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AN EXPLORATION OF STUDENTS' CONCEPTUAL KNOWLEDGE BUILT IN A FIRST ORDINARY DIFFERENTIAL EQUATIONS COURSE (PART I)[1]

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Abstract. This study aims to analyze and document the types of knowledge that university students exhibit to deal with fundamental issues that they had studied in a first ordinary differential equation course. Questions that helped us structure the research included: How do students interpret and deal with the concept of solution to an Ordinary Differential Equation (ODE)? To which extent do students use mathematical concepts they have previously studied to answer basic questions related to ODEs? And, to what extent do the students' answers privilege the use of certain type of representation to explore and examine issues related to ODEs? Results indicate that, in general, students choose one of two methods to verify whether a function represents a solution to a given ODE: a substitution method or by solving directly the given equation. It was observed that they do not rely on concepts associated with the meaning of derivative to make sense and deal with situations that involve basic ODEs' ideas; rather, they tend to reduce their knowledge of ODEs to the search for an algorithm (analytical approach) to solve particular groups of equations. In addition, there is evidence that students do not use graphic representations to explore meanings and mathematical relations and they experience difficulties to move back and forth from one type of representation to another.

ZDM Subject Classification: B45, C75, I45; *AMS Subject Classification*: 97I10, 97I40.

Key words and phrases: Learning of Mathematics; the concept of ordinary differential equation; representations.

1. Introduction

What does it mean to learn or understand a mathematical concept? And how can the learning process involved be documented? Discussion of these types of questions is of importance for formulating a study that aims to analyze university students' resources, representations and strategies when they answer a set of questions related to the concepts embedded in an introductory course of ordinary differential equations. Hiebert & Carpenter [8] state that " ... understanding [a mathematical concept] can be viewed as a process of making connections, or establishing relationships, either between knowledge already internally represented or between existing networks and new information" (p. 80). Thus, in order to analyze

[1]The manuscript has been divided into two parts. The second part will be published in the next issue of the Journal. At the end of the introduction, we explain the logic and general structure of each part.

the students' comprehension of mathematical concepts it is important to document the type of connections and representations that students use in making sense of situations or solving problems.

The concept of ordinary differential equation (ODE) is one of the concepts most widely used to solve applied problems in different disciplines ranging from Physics through Medicine and Economics. Consequently, this concept appears in all types of university scientific, engineering, and social science curricula. Research studies show that when students approach the study of differential equations as mechanisms that describe how functions evolve and change over time, then they develop a conceptual understanding of key themes around the course (Rasmussen, Kwon, Allen, Marrongelle & Burtch, [15]). However, it is common to find approaches that are structured around learning activities that emphasize the classification of equations into various types and the use of specific algorithms to solve each of them. There, the applications of the concept as a means to solve problems appear, in general, at the end of the course and the instruction mainly focuses on presenting a set of standard problems that can be solved by the students through the use of rules and algorithms. As a result, students, in general, associate the analysis of ODEs with the use of a series of algorithms and procedures to find their solutions and to solve the corresponding problems. Indeed, the focus of learning activities shapes to some extent the students comprehension and use of mathematical concepts. Rasmussen and Kwon ([14, p. 191]) pointed out that, in a case study of a differential equations class, "students were learning analytical, graphical, and numerical methods in a compartmentalized manner" and as a consequence, they did not establish proper connections or relate meanings around mathematical properties associated with each approach.

In this context, what will happen when students have to solve a problem related to ODEs but where the statement does not exactly match the type of problems encountered or discussed in the classroom? Will the students figure it out how to identify the tools or concepts they have learnt in order to solve problems in other contexts? What kinds of difficulties will they experience while approaching the problems? To find possible answers to these questions we need to identify some basic features related to the process of problem solving. That is, it is relevant to identify basic components that help explain the students' construction or development of mathematical ideas and problem solving behaviors.

Schoenfeld [18] proposed a framework to explain students' problem solving behaviors in terms of: a) the students use of basic knowledge and resources to comprehend and represent the problem; b) the use of heuristic methods, or general strategies that can help them to solve a concrete problem; c) the use of metacognitive strategies to monitor and self-evaluate the problem solving process, and d) the system of beliefs which takes into account the concepts the individual possesses regarding Mathematics and problem solving. In this perspective, it can be said that the students' construction of a robust concept involves using a wide catalogue of resources and strategies needed to represent and use such concept in a variety of problem situations. Thus, the framework allows describing the students' problem solving process in terms of relevant mathematical process and resources.

Research studies in mathematics education have been concerned about the processes of teaching and learning of the concept of ordinary differential equation and related themes such as the direction fields, the solution methods, equilibrium solutions, or those solutions that involve systems of ordinary differential equations. These studies have allowed researchers to map out some of the difficulties that arise when students learn the concept of ODE[2]. For example, Rasmussen [13] reported that the concept of solution of a differential equation might be difficult for students to comprehend because they are used to identifying solutions of equations with numerical values and not with functions. That is, the fact that the solutions to an ODE are functions means that the obstacles that arise when students learn the concept for algebraic equations are now transferred to the scope of the solutions of an ODE. Similarly, Habre [7] analyzes students' abilities to utilize symbolic and graphical representations to examine direction fields as a means to solve ODEs after a calculus course where the visual approach was emphasized. He notes that the idea of solving an ODE has remained purely algebraic in the minds of all students and none of them succeeded in moving back and forth between the visual and the algebraic aspects of an ODE. Difficulty in articulating algebraic and graphic registers of representation was pointed out too by Moreno & Laborde [11] who designed a didactic engineering for the study of modeling with ODE using the software Cabri II Plus.

What does it make difficult for students to comprehend the concept of ODE and related themes such as solution concept and direction fields? Where should students focus their attention and what types of learning activities should they be engaged in order to develop a comprehensive understanding of those themes? The above considerations have led us to frame a study whose aim is to analyze the type of understanding that students show to deal with concepts involved in the study of ordinary differential equations such as the definition of a ordinary differential equation, its solution, and direction fields. As well, we intend to establish a categorization of some resources that students use in a first ordinary differential equations course. This categorization includes also an analysis of the ways in which the students access and use previous knowledge to deal with basic ODEs questions. To this end, the research questions that helped us guide and orient the development of the study are:

(i) How do students use mathematical concepts they have previously studied to answer questions related to ODEs? In this question, we were interested in analyzing the extent to which students identify and use concepts previously studied such as function, derivative and its meanings, and basic operations (finding derivatives) to make sense of questions statements in order to answer them.

(ii) How do students make sense of, interpret and deal with the concept of solution to an ODE? We aim to document the ways in which students conceptualize and operate the concept of solution to an ODE. In addition, we are interested

[2] Rasmussen and Whitehead [16] present a review about this research.

in examining the types of difficulties students might face during the process of verifying whether a function fulfils the necessary conditions to be the solution to a given ODE.

(iii) What systems of representation[3] do they use to represent and explore the information embedded in those questions in order to answer them? And, to what extent do the students privilege the use of certain type of representation? Here we are interested in identifying the extent to which students display consistent tendencies or preferences to select and use a set of resources to deal with problems or questions associated with ODEs. In this part, we introduce the conceptual framework used to support and structure the development of the study. We also present the design and methods employed to gather data and describe general procedures that appear throughout the research.

In this part of the paper, we focus on presenting results associated with the first research question: How do students use mathematical concepts they have previously studied to answer questions related to ODEs?

In the second part that will be published in next issue of the Journal, we complete the analysis of data to respond and discuss the following questions: How do students make sense of, interpret and deal with the concept of solution to an ODE?, What systems of representation do they use to represent and explore the information embedded in those questions in order to answer them? And, to what extent do the students privilege the use of certain type of representation?

2. Conceptual framework: Focusing on conceptual learning and representations

Lester [10] identifies three types of frameworks used to support and guide research into mathematics education: theoretical, practical, and conceptual. According to Lester, "a conceptual framework is an argument that the concepts chosen for investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem in investigation" (p. 73). In our study, three related themes or issues provided the basis and concepts to frame the research inquiry: A position about what students' mathematics learning involves, the importance of focusing on conceptual understanding, and the problem solving dimensions used to characterize the students process of grasping mathematics ideas. We thus argue that learning mathematics goes beyond memorizing a set of rules, algorithms, and procedures to solve a set of routine problems. It involves students developing a way of reasoning that includes both the development of habits that are consistent with mathematical practices (Couco et al, [3]) and the comprehension of concepts and resources to solve nonroutine problems. The NCTM [12] point out that: "A *reasoning habit* is a productive way of thinking that becomes common in the processes of mathematical inquiry and sense making" (p. 9). Kilpatrick, Swafford, &

[3] Goldin [4] defines a representation system as a system built through primitive entities (e.g., letters, words, symbols, numerals, etc) along with a syntactic structure to combine and operate symbols for forming permissible configurations and moving from one configuration to another.

Findell [9] state that for students to build up mathematical proficiency, they need to develop and show consistent behavior in five intertwined strands:

Conceptual understanding which refers to an integrated and functional grasp of mathematical ideas; *procedural fluency* which refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly accurately, and efficiently; *strategic competence* which involves the students' ability to formulate problems, represent them, and to solve them; *adaptive reasoning* which refers to thinking logically about the relationships among concepts and situations ... [it] includes knowledge of how to justify the conclusions. And a *productive disposition* which is the inclination for students to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics (pp. 115–132). Kilpatrick et al., also highlight that "these strands are not independent; they represent different aspects of a complex wholethe five strands are interwoven and interdependent in the development of proficiency in mathematics. Mathematics proficiency is not a one dimensional trait, and it cannot be achieved by focusing on just one or two of those strands" (p. 116).

How can we evaluate students' mathematical proficiency? To what extent do the students ways of dealing with mathematical tasks provide useful information regarding their types or levels of proficiency? Hiebert & Carpenter state that a useful way to describe a subject's understanding (a key ingredient in mathematical proficiency) is through analysis of the manner in which the subjects internal knowledge representations are structured. Internal representations are not directly accessible; however, they can be expressed and analyzed from the external representations they produce. In the same way that the external representations a student interacts with will affect or influence the way in which a student represents this relationship internally, so the way in which a student generates or relates to an external representation reveals information about how he or she has represented this information internally (Hiebert & Carpenter, [8]).

In the specific case of mathematical concepts, we can only try to gain access to the concepts through semiotic representations used to deal with them (Duval, [4]). These representations play a very important role in students' understanding, making it possible to explain the development of mathematical competences in terms of the use of such representations. Taking into account the *theory of representations*, the process of learning is undertaken by understanding and moving back and forth, in terms of meaning among the registers of representation associated with the concept or problem. That is, making conversions between the various registers and discriminating between them in each faced situation. Therefore, some evidence that learning has been effective can be documented through the analysis of students' answers to tasks where a single mathematical concept is studied through various systems of representation and by connecting different registers (Santos-Trigo & Barrera-Mora, [17]).

In this perspective, the concept of ODE can be studied and examined through the use of algebraic, graphical, numerical or contextual systems of representation.

So, if we wish to observe how robust this concept is in students' knowledge, we have to analyze what systems of representation they use when approaching tasks related to this concept and in what way the students use these systems. Using different systems of representations to carry out mathematical tasks requires students to rely on habits of reasoning such as identifying relevant concepts, procedures, and operations; seeking out patterns, considering special cases, interpreting solutions and refining arguments, etc. The notions of solving an ODE and direction field associated with this equation are some of the meanings that are closely related with the concept of ODE. The graphical nature of the direction field and the traditionally analytic focus from which ODEs are taught, justify the need to analyze the consistency and stability of the constructions used by students to establish relationships between different systems of representation.

We also recognize that there are different ways to develop mathematical proficiency, but each way involves the construction of a web of connections of relationships and meanings associated with concepts and operations. For example, Thurston ([19, p. 163]) identifies several different related ways of developing proficiency in comprehending and using the concept of derivative:

(a) Symbolic: the derivative of x^n is nx^{n-1}, the derivative of $\sin x$ is $\cos x$, etc.

(b) Geometric: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.

(c) Rate: the instantaneous speed of $f(t)$, when t is time,

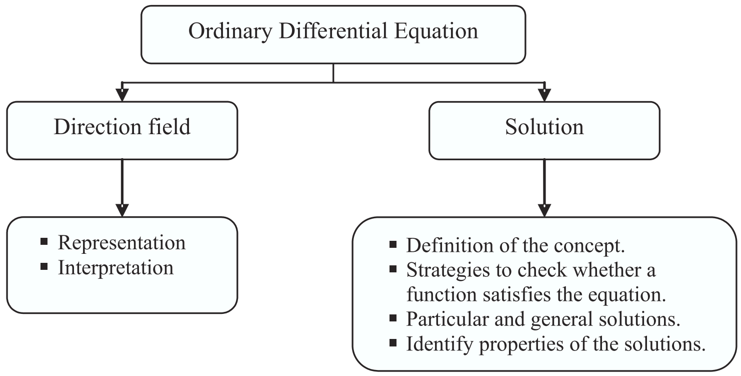
(d) Approximation: The derivative of a function is the best linear approximation to the function near a point.

(e) Logical ... etc.

Thurston suggests that a robust comprehension of the derivative concept takes place when students relate and cognitively integrate the different interpretations and meaning of this concept into a whole.

Similarly, we argue that the close relationship between the concept of derivative of a function and that of an ordinary differential equation implies paying attention to what interpretation of the concept of derivative the students think of and use when carrying out activities related to ODE. Furthermore, understanding of the solution of an ODE demands that students build up a network of relationships and meanings associated with the concept (Camacho, Perdomo & Santos-Trigo, [2]). Some of the processes related to the concept of solution of an ODE are shown in the following diagram. Also, the concept of the direction field associated with an ODE involves two related tasks that we can differentiate from a cognitive point of view: its representation and its interpretation.

Finally, problem-solving activities are essential to develop mathematical proficiency and involve the use of different representations in order to explore connections among mathematical relations. Suitable representation of mathematical objects provides a clearer vision of the intrinsic properties of these objects, which will lead students to develop a deep understanding of the concepts and to their use in problem solving situations (Santos-Trigo & Barrera-Mora, [17]).



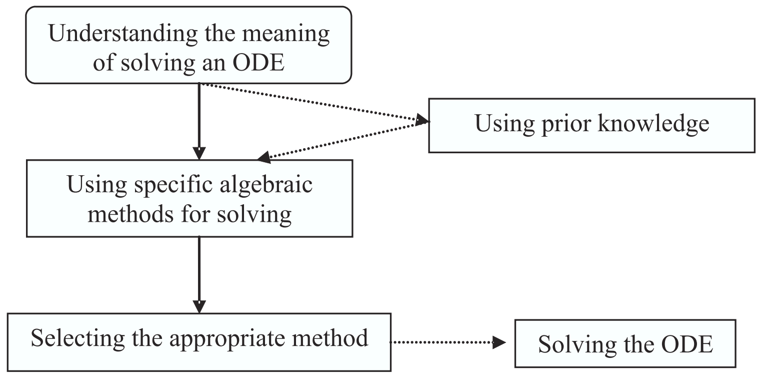
Relevant concepts and processes associated with the study of ODEs

The conjunction of these considerations leads us to argue that the students' development of mathematical proficiency is an ongoing process that goes through various related stages or phases, among which we can find the phase of understanding the definition of the concept itself, the stage where this concept is used or treated algorithmically (procedural fluency), and the stage where the concept is recognized as a tool to formulate and solve problems.

Similarly, solving an ordinary differential equation specifically goes through various stages, the first of which should be the understanding of the meaning of recognizing and solving the equation itself. Following this, two paths might arise: either, use the specific algebraic methods for solving ODEs, whatever the equation to be solved may be, or, attempting to solve it by using concepts, procedures and/or strategies learnt prior to the study of differential equations, in most cases these being related to the concept of derivative of a function. Once some methods to solve certain ODEs are known, the aim is to develop the strategies to decide which method is the most appropriate, taking into account at all times the equation that has to be solved.

3. Methodology and general procedures

Taking part in the study were twenty-one science undergraduate students who took a first ODE course. We were especially interested in analyzing the knowledge and forms of interpretation that the participants exhibited with regard to the concept of ODE and of other concepts related to this, such as the solution of ODEs, and the direction field associated with the equation, as well as documenting if the context in which the problems are presented influences the way in which the concept of differential equation is treated. To this aim, we designed a questionnaire and a task-based interview to gather information in order to answer the research questions.



Stages to solve an ODE

The *questionnaire* consisted of eleven problems (Q1–Q11) that the students had to solve individually, without help from teachers or researchers. The problems were discussed within the research group in detail in order to identify the concepts and resources involved in their solutions. The goal was to relate the features of the tasks to the five strands associated with mathematical proficiency as discussed in the conceptual framework. The tasks came from three different sources: (i) textbook problems that were adjusted to the purpose for the study; (ii) problems used in previous research which were judged to be interesting tasks for the study; and (iii) tasks designed ad-doc to elicit the students use of ideas studied previously to make sense of problem statements and to solve the tasks. The moment chosen for students to answer the questionnaire was defined by the classroom contents they were studying during the course to guarantee that the contents required to answer the questionnaire were covered. The students were asked to work on the questionnaire without any previous warning when they had already studied approximately half the contents of a subject that covers material on the first order ODE such as separate variable, homogeneous, linear, Bernouilli, Ricatti and exact equations. Students had one-hour class time to answer the questions.

Based on the analysis of students' answers we could document a series of general behavior patterns regarding treatment of the problems and the concept of differential equation. Results of this analysis led us to formulate conjectures about some partial answers to our research questions. Interviews with the students could either corroborate or refute our conjectures about the use of different systems of representation and the relationship that students establish between the various elements of mathematical knowledge studied at different moments in their academic courses.

A *task-based interview* (Goldin, [6]) was designed consisting of some problems of the questionnaire (Q1a, Q3, Q6, Q7, Q8, Q9) and four additional questions (Q12, Q13, Q14 and Q15). The interviews were semi-structured as they were designed with a common core whereby all the students chosen to be interviewed had to answer the same problems, but the possibility was left open to ask additional questions that could allow students to specify their actions and permit us to study further the students' reasoning. The additional questions were of the type "What have you done?" or "Why do you do this?", and also some mathematical questions (for example, compute an indefinite or a definite integral).

The interviews were conducted and videotaped two months after students had taken the course and they lasted between one and one-and-a-half hours.

Appendix 1 lists some of the problems used in the questionnaire and the interview, as well as the description of the aims we pursued during their application. There, you can also find information regarding where each problem was used, that is, whether it was part of the questionnaire, the interview or was used in both scenarios. More information regarding the rest of the problems that were given to the students can be found in Camacho, Perdomo & Santos-Trigo [2] and in the second part of this study.

Rational in choosing the problems.

As the main aim of this study is the analysis of the resources shown by students when facing problems containing the concept of ODE, we chose differential equations where the solution might not only be achieved by using algorithms pertaining to this concept but also through arguments related to the concept of derivative of a function and the properties that relate this to the function itself. All the ODEs in the questions, except one, can be solved by separating the variables and in each of these cases, except one, the integrals that have to be solved in order to find the algebraic expression of the solutions to the equation are immediate. There is only one ODE where the solution by separating variables leads to the integral of a rational function (Q8). For this activity the direction field associated with the equation is also given, helping us to determine which representation system students choose to find their solution. If students tend towards solving the equation and fail to solve the integral it would be expected that they try to solve it using the graphical system of representation. In this way students are enticed into needing to use this register to solve the task (Habre, [7]).

Classification of problems

To facilitate analysis of the data the problems used in our study were classified into different groups. Here, we focus on documenting the mathematical knowledge and systems of representation used by the students to solve Type 1 problems only. Results that emerge from analyzing questions types 2 and 3 appear in the second part of this manuscript (next issue to appear). And a preliminary analysis and results of the students' answers to Type 4 questions (Q7, Q9 and Q13) appears in Camacho, Perdomo & Santos-Trigo [2].

Type 1: Solution of this type of question can be approached through use of basic reasoning (making sense of the problem statement) or using simple algebraic methods (Q1, Q2 and Q12, Appendix 1). This type of question implies considering graphic representation of elementary functions, but do not involve either the construction or interpretation of the direction field or the interpretation of data from or towards a mathematical context. One example of this second group is

> Q1. Represent graphically some solutions for the following equations
>
> a) $\dfrac{dy}{dx} = 0;\ x \in [0, 2];$
>
> b) $\dfrac{dy}{dx} = \cos x.$

The aim of using this type of question is to document the extent to which students relate directly the concept of derivative to think of the algebraic expression for the solution of the equation. We are also interested in whether the students use the solution methods for first-order separable variable ODEs and apply integration methods correctly. Furthermore, we aim to document the extent to which they graph correctly the solutions of the equations. A table that describes the research objectives associated with each question of Type 1 is shown in Appendix 1.

Type 2: These questions require knowledge of the concept of solution. This type of question is used to document whether the students recognize an algebraic expression as a particular or general solution to a differential equation and to analyze the extent to which they use some general properties of the solutions to deal with the involved expressions. The questions of this group are Q3, Q4, Q5 and Q11 (Camacho-Machín et al., to appear). For example, in question three (Q3),

> Q3. Say whether the following statements are true or false and give reasons for your answer:
>
> a) The function $y = e^{\int e^{t^2} dt}$ is a solution for the differential equation $\dfrac{dy}{dt} = 4e^{t^2} y$.
>
> b) The functions $y = f(x)$ which satisfy that $-x^3 + 3y - y^3 = C$ are solutions for the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{1 - y^2}$.

the aim is to document the extent to which students recognize different forms of algebraic expressions associated with the solution of an ODE. In this process, we focus on analyzing the types of strategies used to support their answers, including alternative ways of overcoming initial difficulties (when students are faced with them); and the extent to which they select and apply correctly ways to differentiate functions given implicitly and explicitly. And also, whether they are able to identify and access proper algorithms to solve first-order ODEs and separable variable equations and whether they apply them correctly.

Type 3: Questions where answering requires representation and/or interpretation of the direction field of a differential equation (Q6, Q8, Q10, Q14 and Q15)

(Camacho-Machn et al., to appear). For example, the question (Q8)

> Q8. Let us consider the direction field associated with the equation $\dfrac{dP}{dt} = 0.1P(10 - P)$.
>
> Represent the solutions that satisfy that $P(0) = 0$ and $P(-2) = 12$. For what positive values of P are the solutions increasing? For what values are they decreasing? What is the limit of P when t tends to infinity?

was used to analyze the sense given to the statement and the method chosen by the students to approach it. In particular, we were interested in documenting the system of representation they employ and the meaning they give to the direction field associated with the ODE. To this end, we pay attention to the extent to which the students relate the concept of the derivative of a function to the monotony of the function.

4. Data analysis and presentation of results

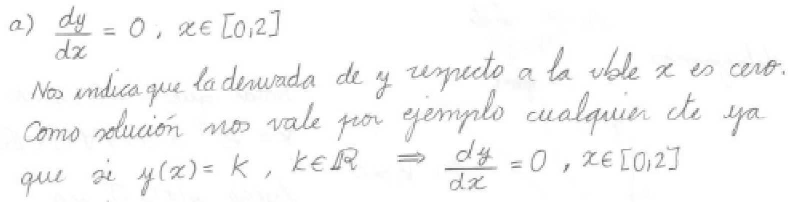
Analysis of the information gathered in our research is divided into two parts. First, we focus on analyzing the answers given by the twenty-one participating students to the problems set in the questionnaire. To this end, each student's answer was analyzed in terms of identifying salient mathematical features. For example, the work shown by the 21 students to deal with questions Q1 and Q2 from the first group is summarized in Appendix 2. This table is used as a heuristic aid to identify initial global patterns in students' work related to their level of proficiency to deal with basic concepts associated with the study of ODEs, This global behavior is complemented with data from interviews with the students. That is, we follow up students' answers to the questionnaire by considering their ideas expressed during the interview. In addition, we also show examples of the students work to illustrate their answers.

On the students' use of previous mathematical knowledge

In developing conceptual understanding (one of the mathematical proficiency strands) students need to "organize their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know" (Kilpatrick et al., [9, p. 118]. In this context, it is relevant to analyze the extent to which students rely on concepts and ideas previously studied to construct and extend knowledge to solve problems that involve ODEs. Thus, answers to Type 1 questions provide us with information on the view or ideas that students possess of the concept of ordinary differential equation and if they relate this concept to any of the meanings that Thurston [19] assigns to the derivative of a function. These questions allow us to see whether students use the basic knowledge acquired prior to studying ODEs in order to solve the simplest cases of ordinary differential equations and also to analyze the methods the students use when solving more complex cases. The mathematical concepts involved in Type 1 questions include explicit functions, fluency in finding derivatives, meaning associated with derivative, and solution of

ODEs. All the equations in these questions can be classified as the separate variable type, permitting us to examine whether or not the students select and apply the standard algorithm for solution and if they do so correctly.

Taking the answers given to Q1 and Q2, we can classify the students' answers into two major groups. On the one hand, there are those students who use their knowledge gained prior to studying differential equations in order to answer some questions; on the other hand, there are those students who solve all the equations by some specific method used to solve ODEs (Appendix 2). The first group of students (14 students) seems to view the concept of ordinary differential equation as an extension of the concept of derivative of a function, that is, they use their interpretation of the concepts of derivative of a function and equation in order to establish the set of solutions for an ODE (Figure 1). The second group of students (7 students), on the other hand, might be thinking of the concept as an independent mathematical entity with its own rules of treatment and solution (Figure 2).



"It shows us that the derivative of y with respect to the variable x is zero.
As a solution we can take for example any constant because if ... "

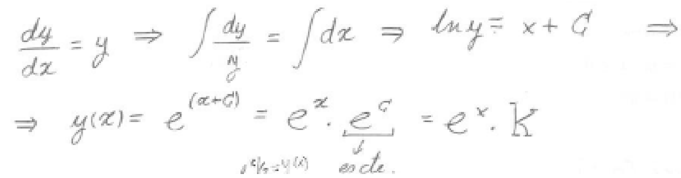


Figure 1. Wanda's answers to Q1a and Q2a

Stella (who belongs to the first group) uses algebraic rules of the derivative to answer questions Q1 and Q2, as is reflected in the remarks she makes during the interview. In the questionnaire, she did not recall any specific method to solve ODEs and she only gives a specific solution for each equation, except for $y'(t) = y^2$. In the interview we observe that she finds it difficult to realize that there are an infinite number of functions whose derivatives coincide. She admits that she does not know how to represent more than one solution to the equation $\frac{dy}{dx} = 0$ and only when the interviewer insists does she manage to express the general solution to the equation $y'(t) = k$. Thus, it is clear that she thinks initially that it is enough to provide only one particular solution to the equation; but when the researcher asked her for another solution, she begins identifying properties of the solution.

Stella (S): Well, if the derivative can be any constant at all, I can suppose that the function is ... degree one, for example. [She writes $y(t) = kt$]

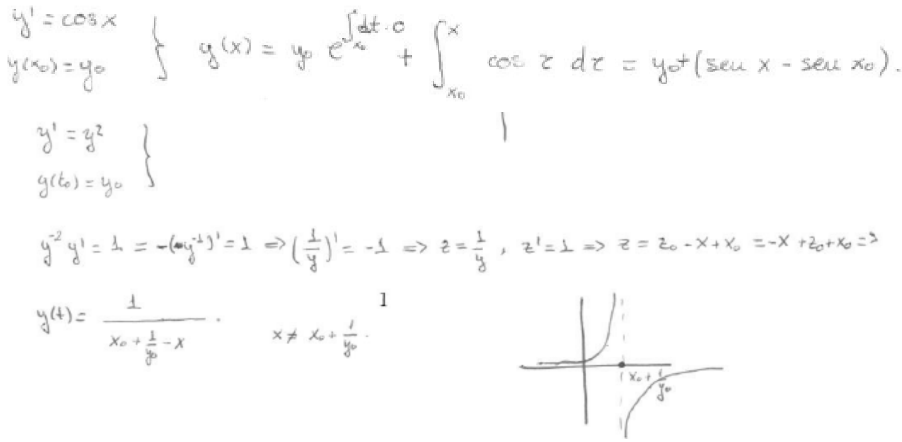


Figure 2. Edna's answers to Q1b and Q2b

[. . .]

Researcher: Could you draw any other solution?

S: After how hard this one was? Another one? Well, the derivative is a constant . . . It would also be $y(t) = kt + p$ . . . This is a constant, too (she indicates p), so the derivative would be this (she indicates the differential equation). Then I can draw it. It would be the same but centred on p, wouldnt it?

Very few students show evidences of using the geometrical meaning of the derivative to answer these questions. The only answer to the questionnaire where this might be happening is one answer provided by Betty who analyzed the sign of the derivative of the function solution, though she failed to provide a complete answer (Figure 3).

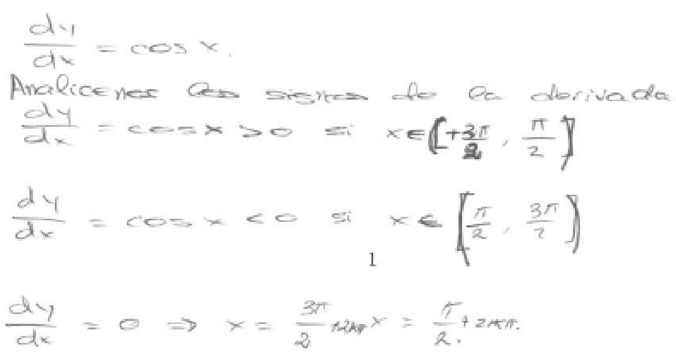


Figure 3. Betty's answer to Q1b

There is evidence that Wanda knows some algorithms for solving ODEs and uses different meanings of the concept of derivative to solve some differential equations in the questionnaire. She uses the relationship between the geometric meaning of the derivative and the differential equations set in the interview questions. Her first action is to solve Q1a finding the derivative and then goes about Q12 by separating the variables. When asked to compare the two tasks she externalizes her geometric meaning of the derivative of a function.

Wanda: ... here what they're asking you (Q1a) is a function so that if the slope were constant on the complete interval, it would always be 0. Here (Q12) what theyre asking you is ... I've made a mistake here! (Q12a) Because it depends on k! What I'm really getting are curves so that the slope, that is, the tangent line to the curve at each point, is then k. If k were positive the lines would go up, but if k were negative the lines would go down as they did for me here (Q12b) ... And if k were 0, I'd get the same as here (Q1a).

We can also see how a more complete view of the concept of derivative has helped enrich the solution to the equation that Wanda first gave and where she only considered positive values of the constant k.

Jordan (who belongs to the second group) uses the method of separating variables to solve all the equations from the questionnaire but experiences difficulties with the integration of elemental functions leading him to solve many of the activities incorrectly (Figure 4). There is no evidence that he relates the differential equations to the concept of derivative.

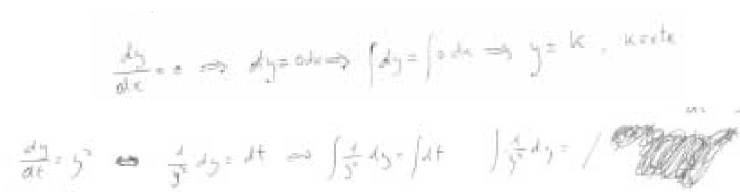
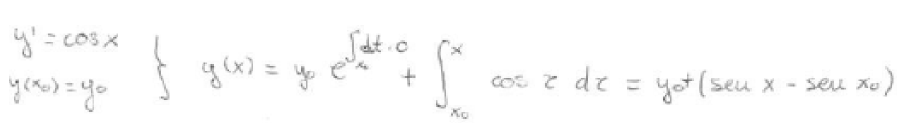
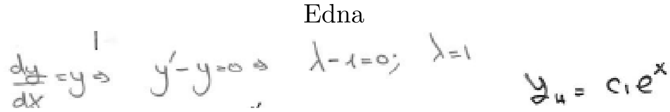


Figure 4. Jordan's answers to Q1a and Q2b in the questionnaire

A third of the students solved all the differential equations we set by using algorithms. Most of them classify these equations as separable variables, but there are also answers where they use solution methods appropriate for other types of ODE (Figure 5).



Edna



Jeremy

Figure 5. Algorithms used to solve Q1b and Q2a

The interviews have allowed us to document that, in some cases, students find the analytic methods of solving ODEs to be meaningless. For example, Wanda correctly uses both her knowledge of the derivative of a function as well as the method of separation of variables to solve the equations in Type 1 questions. In the interview this student shows the same resources, solving the equation $\frac{dy}{dx} = 0$ through separating the variables to be able to approach the equations $y'(t) = k$ and $y'(x) = -1$. However, the student expresses doubts about the mathematical validity of the operations she carries out.

Wanda (W): [...] I again separate the variable. Frankly this might not be totally right but ... I integrate through separation ...

Researcher (R): You've just said that something isn't right.

W: Yes, formally, when I write it, this doesn't seem right (she indicates the expression $dy = -dx$) ... I vaguely remember that you had to look for certain functions that could verify some conditions and the teacher sometimes said "well, like this it would be solved a little bit roughly, you know, a bit too fast." It's like a fast way of doing it [...]

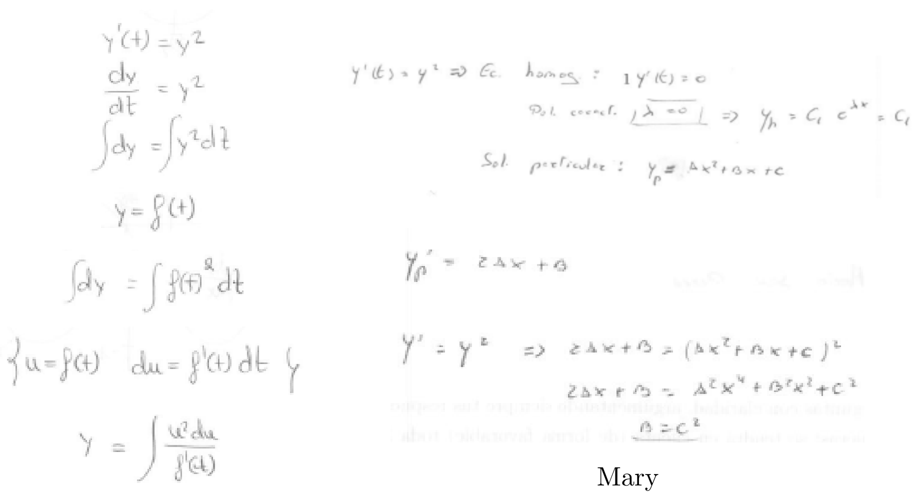
R: And why do you think that separating the differential is meaningless mathematically speaking?

W: I can see this (she indicates dy/dx) and I know that it is a derivative ... that is, a variation of the function y with regard to the variable x. However, the differential of y and the differential of x taken separately ... I can't see any sense when they are separated ...

Wanda's tendency to the use of algorithms leads her to rely on operations that for her are mathematically meaningless as the first option for solving the problems. She relegates to a second plane the knowledge she possesses about the derivative and its relationship with differential equations.

Students' answers to these types of questions also allow us to see that it is not enough to know algorithms to solve differential equations. Certain resources and strategies are also needed to decide when to use them. Four students attempted to solve some equations by separating the variables but failed to solve Q2b correctly. Although they have shown that they know some algorithms for solving equations, they failed to develop the ability to choose the most suitable one in each case (Figure 6). Schoenfeld [18] believes that this might be due to students' deficient use of metacognitive strategies that prevents them from choosing the suitable strategy in order to solve the problem.

Comment. To what extent do students rely on the use of concepts and resources previously studied to make sense of situations and solve problems related to ODEs? There is evidence that even when some students recognize the form of solution functions of some "simple" ODEs, they experience evident difficulties in representing them graphically as solutions of those equations. In addition, they seem to lack the use of problem solving strategies to overcome initial difficulties that can help them systematically re-examine their previous knowledge to deal with completely new situations.



Rosy

Mary

Figure 6. Rosy's and Mary's answers to Q2b

5. Final remarks

In relation to the extent to which the participants used their previous knowledge to deal with basic questions studied in a first ordinary differential equation course. Results show that the participants, in general, did not relate or use the meanings associated with the concept of derivative to make sense of questions or problems that involved differential equations concepts. Instead, they tried to identify or match a particular form of the involved equation with an algorithm to solve it. Their tendency to only identify an analytic approach to answer the questions seems to prevent them from thinking of other possibilities to make sense of and approach the problems. Thus, their approaches to solving problems that involve ODEs were limited, in general, to finding the corresponding algebraic rule that help them to determine the solution. They showed that they had developed some kind of image or referent about the forms of the equations and their corresponding algorithms to solve them; but they often were not able to retrieve the relevant information needed to approach the problem. However, when students (during the interviews) were directly asked to consider the meanings of the concepts involved in the problem statements, in general, they were able to think of the problems in different ways. That is, they recognized, for instance, that in some cases they could think of a function (or family of functions) that was a solution of an ODE without using the solution method they had studied.

In this perspective, the results in this study provide important information to restructure and connect a first calculus course with an introductory differential equation course. For example, to review, extend, and articulate the meanings associated with the concept of derivative studied in a first calculus course, students could also relate the interpretation and/or meaning of the derivative to finding the solution of type of equations that involves $y'(t) = k$ or $\frac{dy}{dx} = \cos x$. Similarly, the geometric interpretation of the derivative could also help students represent the direction fields associated with a particular ODE or to analyze certain solution to an ODE without expressing them algebraically. That is, in a calculus course students can discuss certain types of differential equations in terms of the meaning of the concept of derivative without focusing yet on the use of particular algorithm to solve it. In this context, the use of the derivative to solve this type of problems could be the bridge for students to connect the concept of derivative with the initial concepts that appear in an ordinary differential equation course. Indeed, we argue that discussing this type of ODEs adds another meaning associated to the concept of derivative to the list of meaning proposed by Thurston [19].

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REFERENCES

[1] Camacho, M., Perdomo, J. & Santos-Trigo, M., *Revisiting university students' knowledge that involves basic differential equation questions*, In: Figueras, O. & Seplveda, A. (Eds.), PNA **3** (3) (2009), 123–133.

[2] Camacho-Machín, M., Perdomo-Díaz, J. & Santos-Trigo, M., *An exploration of students' conceptual knowledge built in a first ordinary differential equations course (Part II)*, The Teaching of Mathematics, to appear.

[3] Couco, A., Goldenberg, P.E., & Mark, J., *Habits of mind: An organizing principle for mathematics curriculum*, J. Math. Behavior, **15** (1996), 375–402.

[4] Duval, R., *Registres de représentation sémiotique et fonctionnement cognitif de la penseé*, Ann. Didactique Sci. Cognitives, **5**, IREM de Strasbourg, 37–65.

[5] Goldin, G., *Representational systems, learning, and problem solving in mathematics*, J Math. Behavior **17** (1998), 137–165.

[6] Goldin, G. A., *A scientific perspective on structured, task-based interviews in mathematics education research*, In: Kelly, A. E. & Lesh, R. A. (Eds.), *Handbook of Research Design in Mathematics and Science Education*, Lawrence Erlbaum Associates, London, 1998, pp. 517–545.

[7] Habre, S., *Exploring students' strategies to solve ordinary differential equations in a reformed setting*, J. Math. Behavior **18** (4) (2000), 455–472.

[8] Hiebert, J. & Carpenter, T., *Learning and teaching with understanding*, In: D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, A project of the NCTM (1992), pp. 65–97.

[9] Kilpatrick, J., Swafford, J., & Findell, B. (Eds.), *Adding it up: Helping children learn mathematics*, Washington, DC : National Academy Press, 2001.

[10] Lester, F., *On the theoretical, conceptual, and philosophical foundations for research in mathematics education*, In: B. Sriraman & L. English (Eds.), *Theories of Mathematics Education*, Heidelberg : Springer, 2010, pp. 67–85.

[11] Moreno, J. & Laborde, C., *Articulation entre cadres et registres de représentation des équations différentielles dans un environnement de géométrie dynamique*, Actes du Congrés Européen ITEM, Reims France, 2003.

[12] NCTM, *Focus on high school mathematics: Reasoning and sense making*, Reston VA : The Council, 2009.

[13] Rasmussen, C., *New directions in differential equations. A framework for interpreting students' understandings and difficulties*, J. Math. Behavior **20** (2001), 55–87.

[14] Rasmussen, C. & Kwon, O., *An inquiry-oriented approach to undergraduate mathematics*, J. Math. Behavior **26** (2007), 189–194.

[15] Rasmussen, C., Kwon, O., Allen, K., Marrongelle, K. & Burtch, M., *Capitalizing on advances in mathematics and K-12 mathematics education in undergraduate mathematics: an inquiry-oriented approach to differential equations*, Asia Pacific Education Review **7** (1) (2006), 85–93.

[16] Rasmussen, C. & Whitehead, K., *Learning and teaching ordinary differential equations*, In: A. Selden & J. Selden (Eds.), *MAA Online Research Sampler*, `http://www.maa.org/t_and_l/sampler/rs_7.html`.

[17] Santos-Trigo, M. & Barrera-Mora, F., *Contrasting and looking into some mathematics education frameworks*, Math. Educator **10** (1) (2007), 81–106.

[18] Schoenfeld, A., *Mathematical Problem Solving*, Orlando : Academic Press, 1985.

[19] Thurston, W. P., *On proof and progress in mathematics*, Bull. Amer. Math. Soc. **30** (2) (1994), 161–177.

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Appendix 1: List of activities and aims thereof

Problems of Type 1	***Setting***	***Aims***
Q1. Represent graphically some solutions for the following equations a) $\frac{dy}{dx}=0;\ x\in[0,2]$ b) $\frac{dy}{dx}=\cos x$	Questionnaire Interview	To analyze: ▪ If they transfer knowledge about the concept of derivative of a function in order to obtain the algebraic expression of the solution to the equations. ▪ If they correctly use the solutions methods for first-order, separable variable ODEs, should they in fact do this. ▪ If they integrate correctly. ▪ Graphic representation of the solutions of equations.
Q2. Represent graphically some solutions for the following equations a) $\frac{dy}{dx}=y;\ x\geq 0$ b) $y'(t)=y^2$	Questionnaire	
Q12. Represent graphically some solutions for the following equations a) y'(t)=k b) y'(x)=-1	Interview	

Appendix 2: Procedures used by students in the questionnaire

Type 1 activities

Q1 and Q2		Students
14 students appeared to use their knowledge about derivative of a function to solve some of the simple cases, specially Q1.	7 students did not use algorithms for solving any of these ODEs. Only one student correctly solved the ODE in Q2b, probably tested with different functions. Two students did not answer Q2 and proposed a single solution for Q1a; one of them analyzed the sign of the derivative in Q1b. Two students proposed a single solution for Q1 and Q2a and did not solve correctly or did not answer to Q2b. Two students did not distinguish between the independent and the dependent variables while solving Q2.	Stella, Betty, Laure, Helen, Jason, Eddy, Franklin
	2 students showed that they know methods for solving ODEs, but do not know when or how to use them. One of them used the method of separating variables in Q2a but he did not use it for Q2. The other student tried to use this method in Q2b but she did not apply it correctly.	Rosy, Gaby
	5 students used the solution algorithms properly. Three students used the concept of derivative for solving Q1 and the method of separating variables in Q2. One student used the characteristic equation for solving Q2a and tried to use it in Q2b but it did not work so the student selected the method of separating variables for solving Q2b. Two students did not consider the constant of integration and one of them did not represent correctly the exponential function.	Wanda, Carena, Melvin, Jeremy, Berenice
7 students solves all the ODEs using algorithms	2 students did not select the method to be used correctly. One of them tried to use the characteristic equation in Q2b. The other one used the method of separating variables in Q1 and Q2b but not in Q2b. Neither of them considered the constant of integration and one of them did not represent correctly the sine function.	Mary, Sam
	5 students distinguished between the useful methods in each situation. Only one student solved all the ODEs correctly. One student failed while integrating; another one did not consider the constant of integration; another one did not represent correctly functions of the form $f(x)=k\cdot e^{x}$, where k is constant.	Jordan, Roger, Angie, Edna, Silvana

Strategies for solving problems shown by the students when solving Type 1 tasks