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ON ANGLES AND ANGLE MEASUREMENTS

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Dedicated to Academician Milosav Marjanović[1]

Abstract. In this note, we give a brief discussion on angles and angle measurements from the point of view of our teaching practice. Two things are clarified, namely definition of the notion of angle, and once that is done, various operations and measurements of angles.

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The word "angle" refers to a geometric figure, distinct from other geometric figures as well as to a concept that stands on its own. This dichotomy partly explains difficulties in defining angles and operations on them, such as addition (which is not the same operation as their union), and subtraction, but also in considerations of appropriate ways to measure angles.

It is thus not surprising to see a wide spectrum of understanding this concept as well as presenting it to students in everyday teaching practice. In this note, we give a brief discussion on angles and angle measurements from the point of view of our teaching practice.

Two things need to be clarified, namely definition of the notion of angle, and once that is done, various operations and measurements of angles.

In order to accomplish our task, we first look into a few historical snapshots.

History

[Euclid] refers to angles in his Book one, Definitions 8-12:

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8. A plane angle is the inclination to one another of the lines in a plane which meet one another and do not lie in a straight line.
9. And when the lines containing the angle are straight, the angle is called rectilinear.

[1] Dr. Marjanović was one of my undergraduate professors who has been an excellent example of a practitioner of sound mathematical pedagogy.

10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right angle.
12. An acute angle is an angle less than a right angle.

He then postulates in

POSTULATE 4: All right angles are equal to one another.

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A number of other authors and commentators from antiquity concerned themselves with proper definition of angle, notably Aristotle, Heron, Syrianus (through Proclus), Appolonius, Plutarch, Carpus of Antioch ...

Arguments were made regarding the category (according to Aristotelian scheme) where the notion of angle should belong. Whether angles are a *quantum* (*ποσόν*), *quale* (*ποιόν*) or a *relation* (*πρός τι*). Plutarch (and by his reference perhaps Apollonius) and Carpus saw it as a quantity, Eudemus the Peripatetic considered angles to be in the category of quality (as defined by Aristotle). Syrianus comments that Euclid and all those who called an angle an inclination belong to the group who see angle as a relation (as Euclid characterized angle as region contained between two lines ...). Proclus (as referred to by Syrianus) sees it as if the truth is among these three: The angle is a quantity, as a magnitude that can be compared (equal, greater, less than), a quality by way of its form and, finally, a relation between the lines and/or planes bounding it.

Moving forward to the end of the 19th and the beginning of the 20th century: [Schotten, pp. 94–183], classifies historic development of the notion of angle into three groups of definitions:

1. The angle is the difference of direction between two straight lines. (This group may be compared to Euclid's definition of an angle as an inclination.)
2. The angle is the quantity or amount (or the measure) of the rotation necessary to bring one of its sides from its own position to that of the other side, without of it moving out of the plane containing both.
3. The angle is the portion of a plane included between two straight lines in the plane which meet in a point (or two rays issuing from the same point).

Euclid's commentator Heath states [Euclid] that: "It is remarkable however that nearly of all the text-books which give definitions different from those in group 2 add to them something pointing to a connection between an angle and rotation: a striking indication that the essential nature of an angle is closely connected with rotation, and that a good definition must take account of that connexion."

The first group of definitions seem to already assume a notion of an angle so they seem to be circular.

Italian views from the same period are notable as well. [Amaldi] asserts that the aforementioned second group of definitions are based on the idea of the rotation of a straight line or ray in a plane about a point: " ... an idea which, logically formulated, may lead to a convenient method of introducing the angle. But it must be made independent of metric conceptions, or of the conception of congruence, so as to bring out first the notion of an angle, and afterwards the notion of equal angles.

The third definition may be seen as that angle is the aggregate of the rays issuing from the vertex and comprised in the angular sector."

[Veronese] defines an angle as follows: "We call an angle a part of a cluster of rays, bounded by two rays (as the segment is a part of the straight line bounded by two points). An angle of the cluster, the bounding rays of which are opposite, is called a flat angle."

Veronese has a clever way of bijectively matching segments with angles, after he lays ground for that. Thus he says: "the cluster of rays is a homogeneous linear system in which the element is the ray instead of the point. The cluster being a homogeneous linear system, all the propositions deduced from [Veronese] Post 1, for the straight line apply to it, e.g. that relative to the sum and difference of the segments: it is only necessary to substitute the ray for the point, and the angle for the segment." (Fig.1)

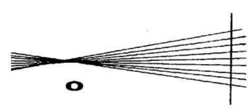

Fig. 1

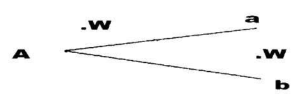

Fig. 2

Hilbert in his lectures on foundations of geometry defines an angle while discussing the congruence axioms (see Hilbert, 2004, pp. 244–247, et seq.). Two half rays a, b passing through a point A (vertex), together with any [all] point W not on a, b define an angle (Fig. 2). Hilbert in fact has in mind both angles defined by the two half-rays (see his diagrams here). The way he defines it, angles have no orientation. It is slightly different in his "Elemente der Euklidischen Geometrie" p. 323, ibid.

[Klein] addresses the question of angles in part III, chapter II on Systematic Discussion of Geometry and Its Foundations, Foundations of Geometry, (part three, Foundations of Geometry, §1 Development of Plane Geometry with Emphasis upon Motions, pp. 160–173). He chooses a coordinate axis (x) and an arbitrary point O on it and an arbitrary unit segment (vector) that he uses to get number 1 and then all other numbers on the real axis. Of course, the Archimedean axiom as well as completeness/continuity axioms are needed to construct the geometric line to real numbers correspondence. Since he also needs another arbitrary axis passing through O, another unit is chosen for that axis and then translation is used to grade that axis, and basic two- or three-dimensional coordinate system for analytic geometry. He then constructs the group of motions (transformations group) by fixing a coordinate origin and considers rotations around that point and the x-axis passing through the point. He notes that there is a unique rotation that will carry the x-axis into an arbitrary given line passing through O. If OA is a given segment, then, by such continuous rotations such segment may be mapped into any other segment starting at O (of the same length)—namely it passes through all the intermediate stages, so that point A describes the circle with center O and radius OA. Klein then wants to introduce a gradation by way of these motions and says that, since OA will eventually get into its initial position after one full rotation, the unit of measurement for the angles may be taken to be the one full rotation and that be matched with real number 1. However, he says, that due to tradition, the unit of measurement of the angles is in fact taken to be one quarter of the full turn, which will be called the right angle and denoted by R. Then any other turn can be measured as ωR, where ω may be any real number, that may be bounded to be in the interval $[0, 4]$ (units), because of periodicity. If another angle $a_1 O_1 a_1'$ has vertex O_1, then denote by T the translation $\overrightarrow{OO_1}$ and Ω the original rotation $a \to a'$ in the starting angle aOa'. In that case the rotation $a_1 \to a_1'$ through the target angle equals $\Omega_1 = T \circ \Omega \circ T^{-1}$. This way rotations ωR from the first angle may be transformed into rotations through the second angle; one consequence is that the angle grading (measurement) scale for the second angle is in fact the same as it is for the starting angle.

Teaching practice

In order to enable operations with segments as it is done with real numbers, a fixed unit interval (length) is needed. It is then identified with number 1. For instance [Hilbert] does it in his *Festschrift* (p. 468), where he introduces segment calculus (Streckenrechnung) on the basis of a Pasch's theorem (p. 463) and develops it into arithmetic of segments which mimics that of real numbers (pp. 473–475 ibid), including introduction of segment corresponding to 0 ... Thus the coordinate system is introduced in one and two dimensions accordingly (via Cartesian approach).

In teaching practice, it is best to keep the dual nature of the notion of angle. In earlier ages (such as elementary school), students can easily identify angle as a plane sector bounded by two half-lines meeting at a common point, as if this sector were made of a peace of cardboard. The "*zero angle*" and the "*full angle*" may be introduced at this stage. Likewise the "*right angle*" as a quarter of the full angle. Other "nice" angles may be introduced through regular polygons, such as triangle, square and hexagon or even octagon (average students at this early age may not be ready for pentagons, heptagons, etc). Even the degree measure is intuitively accepted at this stage—simply put, the full angle is subdivided into 360 equal parts and each of the parts is said to have measure $1°$. Some explanation should be given why 360 is chosen, namely that 360 is the closest number to the number of days in a year, which is at the same time divisible by many factors and that this may have been one of the main reasons for the choice of that number.

Addition of angles can be introduced visually as putting two plane sectors into the appropriate adjacent position and the resulting joint sector as the sum of the two angles. Subtraction will take a tad more effort, but the (physical) visualization helps here. Degree measures can be a companion to this idyllic world of angles.

On the other hand, the problems would arise promptly when addition of angles would result in angles greater than the full angle. At this stage of development (for an "average" student, starting with 7th or 8th grades, or earlier, depending on a particular educational system) the second nature of angles can be brought in by way of rotations and overlaps. For rotations, one of the legs of the angle is fixed (and it becomes an x-axis with 0 identified with the angle vertex). As more intricate measurements of angle are subsequently introduced as well, the already introduced unit of measurement may be denoted by 1 U and should be assigned/identified both with a unit segment (with measure 1 U) and unitless (dimensionless) number 1 on the already established real line, here identified with the x-axis. This is an important point, since the correspondence of the geometric (real) line with the ordered, complete field of real numbers is crucial. In daily practice this correspondence is freely interpreted as identification of numbers with lengths of segments or even with the segments themselves. Thus, 1 U is the geometric (physical) measure of choice, while 1 is the corresponding mathematical (unitless) measure.

Next comes the unit circle of radius 1 U (or 1) whose center O is used as vertex of the angles, while the angles may be prescribed to have the reference leg fixed as the x-axis. This unit circle can provide a reassuring intuition gained through the all familiar clock and its two orientations, "clockwise" and "counterclockwise." Angles can then be identified with the fixed first leg and the rotation needed to move a copy of this half line to the desired position of the second leg. This dynamic definition of angles should emphasize periodicity and the sense that an angle can be described as equal to an appropriate number of full angles plus a positive angle strictly less than the full angle (for negative orientation we would attach the minus sign next to it).

Problem of orientation has to be solved. For the majority of the students "clockwise" and "counterclockwise" metaphors are good enough.

The "rigid sector" view need not be abandoned, rather it may complement the rotation view—the sector may be overlapping itself (once it passes the full circle) and this overlap may be manifold (as in foliations that have no thickness). Adding of angles is by adding rotations of the second leg—when first rotation defining the first angle is finished, the second rotation begins from that position and similarly with overlapping view fig 3. adding the flat angle xOx' to angle xOb results in angle xOa. Implicitly, it would appear that we are relaying on some notion of a measure and thus we can introduce the angle measure at this stage, in a bit more generality than the degree measure introduced earlier on.

Thus, given an angle α, we decide to define

(1) $$m(\alpha) = \text{length of the unit circle arc subtended by the angle,}$$

(2) $$\alpha = n\cdot\text{full angle}+\alpha_0,\ 0 \le \alpha_0 < 2\pi \implies m(\alpha) = n2\pi + m(\alpha_0)$$

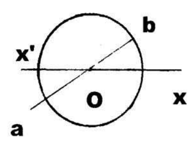

Fig. 3

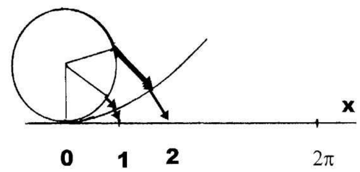

Fig. 4

(mutatis mutandis for negative angles). Visually, measure m is rolling (or unfolding), of the unit circle (as if it were made of wire) on its tangent at 0, in the positive direction of the x axis, if α is positive and in the negative direction, if α is negative (see Fig. 3–4).

It should be noted that *the unit for this measure is the same as the unit we choose to measure intervals, in this case denoted by U.* Historically, this measure has been called the radian measure (Rad) from the second half of the 19th century, which is somewhat confusing for it undermines the fact that this measure is exactly the measure we decide on in metric geometry when we choose a unit segment. Since the unit interval is placed on the real line so that its endpoints represent numbers 0 and 1, the radian measure may be deemed as dimensionless (number). On the other hand, if one wanted to develop geometric (physical) measure (in the arithmetic of segments), this measurement unit would be most definitely the same unit as is used to measure distances on the real line. The extent to which we abstract the latter into the corresponding dimensionless number is the same extent we abstract the "radian" measure into a dimensionless number.[2] Otherwise, the unit of measurement used for segment calculus is the same as the one for the angle measure. Thus, the full angle will have radian measure 2πU (or Rad), but also dimensionless 2π, straight angle is of measure π and the right angle is of measure $\pi/2$. Simply put, we can use geometric or physical measure (as normalized by the unit U) and we can then proceed in accord with the segment calculus, or we can abstract the measure (via the real line) and use analytic (abstract) measure as in abstract measure theory.

Given a non-zero real number t, the map $x \mapsto tx$ is a continuous isomorphism $\mathbb{R} \to \mathbb{R}$ and thus $\alpha \mapsto m(t\alpha)$ are also measures induced by m and t. Thus an angle measure may be obtained from radian measure, by choosing $t = 360/2\pi$ and the resulting measure is called the degree measure.

Given a circle of radius r, and its central angle α in the radian measure $m(\alpha)$, then the subtended arc length is easily seen to be equal to $\ell = m(\alpha)r$, where $m(\alpha)$ in the latter formula is the dimensionless number. This follows from the proportion: $2\pi rU : 2\pi\,\mathrm{Rad} = \ell U : \alpha\,\mathrm{Rad}$. The proportion is readily acceptable

[2]One can define the "radian" measure of a central angle α_0 in a circle of radius r defined as the ratio of the length of its subtended arc and the radius of the circle. A proof has to be made of the fact that the measure is independent of the radius of the circle. In this way, "radian" measure is prescribed to be dimensionless by definition. It should be noted that defining this measure dimensionless while using the unit of measurement "Rad" is problematic.

by most students, and most advanced students may be presented by triangulation argument to justify this proportionality.

Angles for more sophisticated students

For the more advanced students, group theory may be utilized for group of rotations may be used, in a similar manner that, for instance, Klein and Choquet used it.

One can resolve the question of orientation by stipulating that angles have the same orientation if an even isometry preserves orientation of all the angles in that orientation class. Rotation (clockwise vs. counterclockwise) subdivision into two orientation classes seems to work well with students. It seems that axioms of incidence and order are the only axioms needed to define orientation.

In a more rigorous presentation where level of mathematical culture of the audience is higher one should prove that this map m, as defined by (1) and (2) is indeed a measure (that it is a homomorphism and that it is σ-additive follows from the same property for the interval length measure on the real line). Furthermore, one can show that this measure is continuous, namely that $\lim_{\alpha\to 0} m(\alpha) = 0$.

In the unit circle the arc length is the same as the corresponding angle measure and the arc length formula $\ell = rx$ corresponding to angle x may be obtained by triangulating the unit circle arc and using similarity of triangles in the corresponding triangulation of the arc in the circle of radius r concentric to the unit circle.

Klein does not forget to utilize complex analysis, namely Euler's formula $e^{iz} = \cos z + i \sin z$ (relying initially only on analytic, not geometric meaning for this formula); he promptly derives the rotations group formulas for this approach. Here for instance, $i = e^{i\pi/2}$ which prompts him to change the idea that the right angle should be the unit of (angle) measurement, but instead suggests that the measure of the right angle be $\pi/2$ and that this scale of measurement be named the "natural scale" similarly to the natural logarithm name. Then the angle $\omega R = \omega\pi/2$ will have simply measure ω and that is how it will be written as well (in calculations). Then the group of rotations formulas

$$x' = x\cos\omega - y\sin\omega, \qquad y' = x\sin\omega + y\cos\omega$$

holds and they encode the usual geometric theorems.

[Choquet] devotes Chapters V–VII to angles.

Choquet identifies angles with rotations: For a given point O in plane, a rotation around O is called an angle with vertex at O. Furthermore, if (A_1, A_2) is a pair of half-lines with origin O, then the rotation about O taking A_1 to A_2 is called the angle formed by the pair, denoted by $\widehat{A_1A_2}$. This way, the set of rotations $\mathcal{A}$ is a commutative group (written additively). The neutral element is the zero angle. Translations then take care of angles with different vertices. Notice that periodicity here is incorporated and that an angle α is not distinguished from multiples of 2π added to α. Because of this periodicity, a measure catering from our acquired intuition about angle measurements cannot be constructed from such rotation group into the reals.[3] This is why Choquet defines a measure $\phi : \mathbb{R} \to \mathcal{A}$ (reversal of the measure map in our definition (1) and (2) above) as a continuous additive morphism from the reals onto the additive group of angles $\mathcal{A}$, $\phi(x + y) = \phi(x) + \phi(y)$. $\mathcal{A}$ is isomorphic to the multiplicative group T of complex numbers and topology in question of continuity is that one mapped by this identification.

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[3]However, there is somewhat related measure from the unit circle (of total measure 1) into the reals, namely the so-called Haar measure, invariant under group multiplication.