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GENERALIZATIONS OF SOME REMARKABLE INEQUALITIES

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Abstract. In this paper, we present a generalization of Mitrinović's inequality for polygons, and triangles, a generalization of J. Radon's inequality and a generalization of Nesbitt's inequality. The main tool in the proofs is the inequality of Jensen.

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1. Generalizations of Mitrinović's inequality for convex polygons and triangles

If $A_1A_2\dots A_n$, $n\geqslant 3$ *is a convex polygon, and* $M\in \operatorname{int}(A_1A_2\dots A_n)$, *with* $\operatorname{pr}_{A_kA_{k+1}} M = T_k\in[A_kA_{k+1}]$ *for* $k\in\{1,2,\dots,n\}$, $A_{n+1}=A_1$, *then*

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}\geqslant 2n\tan\frac{\pi}{n}.$$

Proof. We first prove the following

LEMMA. *Let* A,B, $A\neq B$ *be points in the plane and* $M\notin AB$, $T=\operatorname{pr}_{AB} M$. *Then*

$$\frac{AB}{MT}=\tan u+\tan v,$$

where $u=\mu(\angle AMT)$, $v=\mu(\angle TMB)$ *are the measures of the angles* $\angle AMT$ *and* $\angle TMB$.

Proof of the Lemma. We have the following cases:

i) $T\in(AB)$. Then $\tan u=\dfrac{AT}{MT}$ and $\tan v=\dfrac{BT}{MT}$, hence $\tan u+\tan v=\dfrac{AB}{MT}$.

ii) $T\equiv A$. Then $\tan u=\dfrac{AT}{MT}=\dfrac{AA}{MT}=0$ and $\tan v=\dfrac{BT}{MT}$, hence $\tan u+\tan v=\dfrac{AB}{MT}$. Similarly if $T\equiv B$.

From Lemma, we have $\dfrac{A_kA_{k+1}}{MT_k}=\tan u_k+\tan v_k$, for $k=\overline{1,n}$, where $u_k=\mu(\angle A_kMT_k)$, $v_k=\mu(\angle T_kMA_{k+1})$. It follows that

$$\sum_{k=1}^{n}\frac{A_kA_{k+1}}{MT_k}=\sum_{k=1}^{n}(\tan u_k+\tan v_k).$$

Since the function $f\colon [0,\pi/2) \to [0,+\infty)$, $f(x) = \tan x$ is convex on $[0,\pi/2)$, we can apply Jensen's inequality and obtain that

$$\sum_{k=1}^{n} \frac{A_k A_{k+1}}{MT_k} \geqslant 2n \tan\left(\frac{1}{2n}\sum_{k=1}^{n}(u_k + v_k)\right).$$

Since, $\sum_{k=1}^{n}(u_k + v_k) = 2\pi$, we deduce that

$$\sum_{k=1}^{n} \frac{A_k A_{k+1}}{MT_k} \geqslant 2n \tan\frac{2\pi}{2n} = 2n\tan\frac{\pi}{n},$$

and we are done. ∎

Remark 1.1. If $A_1A_2\dots A_n$ is circumscribed about the circle $C(I;r)$ and $M \equiv I$, we have $MT_k = r$, $k = \overline{1,n}$, and the obtained inequality becomes $\frac{1}{r}\sum_{k=1}^{n} A_kA_{k+1} = \frac{2s}{r} \geqslant 2n\tan\frac{\pi}{n}$, wherefrom,

(1) $$s \geqslant nr\tan\frac{\pi}{n}.$$

The inequality (1) is a generalization of Mitrinović's inequality for triangles

(M) $$s \geqslant 3r\sqrt{3}.$$

Remark 1.2. If $A_1A_2A_3$ is a triangle, then the obtained inequality becomes

$$\frac{A_1A_2}{MT_1} + \frac{A_2A_3}{MT_2} + \frac{A_3A_1}{MT_3} \geqslant 6\tan\frac{\pi}{3} = 6\sqrt{3}.$$

For $M \equiv I$, we obtain (M). For more results see [1].

2. A generalization of J. Radon's inequality

In what follows, we denote $\mathbb{R}_+ = [0,+\infty)$ and $\mathbb{R}_+^* = (0,+\infty)$.

Let $a,b,c,d,x_k,y_k \in \mathbb{R}_+^*$, $k = \overline{1,n}$ and $X_n = \sum_{k=1}^{n} x_k$, $Y_n = \sum_{k=1}^{n}$. If $m,p,q,s \in \mathbb{R}_+$, $r \in [1,+\infty)$ are such that $cY_n^s > d\max_{1\leqslant k\leqslant n} y_k^s$, $k = \overline{1,n}$, then

(2) $$\sum_{k=1}^{n} \frac{(aX_n^p + bx_k^q)^{m+1}x_k^{r(m+1)}}{(cY_n^s - dy_k^s)^m y_k^m} \geqslant \frac{(an^qX_n^{p+r} + bX_n^{q+r})^{m+1}}{(cn^s - d)^m Y_n^{m(s+1)}} \cdot \frac{1}{n^{(m+1)(q+r-1)-ms}}.$$

Proof. Denote $u_k = (aX_n^p + bx_k^q)x_k^r$, $v_k = (cY_n^s - dy_k^s)y_k$, $k = \overline{1,n}$, $V_n = \sum_{k=1}^{n} v_k$ and the left-hand side of (2) becomes

$$\sum_{k=1}^{n}\frac{u_k^{m+1}}{v_k^m} = \sum_{k=1}^{n} v_k\left(\frac{u_k}{v_k}\right)^{m+1} = V_n\sum_{k=1}^{n}\frac{v_k}{V_n}\left(\frac{u_k}{v_k}\right)^{m+1}.$$

Since the function $f\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$, $f(x) = x^{m+1}$ is convex, we use Jensen's inequality and we obtain that

$$\sum_{k=1}^{n}\frac{v_k}{V_n} f\left(\frac{u_k}{v_k}\right) \geqslant f\left(\sum_{k=1}^{n}\frac{v_k}{V_n}\cdot\frac{u_k}{v_k}\right) = f\left(\sum_{k=1}^{n}\frac{u_k}{V_n}\right) = \frac{1}{V_n^{m+1}}\left(\sum_{k=1}^{n} u_k\right)^{m+1}.$$

Therefore,

$$\sum_{k=1}^{n}\frac{u_k^{m+1}}{v_k^m}\geqslant\frac{V_n}{V_n^{m+1}}\left(\sum_{k=1}^{n}u_k\right)^{m+1}=\frac{1}{V_n^m}\left(\sum_{k=1}^{n}u_k\right)^{m+1},$$

i.e.,

$$\sum_{k=1}^{n}\frac{(aX_n^p+bx_k^q)^{m+1}x_k^{r(m+1)}}{(cY_n^s-dy_k^s)^m y_k^m}\geqslant\frac{\left(\sum\limits_{k=1}^{n}(aX_n^p+bx_k^q)x_k^r\right)^{m+1}}{\left(\sum\limits_{k=1}^{n}(cY_n^s-dy_k^s)y_k\right)^m}$$

$$=\frac{\left(aX_n^p\sum\limits_{k=1}^{n}x_k^r+b\sum\limits_{k=1}^{n}x_k^{q+r}\right)^{m+1}}{\left(cY_n^s\sum\limits_{k=1}^{n}y_k-d\sum\limits_{k=1}^{n}y_k^{s+1}\right)^m}=\frac{\left(aX_n^p\sum\limits_{k=1}^{n}x_k^r+b\sum\limits_{k=1}^{n}x_k^{q+r}\right)^{m+1}}{\left(cY_n^{s+1}-d\sum\limits_{k=1}^{n}y_k^{s+1}\right)^m}.$$

Since the functions $g,h,k\colon\mathbb{R}_+^*\to\mathbb{R}_+^*$, $g(x)=x^r$, $h(x)=x^{q+r}$, $k(y)=y^{s+1}$ are convex, also by Jensen's inequality, we have:

$$\sum_{k=1}^{n}x_k^r=\sum_{k=1}^{n}g(x_k)\geqslant ng\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)=n\cdot\frac{X_n^r}{n^r}=\frac{X_n^r}{n^{r-1}},$$

$$\sum_{k=1}^{n}x_k^{q+r}=\sum_{k=1}^{n}h(x_k)\geqslant nh\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)=n\cdot\frac{X_n^{q+r}}{n^{q+r}}=\frac{X_n^{q+r}}{n^{q+r-1}},$$

$$\sum_{i=1}^{n}y_i^{s+1}=\sum_{i=1}^{n}k(y_i)\geqslant nk\left(\frac{1}{n}\sum_{i=1}^{n}y_i\right)=n\cdot\frac{Y_n^{s+1}}{n^{s+1}}=\frac{Y_n^{s+1}}{n^s}.$$

Then, we deduce that

$$\sum_{k=1}^{n}\frac{(aX_n^p+bx_k^q)^{m+1}x_k^{r(m+1)}}{(cY_n^s-dy_k^s)^m y_k^m}\geqslant\frac{\left(a\cdot\frac{X_n^{p+r}}{n^{r-1}}+b\cdot\frac{X_n^{q+r}}{n^{q+r-1}}\right)^{m+1}}{\left(cY_n^{s+1}-\frac{dY_n^{s+1}}{n^s}\right)^m}$$

$$=\frac{(an^qX_n^{p+r}+bX_n^{q+r})^{m+1}}{(cn^s-d)^mY_n^{m(s+1)}}\cdot\frac{n^{ms}}{m^{(m+1)(q+r-1)}}$$

$$=\frac{(an^qX_n^{p+r}+bX_n^{q+r})^{m+1}}{(cn^s-d)^mY_n^{m(s+1)}}\cdot\frac{1}{n^{(m+1)(q+r-1)-ms}},$$

and we are done. ∎

Remark 2.1. If $p=q=s=0$ then (2) becomes

$$\sum_{k=1}^{n}\frac{(a+b)^{m+1}x_k^{r(m+1)}}{(c-d)^m y_k^m}\geqslant\frac{(a+b)^{m+1}X_n^{r(m+1)}}{(c-d)^mY_n^m}\cdot\frac{1}{n^{(m+1)(r-1)}},$$

i.e.,

$$\sum_{k=1}^{n}\frac{x_k^{r(m+1)}}{y_k^m}\geqslant\frac{X_n^{r(m+1)}}{Y_n^m n^{(m+1)(r-1)}}.\tag{2'}$$

If we consider $r=1$ then by (2′) we obtain

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{y_k^m}\geqslant\frac{X_n^{m+1}}{Y_n^m}, \tag{R}$$

i.e., just the inequality of J. Radon (see, e.g., [2]), with equality if and only if there exists $t\in\mathbb{R}_+^*$ such that $x_k=ty_k$, $k=\overline{1,n}$.

Remark 2.2. If $m=1$ then (2) becomes

$$\sum_{k=1}^{n}\frac{(aX_n^p+bx_k^q)^2x_k^{2r}}{(cY_n^s-dy_k^s)y_k}\geqslant\frac{(an^qX_n^{p+r}+bX_n^{q+r})^2}{(cn^s-d)Y_n^{s+1}}\cdot\frac{1}{n^{2(q+r-1)-s}}. \tag{2''}$$

If we take $p=q=s=0$, $r=1$ then by (2″) we obtain

$$\sum_{k=1}^{n}\frac{x_k^2}{y_k}\geqslant\frac{X_n^2}{Y_n}. \tag{B}$$

But, that is just the inequality of H. Bergström.

3. A generalization of Nesbitt's inequality

If $a\in\mathbb{R}_+$, $b,c,d,x_k\in\mathbb{R}_+^*$, $k=\overline{1,n}$, $X_n=\sum_{k=1}^n x_k$, $m\in[1,+\infty)$ *and* $cX_n^m>d\max_{1\leqslant k\leqslant n}x_k^m$, *then*

$$\sum_{k=1}^{n}\frac{aX_n+bx_k}{cX_n^m-dx_k^m}\geqslant\frac{(an+b)n^m}{cn^m-d}X_n^{1-m}. \tag{OG}$$

Proof. We have

$$U_n=\sum_{k=1}^{n}\frac{aX_n+bx_k}{cX_n^m-dx_k^m}=\sum_{k=1}^{n}\frac{(aX_n+bx_k)^2}{(aX_n+bx_k)(cX_n^m-dx_k^m)}$$
$$=\sum_{k=1}^{n}\frac{(aX_n+bx_k)^2}{acX_n^{m+1}-adX_nx_k^m+bcX_n^mx_k-bdx_k^{m+1}},$$

where we apply H. Bergström inequality (B) and we deduce that

$$U_n\geqslant\frac{\left(\sum\limits_{k=1}^{n}(aX_n+bx_k)\right)^2}{acnX_n^{m+1}-adX_n\sum\limits_{k=1}^{n}x_k^m+bcX_n^m\sum\limits_{k=1}^{n}x_k-bd\sum\limits_{k=1}^{n}x_k^{m+1}}$$
$$=\frac{(anX_n+bX_n)^2}{(acn+bc)X_n^{m+1}-adX_n\sum\limits_{k=1}^{n}x_k^m-bd\sum\limits_{k=1}^{n}x_k^{m+1}}.$$

Using convexity of the functions $f,g\colon\mathbb{R}_+^*\to\mathbb{R}_+^*$, $f(x)=x^m$, $g(x)=x^{m+1}$, by Jensen's inequality we have:

$$\sum_{k=1}^{n}x_k^m\geqslant\frac{X_n^m}{n^{m-1}}\quad\text{and}\quad\sum_{k=1}^{n}x_k^{m+1}\geqslant\frac{X_n^{m+1}}{n^m},$$

and we obtain that

$$\begin{aligned}U_n &\geqslant \frac{(an+b)^2X_n^2}{(acn+bc)X_n^{m+1}-\dfrac{adX_n^{m+1}}{n^{m-1}}-\dfrac{bdX_n^{m+1}}{n^m}}\\ &=\frac{(an+b)^2n^m}{acn^{m+1}+bcn^m-adn-bd}X_n^{1-m}\\ &=\frac{(an+b)^2n^m}{(an+b)(cn^m-d)}X_n^{1-m}=\frac{(an+b)n^m}{cn^m-d}X_n^{1-m},\end{aligned}$$

which completes the proof. ■

REMARK 3.1. If $a=0$ and $b=c=d=m=1$ then we obtain the Nesbitt's inequality for n variables, i.e.,

$$\text{(N)}\qquad \sum_{k=1}^{n}\frac{x_k}{X_n-x_k}\geqslant\frac{n}{n-1}.$$

If we take $n=3$ then by (N) we obtain

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_1}+\frac{x_3}{x_1+x_2}\geqslant\frac{3}{2},$$

i.e., Problem 15114, proposed by A.M. Nesbitt to Educational Times **3** (1903), 37–38.

REMARK 3.2. A generalization of (OG) was published in [3], i.e., if $a,m\in\mathbb{R}_+$, $b,c,d,x_k\in\mathbb{R}_+^*$, $k=\overline{1,n}$, $X_n=\sum\limits_{k=1}^{n}x_k$, $p\in[1,+\infty)$, and $cX_n^m>d\max_{1\leqslant k\leqslant n}x_k^m$, then

$$\text{(AMM)}\qquad \sum_{k=1}^{n}\frac{aX_n+bx_k}{(cX_n^m-dx_k^m)^p}\geqslant\frac{(an+b)n^{mp}}{(cn^m-d)^p}X_n^{1-mp}.$$

REMARK 3.3. A generalization of (AMM) appeared in [4], i.e., if $n\in\mathbf{N}^*\setminus\{1\}$, $a\in\mathbb{R}_+$, $b,c,d,x_k\in\mathbb{R}_+^*$, $k=\overline{1,n}$, $X_n=\sum\limits_{k=1}^{n}x_k$, $m,p,r,s\in[1,+\infty)$, such that $cX_n^m>d\max_{1\leqslant k\leqslant n}x_k^m$, then

$$\sum_{k=1}^{n}\frac{(aX_n^r+bx_k^r)^s}{(cX_n^m-dx_k^m)^p}\geqslant\frac{(an^r+b)^s}{(cn^m-d)^p}n^{mp-rs+1}X_n^{rs-mp}.$$

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