them or by recognition of these numbers at the first glance. It is generally maintained that such recognition does not work when the numbers are larger than 3 (or 4) or if the sets are not arranged so that such arrangements have recognizable shapes. Dominos are an example of such arrangement, the others are so called number pictures which are designed for this purpose. In Fig. 6 the number pictures that we use here are represented:

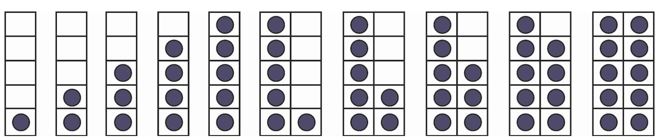


Fig. 6. Number pictures representing numbers 1—10

For example, looking at the pictures in the following exercise:

Fig. 7. An additive scheme followed by two different subtraction tasks

in the first case minuend and subtrahend and in the second minuend and difference are recognized at the first glance.

Hence, following this approach of elaboration of arithmetic, the equal sign has the function of equating and it is not understood as a command for calculation.

6. *Interchange of places of summands.* When an additive scheme consists of sets A and B, having m and n elements respectively and when A, (B) is considered to be the first set, then $m+n$, $(n+m)$ is used to denote the number of elements of $A \cup B$. As this order of looking does not have an effect on the total number of elements, by equating two deferent denotations, the equality $m + n = n + m$ is obtained. Termed didactically, this is the *rule of interchange of summands* and in theoretical mathematics it is a fundamental property of addition called *commutative law*. Stated in words this rule reads: *summands may be interchanged without altering the sum.*

At this stage of learning, it is spontaneously accepted that the number of elements of $A \cup B$ does not depend on the way of counting them—starting with counting elements of A first or those of B. Of course, this means that activity of attaching both sums alternatively will run without any suspicion that such procedure could be incorrect. But if this underlying intuition has to be expressed symbolically, then some exemplary exercises have to be done first:

When we look at the picture

we see 3 *red counters and* 4 *blue counters. Altogether, it is* 3 + 4 *counters. Looking again at the picture, we see* 4 *blue counters and* 3 *red counters. Altogether, it is* 4 + 3 *counters. Equating two denotations for the same number, we can write* 3 + 4 = 4 + 3.
Etc.

Fig. 8 An additive scheme to be seen from left to right and in the reversed direction

Doing similar exercises children will accept this rule and will be successful in completing equations like the following:

$$3+6=6+\underline{\quad}, \quad 3+6=\underline{\quad}+3, \quad \underline{\quad}+8=8+2, \quad 2+\underline{\quad}=8+2, \quad \text{etc.}$$

But it will be better if they also have a motive to do such interchanges. The values (denotations by digits) of all sums within this block are left to be memorized spontaneously. But, in fact, these values are found by a quick mental "putting more" which could be supported by inner representations of number images. We leave this topic aside and we note that children become easily aware that it is easier to complete, for example, an arrangement of 9 counters, starting with a group of 6 of them than with a group of 3 of them. Finding the value of 6 + 3 is easier as it corresponds to that easier way of arranging.

This is an instance of interchange of places of summands with a clear motivation. To practice this interchange, exercises supplied with place holders should be given:

a) $2+6=6+\_\_=\_\_$, $3+7=7+\_\_=\_\_\_$, $3+4=4+\_\_=\_\_$, etc.

b) $2+4=\_\_=\_\_=\_\_$, $1+8=\_\_+\_\_=\_\_$, $4+5=\_\_+\_\_=\_\_$, etc.

Reacting to the same additive scheme, consisted of the sets A and B, having m and n elements respectively and with their union $A \cup B$ having s elements, in case of a subtraction task the equalities $s-m=n$ or $s-n=m$ are written. But when one of these equalities is checked to be true, the other one will be true without additional checking. In this way the *rule of interchange of subtrahend and difference* is expressed and that rule is correspondent to the rule of interchange of places of summands in the case of addition. Corresponding exercises that can be given are:

When you check:	*without checking you write*:
$8-3=5$	$8-5=\_\_$
$7-4=3$	$7-3=\_\_$, etc.

7. *Interdependence of addition and subtraction.* When an additive scheme consists of the sets A, B and $A \cup B$ having m, n and s elements respectively, then the following four equalities

$$m+n=s, \quad n+m=s, \quad s-m=n, \quad s-n=m$$

could be attached to it, dependently on the accompanying task. When one of the first two equalities is checked to be true, then one of the last two can be taken without checking and vice versa. In this way interdependence of addition and subtraction is expressed. At this early stage, children should have these equalities grouped together and let to get aware of this interdependence through performing activities. For example:

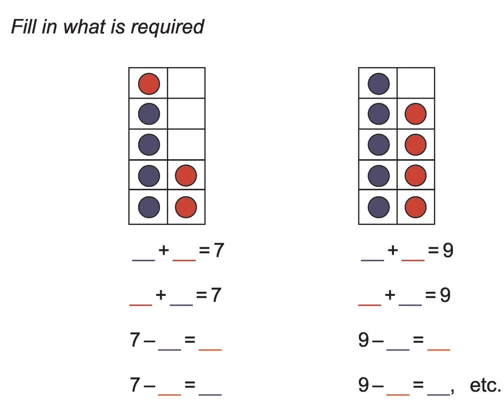


Fig. 9. Two additive schemes. Place holders suggest addition and subtraction tasks, each performed in two different ways

Let us note that traditionally this interdependence was expressed each time when a subtraction, considered as being a more difficult operation, was checked by its corresponding addition.

Here we have analyzed all instances of application of early algebra, following the task of building number system up to 10. It is, of course, a matter of pedagogical estimation which of them and in which extent and form will be present in the frame of a didactical transformation of this content, intended to be a material for learning. Instead of a too early expressing of properties in symbols, their illustration by means of iconic representations is often more acceptable. We respect, of course, all pedagogical requirements, but here we have been focused on structural analysis as our main aim.

6. Block of numbers up to 20

Some of the tasks listed in previous section continue to be done in this new block even more intensively. But we will single out here only those of them which have not already been covered before.

1. *Introduction of numbers* 11, 12, . . . , 20. Sums $10 + 1$, $10 + 2$, . . . , $10 + 10$ have their meaning established in the block of numbers up to 10 – one group of 10 objects and another one of 1, 2, . . . , 10 objects, but their values surpass the range of this block. Symbols 11, 12, . . . , 20 are used to be their shortened denotations and children are told to read them: eleven, twelve, . . . , twenty. Taken accurately, now the procedure of counting is also extended up to 20 (though, in fact, children will already have it picked up as an ordered reciting of number names). Thus, sums are used to extend the set of numbers 0, 1, 2, . . . , 10 and their iconic representation should be appropriately designed:

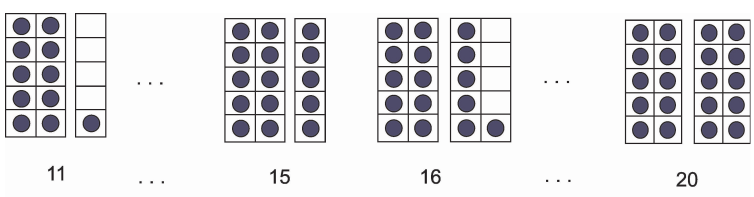


Fig. 10. Number pictures representing numbers 11—20

Remarking on the equalities

$$10 + 1 = 11, \quad 10 + 2 = 12, \quad \ldots, 10 + 10 = 20,$$

we point out that they result from the equating of different denotations for the same numbers.

2. *Rule of association of summands.* When the objects of a set are sorted into three groups A, B and C, having l, m and n objects respectively, then the triple sum $l+m+n$ is attached to that set to denote the total number of its elements and l, m and n are called first, second and third summand of this sum, respectively.

As two of these groups, say A and B, make an additive scheme and their union together with the third one, say $A \cup B$ with C, make another additive scheme and following this way of association, the sum $(l+m)+n$ will also be attached to these collection of three sets. Associating B and C and then A with $B \cup C$, the sum $l+(m+n)$ will also be attached to this collection of sets. As both of these sums denote the number of elements of one and the same set, the equality

$$(l+m)+n = l+(m+n)$$

can be written, which, in terms of didactics, is called the *rule of association of summands.* and in theoretical mathematics *associative law.*

In the process of learning, children have to accept the function of brackets first. An illustration can be used as it is the following one:

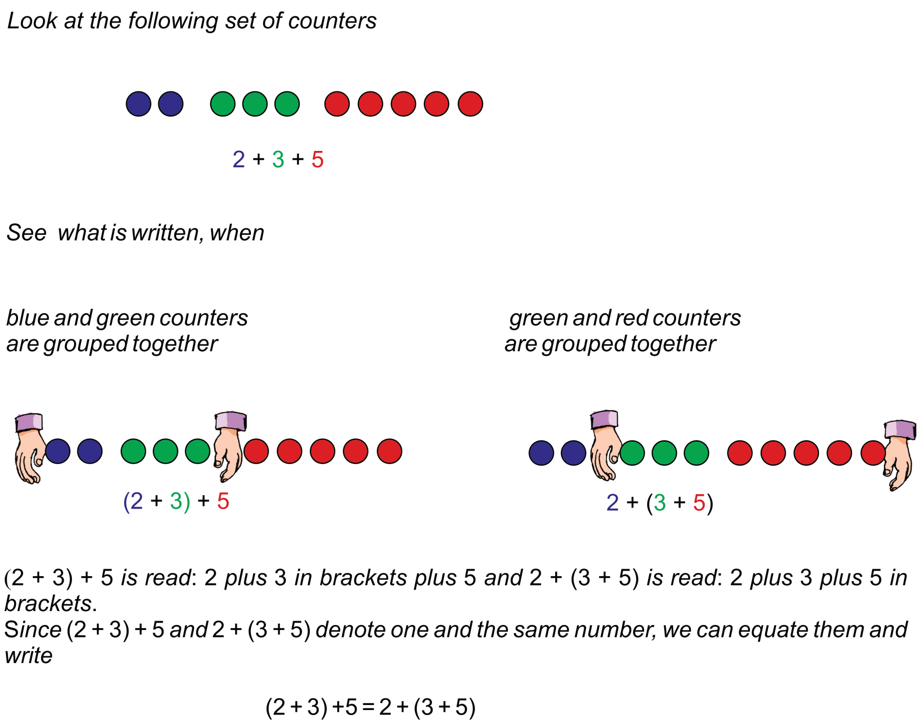


Fig. 11. A three set additive scheme. Two ways of association of sets with the hands in the role of grouping together. Brackets have an analogous role when numbers are associated

Doing a number of similar examples, children will grasp the rule of association of summands what means that *they apply it without relying on iconic representations.* This rule is also expressed in words: *the summands may be interchanged without altering the sum.*

Children should know that the expression "to do a sum" means to replace it with decimal denotation of the number which that sum stands for. Then they are told to do first what is in brackets and some exercises as the following ones can be assigned to them:

Find the sums

$$(4+3)+3 = \_\_ + \_\_ = \_\_, \qquad 10+(4+3) = \_\_ + \_\_ = \_\_,$$
$$10+(5+5) = \_\_ + \_\_ = \_\_, \quad \text{etc.}$$

As an application of this rule we take the sums when the first summand surpasses ten. For example,

$$11+4 = (10+1)+4 = 10+(1+4) = 10+5 = 15$$
$$13+3 = (10+\_\_)+3 = 10+(\_\_+\_\_) = 10+\_\_ = \_\_\_\_$$
$$14+5 = (10+\_\_)+\_\_ = 10+(\_\_+\_\_) = 10+\_\_ = \_\_\_\_$$

To find differences like $13-2$, $17-4$, etc. and to do it without relying on iconic representations, the rule $(l+m)-n = l+(m-n)$, $(m>n)$ should be established first. Usually, in the school practice, it is not done, but the differences are found in this way:

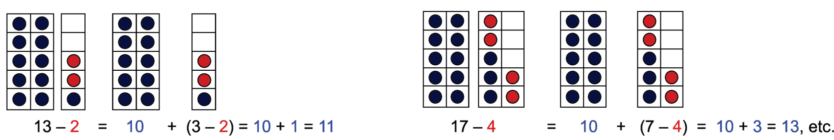


Fig. 12. Subtractions reduced to an easy subtracting of one digit numbers and an easy addition with 10 as the first summand

4. *Addition and subtraction by crossing the ten-line.* These are examples of addition, when a sum, say $8+6$, is found by performing two easy additions, say $8+2=10$ and $10+4=14$ and of subtraction when a difference, say $14-6$, is found by performing two easy subtractions, say $14-4=10$ and $10-2=8$. In the former case first we add up to 10 and in the latter we subtract down to 10 – continuing in both cases the ten-line is crossed. Also, in both cases the step of breaking up the second summand or the subtrahend is involved. Thus, children practice first to do this breaking, which is a procedure opposite to addition:

Fill in, what is required

$$8 = 5+\_\_, \quad 8 = \_\_+6, \quad 7 = 4+\_\_, \quad 7 = \_\_+5, \quad 9 = 6+\_\_, \quad 9 = \_\_+5, \quad \text{etc.}$$

Let us note that this is an instance when the equal sign is used to connect two expressions in the way which is reversed to that in the case of addition.

The addition when the ten-line is crossed begins with a number of examples supported by iconic representation:

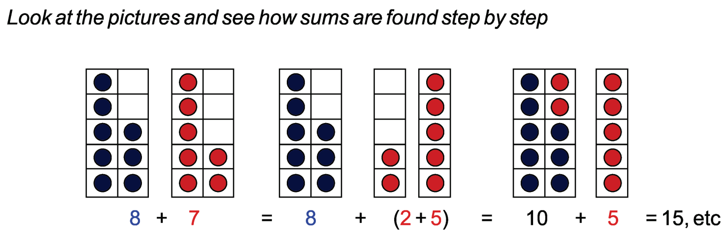


Fig. 13. Illustrations of regroupings supporting the method of addition when the ten-line is crossed

Teacher should turn children's attention to the fact that 10 is completed first and after a number of such exercises is done, children are let to do such examples of addition helped by place holders which indicate all intermediate steps. Such examples are:

$$6+5=6+(4+\underline{\quad})=(6+\underline{\quad})+\underline{\quad}=10+\underline{\quad}=\underline{\qquad}.$$

$$8+7=8+(2+\underline{\quad})=(\underline{\quad}+\underline{\quad})+\underline{\quad}=\underline{\qquad}+\underline{\quad}=\underline{\qquad},\quad \text{etc.}$$

This kind of programmed exercises is intended to help children understand the method but they are not required to form such sequences of steps by themselves. We would just mention that displaying of all these intermediate steps contributes to a thorough understanding of the method, but the ultimate aim is automatic performance which is achieved when such steps are completely suppressed.

A number of exercises of subtraction is done in the similar way:

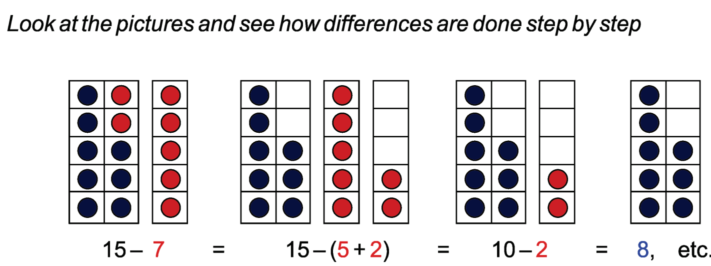


Fig. 14. Illustrations of regroupings supporting the method of subtraction when the ten-line is crossed

When children have grasped this method of subtraction through doing a number of similar exercises, then examples supplied by place holders are given:

Fill in what is required

$$14-6=(14-4)-\underline{\quad}=10-2=\underline{\quad},\quad 16-8=(16-6)-\underline{\quad}=\underline{\qquad}-\underline{\quad}=\underline{\quad},\quad \text{etc.}$$

5. *Building of addition (subtraction) table.* The block of numbers up to 20 is a natural range within which the addition (subtraction) table is formed. The cases of addition and subtraction that correspond each to other should be often done together. For example,

$$9+6=15,\quad 15-9=\_\_,\quad 15-6=\_\_,$$
$$9+8=\_\_\_\_,\quad 17-9=\_\_,\quad 17-8=\_\_,\quad \text{etc.}$$

By doing a great number of such exercises all entries in the tables are spontaneously remembered or they are found by a quick mental calculation.

If display of intermediate steps contributes to the thorough understanding of the way how these methods work, their application to quick calculation requires their suppressing. For instance,

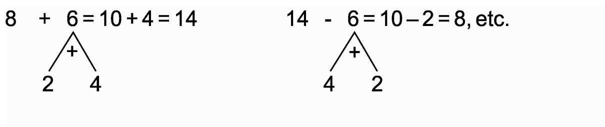


Fig. 15. Examples of suppressing intermediate steps

Let us note that this is the first situation when a task of finding sums and differences is reduced to easier ones and when such procedures deserve to be called calculation.

7. Comments

Let us recall that Descartes considered algebra to be a technique of calculation that precedes the other branches of mathematics. Analyzing the way how the number systems are developed and properly structured in school mathematics, a prominent role of algebra can be discerned. To study thoroughly this role is an important task of didactics of mathematics and we consider this paper to be just a small initial step.

As it is seen, this paper contains a series of exercises worked out with an evident care of detail. Thus, they are shaped in a way appropriate for actual learning in the classroom. It is done deliberately, of course, because we consider such exemplification to be the best way of verification of theoretical stands. On the other hand, it is also a way to show that a suggested innovation is feasible with respect to the developmental norms.

Sets at the sensory level are the intuitive basis of arithmetic and the mental operations with these sets which are corresponding to the set theoretic ones, provide the meaning for arithmetic operations. Various illustrations (pictograms and ideograms) are used to represent sets and to suggest these operations. Children are instructed what and how to observe but the observed material needs no explanation. What children learn is the way how the seen is codified in symbols correctly. Then, symbolic activities are understood properly in their dependence on the corresponding intuitive means.

It is often seen that primary school mathematical curricula include the items "commutative law" and "associative law". Majority of teachers wonder why such evident properties are emphasized in that way and they do not know how to treat them properly. A fundamental misunderstanding about the real meaning of these rules is seen in some text-books when their validity is based on verification in a number of specific cases. (For example, $3+6=9$, $6+3=9$, $9+7=16$, $7+9=16$, etc.). Being principles these rules are accepted on the basis of evidence—not proved and their significance is related to the syntactic apparatus of algebra. At the level of visual intuition these properties are evident and they are just specific aspects of the Cantor principle of invariance of number. When stated, these rules should serve for the transformation of expressions without reliance on any iconic material.

It is worth noting that a similar situation existed in mathematics and until the beginning of nineteenth century these properties had been taken by mathematicians as a matter of course. Thus, F.-J. Servois (Ann. de Math., 5, 1814/15) was the first to introduce terms denoting these laws. G. Peacock and his followers (D. F. Gregory, A. De Morgan, et al.) formed the systems of axioms, laying so the foundation of symbolic algebra. Using the language of contemporary algebra, the axioms for an Abelian group are the minimum of properties which can be used to derive all other properties of addition and subtraction. As the commutative and associative laws are included in the list of axioms for an Abelian group, they are normally considered to be of fundamental importance. But they, taken alone, are not sufficient for derivation of many other properties on which arithmetic procedures are based. On the other hand, this derivation is based on algebraic skills which are not seen to be systematically treated in school mathematics at all. (A reader who would try to use the axioms of the Abelian group to derive the relation $l-(m+n)=(l-m)-n$ will see that it is not a very simple, routine task). Let us also add that when a rule is expressed in the form of a literal relation, its permanence is immediate in the cases of transition from one number system to the following one.

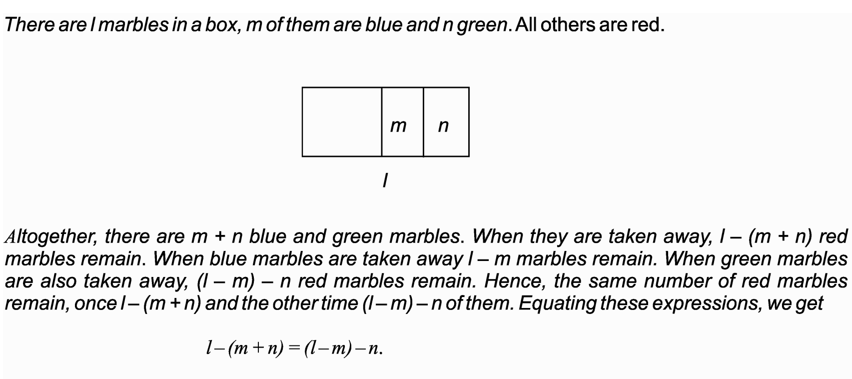


Fig. 16. Model of a box for illustration of a three set additive scheme and associated tasks which lead to the establishment of the rule of subtracting a sum

Primary school teachers should not look at the literal algebra to find rules for all explanations of arithmetic procedures. As a matter of fact, for that purpose several rules should be established in the courses of didactics of mathematics without any care if they are logically independent each of others. To stress the general validity of such rules, letters should be used to denote any number and the method of equating different expressions that denote the same number applied. An example to illustrate this method is given in Fig. 16.

Thus, we have derived a rule which is called in didactics of mathematics the rule of subtracting a sum. This rule is independent of commutative and associative laws and cannot be derived using them alone.

The cases of subtraction when the ten-line is crossed which are based on this rule and done independently of interpretation, are as follows:

$$15-8=15-(5+3)=(15-5)-3=10-3=7, \quad \text{etc.}$$

There are several other rules which serve as the explanations for arithmetic procedures and with which teachers should be acquainted. But would children use these rules at all, when and how is an open problem of early algebra whose solution we are not inclined to suggest here.

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