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DEFINITION OF THE DEFINITE INTEGRAL

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Abstract. In this paper we suggest a new approach for a definition of definite integral of a real function in the first course in Mathematical Analysis. The definite integral exists if for any sequence of partitions, the upper sum and the lower sum of Darboux have the same limit. If the definite integral of a real function exists, then we can simply compute it, as a limit of a sequence of integral sums of Riemann.

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The most usual way of defining definite integral of a real function $f\colon [a,b]\to\mathbb{R}$ in the contemporary literature is by integral sums of Riemann.

If $T=\{x_0,x_1,\dots,x_n\}$, $a=x_0<x_1<\dots<x_n=b$ is a partition of $[a,b]$, $\Delta x_i=x_{i+1}-x_i$, and $u_i\in[x_i,x_{i+1}]$ are arbitrarily chosen points, for $i=0,1,\dots,n-1$, the sum

$$R(T)=\sum_{i=0}^{n-1} f(u_i)\Delta x_i$$

is called the integral sum of Riemann of the function f on the interval $[a,b]$ (for the given partition T and for the chosen points u_i).

Let $h(T)=\max\{\Delta x_i \mid i=0,1,\dots,n-1\}$. The definite integral is defined in the following way.

Definition. The real function $f\colon[a,b]\to\mathbb{R}$ is integrable (in the sense of Riemann) if there exists a real number I such that for every $\varepsilon>0$, there exists $\delta>0$ such that for any partition $T=\{x_0,x_1,\dots,x_{n-1}\}$, with and any choice of points $u_i\in[x_i,x_{i+1}]$, for $i=0,1,\dots,n-1$,

$$\Big|\sum_{i=0}^{n-1} f(u_i)\Delta x_i\Big|<\varepsilon.$$

The real number I is called the *definite integral* of f in the sense of Riemann and we put $I=\int_a^b f(x)\,dx$.

An easy theorem is that if a function is integrable in the sense of Riemann then it is bounded.

Proving the properties of the definite integral just using the definition of Riemann is a very difficult task, because the definition contains too many variables. In older books there appears mainly a definition of definite integral as a limit

$$\int_a^b f(x)\,dx = \lim_{h(T)\to 0} \sum_{i=0}^{n-1} f(u_i)\Delta x_i.$$

This limit is not exactly described and in fact the existence of this limit is the same as the definition of definite integral (in the sense of Riemann) by the first definition.

In this paper we will show how the existence of the definite integral $\int_a^b f(x)\,dx$ can be checked as a limit of sequence. If the definite integral $\int_a^b f(x)\,dx$ exists, then we can simply compute it as a limit of a sequence of integral sums of Riemann.

Suppose the function f is bounded on the segment $[a,b]$, and
$T = \{x_0, x_1, \dots, x_n\}$, $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a,b]$. As usual, we introduce the notations

$$M_i = \sup\{\, f(x) \mid x \in [x_i, x_{i+1}]\,\}, \quad \text{and} \quad m_i = \inf\{\, f(x) \mid x \in [x_i, x_{i+1}]\,\}$$

and

$$S(T) = \sum_{i=0}^{n-1} M_i \Delta x_i, \quad s(T) = \sum_{i=0}^{n-1} m_i \Delta x_i.$$

The sums $S(T)$ and $s(T)$ are known as the upper and the lower sum of Darboux.

THEOREM 1. *If (T_k) is a sequence of partitions of $[a,b]$ such that*

$$T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k \subseteq \cdots,$$

then the limits $\lim_{k\to\infty} s(T_k)$ and $\lim_{k\to\infty} S(T_k)$ exist and are finite.

Proof. Since $s(T_1) \leqslant s(T_2) \leqslant \cdots$ is an increasing sequence and since $s(T_k) \leqslant S(T_1)$, the sequence is bounded from above. It follows that there exists a finite $\lim_{k\to\infty} s(T_k)$.

Similarly, since $S(T_1) \geqslant S(T_2) \geqslant \cdots$, there exists a finite $\lim_{k\to\infty} S(T_k)$. ∎

The following theorem is well known and it is easy to prove.

THEOREM 2. *If $T \subseteq T'$, and T' is obtained from T by adding of p new points, then:*

$$0 \leqslant S(T) - S(T') \leqslant p(M-m)h(T)$$

and

$$0 \leqslant s(T') - s(T) \leqslant p(M-m)h(T),$$

where $M = \sup\{\, f(x) \mid x \in [a,b]\,\}$, $m = \inf\{\, f(x) \mid x \in [a,b]\,\}$.

DEFINITION. The function f defined on the interval $[a,b]$, is *integrable* on $[a,b]$, if for any $\varepsilon > 0$, there exists a partition T such that $S(T) - s(T) < \varepsilon$.

THEOREM 3. *If the function $f\colon [a,b] \to \mathbb{R}$ is integrable, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $h(T) < \delta$ implies $S(T) - s(T) < \varepsilon$.*

Proof. We choose T' such that $S(T') - s(T') < \varepsilon/3$. If T' has p points in (a, b), we choose $\delta < \dfrac{\varepsilon}{3(M-m)p}$ and we choose a partition T such that $h(T) < \delta$. Then

$$S(T) - S(T \cup T') < p(M-m)h(T) < p(M-m)\frac{\varepsilon}{3(M-m)p} = \frac{\varepsilon}{3},$$

i.e., $S(T) - S(T \cup T') < \dfrac{\varepsilon}{3}$, and similarly $s(T \cup T') - s(T) < \dfrac{\varepsilon}{3}$. Adding the two previous inequalities we obtain

$$S(T) - s(T) < S(T \cup T') - s(T \cup T') + \frac{2\varepsilon}{3}.$$

On the other hand,

$$S(T \cup T') - s(T \cup T') < S(T') - s(T') < \frac{\varepsilon}{3},$$

and it follows that $S(T) - s(T) < \dfrac{\varepsilon}{3} + \dfrac{2\varepsilon}{3} = \varepsilon$. ■.

COROLLARY. *If the function $f\colon [a, b] \to \mathbb{R}$ is integrable then $\lim_{k\to\infty} h(T_k) = 0$ implies that $\lim_{k\to\infty}(S(T_k) - s(T_k)) = 0$.*

Proof. Since the function f is integrable, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $h(T) < \delta$ implies $S(T) - s(T) < \varepsilon$. $\lim_{k\to\infty} h(T_k) = 0$ implies that there exists an integer k_0 such that $h(T_k) < \delta$, for all $k \geqslant k_0$. It follows that $S(T_k) - s(T_k) < \varepsilon$, i.e.

$$\lim_{k\to\infty}(S(T_k) - s(T_k)) = 0. \quad ■$$

THEOREM 4. *If (T_k) is a sequence of partitions of $[a, b]$ such that $\lim_{k\to\infty} h(T_k) = 0$ and $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k \subseteq \cdots$, then the following conditions are equivalent for the function f:*

1) *f is integrable on $[a, b]$,*
2) *$\lim_{k\to\infty} S(T_k) = \lim_{k\to\infty} s(T_k)$.*

Proof. 1) $\Longrightarrow$ 2). By Theorem 1, there exist finite $\lim_{k\to\infty} S(T_k)$ and $\lim_{k\to\infty} s(T_k)$. By the previous corollary,

$$\lim_{k\to\infty} S(T_k) - \lim_{k\to\infty} s(T_k) = 0, \quad \text{i.e.,} \quad \lim_{k\to\infty} S(T_k) = \lim_{k\to\infty} s(T_k).$$

2) $\Longrightarrow$ 1). Suppose that $\lim_{k\to\infty} S(T_k) = \lim_{k\to\infty} s(T_k)$, i.e., $\lim_{k\to\infty} S(T_k) - \lim_{k\to\infty} s(T_k) = 0$. Then for a given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $S(T_k) - s(T_k) < \varepsilon$ for all $k \geqslant k_0$, i.e., the function f is integrable. ■

The above theorems allow us to define the definite integral $\int_a^b f(x)\,dx$ of an integrable function f on $[a, b]$, as a real number introduced in the following way:

Definition. Let f be an integrable function on $[a,b]$, and let (T_k) be a sequence of partitions of $[a,b]$, such that $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k \subseteq \cdots$ and $\lim_{k\to\infty} h(T_k) = 0$. The definite integral is the real number

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} S(T_k) = \lim_{k\to\infty} s(T_k).$$

Remark. Instead of a sequence (T_k) of partitions of $[a,b]$ such that $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k \subseteq \cdots$ and $\lim_{k\to\infty} h(T_k) = 0$, we can require the sequence (T_k) to satisfy only the condition $\lim_{k\to\infty} h(T_k) = 0$. However, in this case the proofs of theorems become more complicated.

We have to show that the definite integral is well defined, i.e., it does not depend on the choice of a sequence (T_k). We consider another sequence (T'_k) of partitions of $[a,b]$ such that $T'_1 \subseteq T'_2 \subseteq \cdots \subseteq T_k \subseteq \cdots$ and $\lim_{k\to\infty} h(T'_k) = 0$. Then

$$\lim_{k\to\infty} s(T'_k) \leqslant \lim_{k\to\infty} S(T_k) = \int_a^b f(x)\,dx = \lim_{k\to\infty} s(T_k) \leqslant \lim_{k\to\infty} S(T'_k).$$

Since the function f is integrable, the equality $\lim_{k\to\infty} s(T'_k) = \lim_{k\to\infty} S(T'_k)$ is satisfied, and we conclude that the definition does not depend on the choice of a sequence of partitions.

Theorem 5. *Let the function f be integrable on the interval $[a,b]$ and let (T_k) be a sequence of partitions of $[a,b]$ such that $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k \subseteq \cdots$ and $\lim_{k\to\infty} h(T_k) = 0$. Then*

$$\lim_{k\to\infty} R(T_k) = \int_a^b f(x)\,dx. \tag{1}$$

Proof. Since $s(T_k) \leqslant R(T_k) \leqslant S(T_k)$ and since f is integrable, we obtain that (1) holds true. ■

Remark. By the previous theorem with a little use of computer programming skills we can approximately compute any definite integral $\int_a^b f(x)\,dx$ of an integrable function. By taking a sequence of partitions (T_k) of $[a,b]$ such that $\lim_{k\to\infty} h(T_k) = 0$, we can illustrate that the sequence of Riemann sums converges.

The advantage of the definition above is illustrated by the following examples.

Examples. **1.** (Newton-Leibnitz formula) If the function f is integrable on the segment $[a,b]$ and f has a primitive function F on $[a,b]$ (i.e., $f(x) = F'(x)$ for $x \in [a,b]$), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Let $T = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. By the Mean Value Theorem, there exist points $v_i \in [x_i, x_{i+1}]$, such that

$$F(x_{i+1}) - F(x_i) = F'(v_i)(x_{i+1} - x_i) = f(v_i)(x_{i+1} - x_i),$$

for $i = 0, 1, \ldots, n-1$. It follows that

$$F(b) - F(a) = \sum_{i=0}^{n-1}(F(x_{i+1}) - F(x_i)) = \sum_{i=0}^{n-1} f(v_i)\Delta x_i,$$
$$s(T) \leqslant F(b) - F(a) \leqslant S(T).$$

If (T_k) is a sequence of partitions of $[a, b]$ such that $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k \subseteq \cdots$ and $\lim_{k\to\infty} h(T_k) = 0$, then

$$s(T_k) \leqslant F(b) - F(a) \leqslant S(T_k).$$

Since f is integrable we obtain $\lim_{k\to\infty} s(T_k) = F(b) - F(a) = \lim_{k\to\infty} S(T_k)$ and

$$\int_a^b f(x)\,dx = F(b) - F(a). \quad \blacksquare$$

2. If $a < c < b$, and f is integrable on $[a, c]$ and $[c, b]$ then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. We choose an increasing sequence (T'_k) of partitions of the segment $[a, c]$ and an increasing sequence (T''_k) of partitions of the segment $[c, b]$, such that

$$T'_1 \subseteq T'_2 \subseteq T'_3 \subseteq \cdots, \qquad T''_1 \subseteq T''_2 \subseteq T''_3 \subseteq \cdots,$$
$$\lim_{k\to\infty} h(T'_k) = 0, \qquad \lim_{k\to\infty} h(T''_k) = 0.$$

Then,

$$\lim_{k\to\infty} S(T'_k) = \int_a^c f(x)\,dx = \lim_{k\to\infty} s(T'_k),$$
$$\lim_{k\to\infty} S(T''_k) = \int_c^b f(x)\,dx = \lim_{k\to\infty} s(T''_k).$$

If we put $T_k = T'_k \cup T''_k$ then

$$S(T_k) = S(T'_k) + S(T''_k), \quad \text{and} \quad s(T_k) = s(T'_k) + s(T''_k).$$

(T_k) is an increasing sequence of partitions of the segment $[a, b]$, and such that $\lim_{k\to\infty} h(T_k) = 0$. Moreover, we have

$$\lim_{k\to\infty} S(T_k) = \lim_{k\to\infty} S(T'_k) + \lim_{k\to\infty} S(T''_k) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
$$= \lim_{k\to\infty} s(T'_k) + \lim_{k\to\infty} s(T''_k) = \lim_{k\to\infty} s(T_k).$$

It follows that $\int_a^b f(x)\,dx$ exists, and

$$\int_a^b f(x)\,dx = \lim_{k\to\infty} S(T_k) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \quad \blacksquare$$

3. (Length of the graph of a continuous function) The length of the graph of a continuous real function $f\colon [a,b] \to \mathbb{R}$ from the point $(a, f(a))$ to the point $b, f(b))$ is defined in the following way.

If $T = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ we define

$$Q(T) = \sum_{i=0}^{n-1} \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}.$$

Then the length L of the graph of function f is defined by

$$L = \sup\{\, Q(T) \mid T \text{ is a partition of } [a,b] \,\}.$$

Now, suppose that $f\colon [a,b] \to \mathbb{R}$ has a continuous derivative f'. If $T = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$, by the theorem of Lagrange, there exist points $v_i \in (x_i, x_{i+1})$ such that

$$f(x_{i+1}) - f(x_i) = f'(v_i)\Delta x_i,$$

for $i = 0, 1, \ldots, n-1$. It follows that the length of the segment connecting the points $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$ is $\sqrt{1 + f'(v_i)^2}\,\Delta x_i$. The sum of these lengths for $i = 0, 1, \ldots, n-1$ is

$$Q(T) = \sum_{i=0}^{n-1} \sqrt{1 + f'(v_i)^2}\,\Delta x_i.$$

On the other hand, this is the sum of Riemann for the function $\sqrt{1 + f'(x)^2}$ and for the partition T, i.e.,

$$R(T) = \sum_{i=0}^{n-1} \sqrt{1 + f'(v_i)^2}\,\Delta x_i.$$

If (T_k) is an increasing sequence of partitions of $[a,b]$ such that $\lim_{k\to\infty} h(T_k) = 0$, and $\lim_{k\to\infty} Q(T_k) = L$, then

$$L = \lim_{k\to\infty} R(T_k) = \int_a^b \sqrt{1 + f'(x)^2}\,dx.$$

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