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LET'S GET ACQUAINTED WITH MAPPING DEGREE!

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Dedicated to Professor Milosav Marjanović on the occasion of his 80th birthday

Abstract. Given a continuous map $f : M \to N$ between oriented manifolds of the same dimension, the associated *degree* $\deg(f)$ is an integer which evaluates the number of times the domain manifold M "wraps around" the range manifold N under the mapping f. The mapping degree is met at almost every corner of mathematics. Some of its avatars, pseudonyms, or close relatives are "winding number", "index of a vector field", "multiplicity of a zero", "Milnor number of a singularity", "degree of a variety", "incidence numbers of cells in a CW-complex", etc. We review some examples and applications involving this important invariant. One of emerging guiding principles, useful for a mathematical student or teacher, is that the study of mathematical concepts which transcend the boundaries between different mathematical disciplines should receive a special attention in mathematical (self)education.

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Key words and phrases: Mapping degree; winding number.

1. Introduction

- " ... Intuitively, the degree represents the number of times that the domain manifold wraps around the range manifold under the mapping. The degree is always an integer, but may be positive or negative depending on the orientations."

Wikipedia

- " ... In mathematics, the winding number of a closed curve in the plane around a given point is an integer representing the total number of times that curve travels counterclockwise around the point. The winding number depends on the orientation of the curve, and is negative if the curve travels around the point clockwise."

Wikipedia

The title of this article provides one of possible advices to those who want to be initiated in algebraic topology or global analysis but have neither time nor enthusiasm to follow closely some of the more standard academic paths. This applies to a student who wants to get some flavor of these fields without committing herself to a careful study of some of the available textbooks. This applies as well to a non-specialist who would like to have a quick grasp of what topology is all about and how it can be applied in different areas. Finally the mapping degree provides a concept which ties together many mathematical areas and illustrates an important methodical principle which can be summarized as follows.

- Follow an interesting mathematical concept through all its metamorphoses and incarnations. Don't pay attention to artificial boundaries between different mathematical fields. Instead, use different points of view to achieve deeper understanding of the inner structure of the concept itself and all mathematical places where it resides. And above all, enjoy in the sightseeing of mathematics along the way.

2. Winding number of a curve

Counting the *winding number* of a closed curve, defined as the number of times a closed curve winds around a given point in the plane or around a line in space, is a task which can be easily performed by a layman. This is an example of a mathematical concept which is deeply rooted in our every day experience and can be used to motivate many related mathematical concepts and facts.

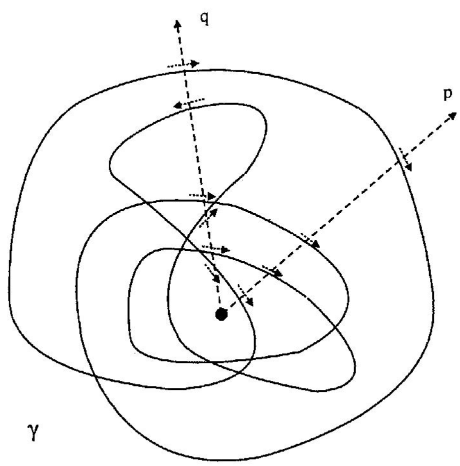

Fig. 1. The winding number as the number of signed intersections with a half-ray.

Figure 1 depicts a smooth curve with winding number -4. One way of seeing this is by evaluating the number of *signed intersections* of an oriented curve γ with a half-ray emanating from the given point. The intersection is evaluated as "$+1$", respectively "-1", if the intersection is positive (respectively negative), i.e. if the curve intersects the half-ray from right to left (or the other way around).

For this calculation one can use any half-ray which is *transverse* to γ in the sense that at meeting points they cross each other, rather than being tangent to each other at that point. For example both half-rays p and q depicted in Figure 1 are transverse to γ.

Observation 1. Suppose that γ is a closed curve and r a half-ray emanating from a point O which is not on γ. Then neither the number of positive intersections

$n^+(\gamma, r)$ nor the number of negative intersections $n^-(\gamma, r)$ of the curve γ with r are *constant numbers* independent of the half-ray r. For example in Figure 1,

$$0 = n^+(\gamma, p) \neq n^+(\gamma, q) = 1 \quad \text{and} \quad +4 = n^-(\gamma, p) \neq n^-(\gamma, q) = +5.$$

However, the winding number $w(\gamma) = n^+(\gamma, p) - n^-(\gamma, p)$ is a well defined number which does not depend on the half-ray r.

The winding number of a curve is one out of many manifestations of the concept of the *mapping degree* $\deg(f)$ of a smooth map $f : M \to N$. Both numbers belong to a family of closely related numerical invariants which include "index of a vector field", "multiplicity of a zero", "Milnor's number of a singularity", "degree of a variety", "dimension of a local algebra", and many more as close relatives. This multitude of manifestations (avatars) of the same concept provides an excellent opportunity for a teacher and student alike to explore passages connecting different areas of mathematics. This in turn leads to deeper understanding of some of the fundamental mathematical ideas unifying different areas of geometry, topology, analysis, algebra, mathematical physics, and combinatorics.[1]

2.1. Radial invariance of the winding number

Let $f : S^1 \to \mathbb{R}^2 \setminus \{0\}$ be a (parametrization of a) curve γ in the plane which does not pass through the origin. Here we observe some immediate consequences of the definition of the winding number $w(\gamma) = w(f)$ as the difference $n^+(f, r) - n^-(f, r)$ of two integers evaluating signed intersections of the curve with a given half-ray r.

Let us introduce a "radial modification" of the curve γ. Given a strictly positive smooth function $\lambda : S^1 \to \mathbb{R}^+$ let γ' be the curve parameterized by the function $f'(t) = \lambda(t)f(t)$. In particular if $\lambda(t) = 1/|f(t)|$ we obtain a function $f' : S^1 \to S^1 \subset \mathbb{R}^2$. Since $\lambda(t)$ is a positive scalar we observe that

$$n^+(f', p) = n^+(f, p) \quad \text{and} \quad n^-(f', p) = n^-(f, p) \tag{1}$$

for each half-ray p. As a consequence $w(\gamma) = w(\gamma')$ and we established a principle of "radial invariance" of the winding number. Encouraged by this example let us see what other modifications of curves have no effect on the winding number. For example let us alow modification of the curve γ outside an open angle $(r^{-\epsilon}, r^{+\epsilon}) \subset \mathbb{R}^2$ containing a chosen half-ray r. Again by the formula $w(\gamma) = n^+(\gamma, r) - n^-(\gamma, r)$ we observe that the winding number remains the same.

2.2. The "dog on a leash" theorem

There are other useful principles that allow a modification of a curve γ without changing its winding number. A classical *Rouché's theorem* claims that if $f : S^1 \to \mathbb{R}^2$ is obtained by a "small perturbation" $f = g + h$ of a map $g : S^1 \to \mathbb{R}^2 \setminus \{0\}$ by

[1] Milosav Marjanović, my Belgrade thesis adviser, my topology teacher, dear friend and coauthor of joint mathematical papers, was the first to introduce me to this point of view about mathematics. I hope that some examples selected for this brief exposition may convey the spirit of his lectures and his conviction that topology is a wonderful and powerful mathematical tool!

a map $h : S^1 \to \mathbb{R}^2$ then $w(f) = w(g)$ provided $|h(t)| < |g(t)|$ for each $t \in S^1$. One popular, informal way to paraphrase this result is the following.

($\bullet_1$) If a person were to walk a dog on a leash around and around a tree, and if the length of the leash is at all times kept shorter than the distance of the person from the tree, then the person and the dog go around the tree an equal number of times.

2.3. Homotopy invariance of the winding number

Provably the most general result which guarantees the invariance of the winding number is the following principle which introduces the concept of a *homotopy* of continuous maps. Recall that two maps $f, g : S^1 \to \mathbb{R}^2 \setminus \{0\}$ are homotopic $f \simeq g$ if there is a continuous family (homotopy) $f_\alpha : S^1 \to \mathbb{R}^2 \setminus \{0\}$ of maps, indexed by a parameter $\alpha \in [0, 1]$, such that $f_0 = f$ and $f_1 = g$. By definition the value $f_\alpha(x)$ depends on two input parameters (α and x) so homotopies are most often described as continuous maps $F : X \times [0, 1] \to Y$ where $(x, \alpha) \in X \times [0, 1]$. In our case $X = S^1$ and $Y = \mathbb{R}^2 \setminus \{0\}$ but the concept is used for the classification "up to homotopy" of arbitrary continuous maps $f, g : X \to Y$ with a lasting impact to virtually all branches of mathematics.

THEOREM 2. (*Homotopy invariance of the degree*) *Homotopic maps $f \simeq g$ have the same winding number $w(f) = w(g)$. Conversely, if two maps $f, g : S^1 \to \mathbb{R}^2 \setminus \{0\}$ have the same winding number they are homotopic in the sense that they are connected by a homotopy $F : S^1 \times [0, 1] \to \mathbb{R}^2 \setminus \{0\}$ such that $f(t) = F(t, 0)$ and $g(t) = F(t, 1)$ for each $t \in S^1$.*

Some readers may find the concept of homotopy somewhat overwhelming on the first encounter so let us see how it can be used in practice. For starters let us demonstrate how *Rouché's theorem* follows as a simple consequence. Indeed, given two maps $f, g : S^1 \to \mathbb{R}^2 \setminus \{0\}$ one can try to construct a "linear homotopy"

$$F(t, \alpha) = (1 - \alpha)f(t) + \alpha g(t) \tag{2}$$

between f and g. The map $F : S^1 \times [0, 1] \to \mathbb{R}^2$ defined by (2) is always continuous so what can go wrong with this homotopy? The answer is simple. We *must not* overlook the condition that $F(t, \alpha) \neq 0$ for each α and t! Since $F(t, \alpha)$ is a point on the segment connecting $f(t)$ and $g(t)$ we have the following immediate consequence of Theorem 2:

($\bullet_2$) Suppose that $f, g : S^1 \to \mathbb{R}^2 \setminus \{0\}$ are two maps such that the origin is never in the line segment $[f(t), g(t)]$ for some $t \in S^1$, or equivalently that $(1 - \alpha)f(t) + \alpha g(t) \neq 0$ for each α and t. Then f and g have the same winding number.

EXERCISES:

E_1 : Show that if f and g satisfy the condition $|f(t) - g(t)| < |g(t)|$ for each $t \in S^1$ then they also satisfy the condition ($\bullet_2$). Use this to deduce *Rouché's theorem* from the homotopy invariance of the winding number.

E_2 : Prove a symmetric version of *Rouché's theorem* which claims that if $|f(t) - g(t)| < |f(t)| + |g(t)|$ for each $t \in S^1$ then $w(f) = w(g)$.

Hint: Use the same strategy as in the first exercise.

E_3 : Show that a person and a dog on a leash will go around a tree an equal number of times even if the leash is of variable length, provided the master never looses the dog out of his sight.

Hint: It is assumed that the visibility of the dog can be obstructed only by the tree!

E_4 : Why could we say that the homotopy invariance is provably the strongest result that guarantees the invariance of the winding number?

Hint: Read carefully the second sentence in the statement of Theorem 2.

2.4. A second look at "dog-walking theorems"

The "dog-walking theorem" ($\bullet_1$) provides an appealing reformulation of Rouché's theorem suitable both for the classroom use and for communicating mathematical ideas to non-specialists. An interested reader may try to trace the origin of this reformulation by a *Google*-search involving phrases like "dog on leash theorem", "dog-walking theorem", or equivalent. Among the hits are the Wikipedia article `http://en.wikipedia.org/wiki/Rouch%C3%A9%27s_theorem` (where the exercise E_2 is attributed to T. Estermann) and the book of R.B. Ash and W.P. Novinger `http://www.math.uiuc.edu/~r-ash/CV.html`, which points to W.A. Veech, A Second Course in Complex Analysis, page 30. The site `www.numericana.com`, has in the topology section a "dog on a leash"-lemma, `http://www.numericana.com/answer/topology.htm#leash`, in a form very close to our statement ($\bullet_2$), etc.

Here we modestly observe that Theorem 2 can be also rephrased as a "dog on leash"-type result. Actually this reformulation appears to be even more natural considering that the leash itself, understood as a curve of variable length, plays here a much more explicit and significant role.

Suppose that a person P walks from a point P_0 to a point P_1 along a path γ_P. P is accompanied with a dog D who walks along a path γ_D from a point D_0 to a point D_1. Paths are naturally interpreted as maps $\gamma_P, \gamma_D : [0,1] \to \mathbb{R}^2$ or, in the case of a circular motion when $P_0 = P_1$ and $D_0 = D_1$, as maps $\gamma_P, \gamma_D : S^1 \to \mathbb{R}^2$.

The leash itself is, for a given moment in time $t \in [0,1]$, a curve $\omega_t : [0,1] \to \mathbb{R}^2$ which connects the person $P(t) = \gamma_P(t)$ and the dog $D(t) = \gamma_D(t)$. Summarizing we observe that the continuous motion of a system $(person, dog, leash)$ is nothing but a homotopy $F : [0,1] \times [0,1] \to \mathbb{R}^2$, or for circular motion a homotopy $F : S^1 \times [0,1] \to \mathbb{R}^2$. Observe that for a given pair of parameters $(t, \alpha) \in S^1 \times [0,1]$, $F(t, \alpha)$ describes the position of the point on the leash which at the time t divides the leash in the ratio $\alpha : (1 - \alpha)$.

From here we easily obtain a "dog on leash"-type reformulation of Theorem 2.

($\bullet_3$) If a person walks a dog on a leash of variable length around and around a tree, and if after some time both the person, the dog, and the leash return to their initial positions, then the person and the dog went around the tree an equal number of times.

2.5. Useful formulas for the degree of a map $f : S^1 \to S^1$

There are many ways how to define or calculate the degree $\deg(f)$ of a continuous map $f : S^1 \to \mathbb{R}^2 \setminus \{0\}$, or equivalently the degree of its radial normalization $g = f/|f| : S^1 \to S^1$. Here is a short list of examples each carrying the trademark of the mathematical discipline where it naturally belongs. The reader is not expected to master all of them at once but should notice the variety of different *key words and phrases* involved in these definitions including the *homology groups, differential forms, regular value, tangent space* $T_x(M)$ *etc.*

(1) The degree $\deg(\phi)$ of $\phi : S^1 \to S^1$ is the *signed count* of points in the inverse image $\phi^{-1}(p)$ where $p \in S^1$ is a *regular value* of the function ϕ and $x \in \phi^{-1}$ is counted with the sign " $+$ " (alternatively " $-$ ") if the derivative (differential) $d\phi_x : T_xS^1 \to T_pS^1$ preserves (alternatively changes) the orientation.

(2) Suppose that $f(t) = (x(t), y(t))$, $t \in [0, 2\pi]$ is a smooth parametrization of a curve in the plane $\mathbb{R}^2$. Then,
$$\deg(f) = \frac{1}{2\pi}\int_{S^1} \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

(3) If $f : S^1 \to \mathbb{C} \setminus \{0\}$ is a smooth map then
$$\deg(f) = \frac{1}{2\pi i}\int_{S^1} \frac{df}{f}.$$

(4) If $\phi : S^1 \to S^1$ is a smooth map and $\omega \in \Omega^1(S^1)$ a differential 1-form, then
$$\int_{S^1} f^*(\omega) = \deg(f)\int_{S^1} \omega.$$

(5) The degree $\deg(\phi)$ is the unique number $k \in \mathbb{Z}$ such that $\phi_*(x) = kx$ where
$$\phi_* : H_1(S^1; \mathbb{Z}) \to H_1(S^1; \mathbb{Z})$$
is the induced map of the associated homology groups where $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$.

2.6. Index of a vector field near a singular point

One of standard applications of the mapping degree is the classification of isolated singular points of vector fields, Figure 2. Let us imagine that each of the six diagrams depicts flow lines of a "magnetic field" vanishing in the center (the singular point of the field). Suppose that an observer is moving with a "magnetic compass" in his hand along a small circle centered at the singular point in the counterclockwise direction. The number of rotation of the magnetic needle is called the *index* of the vector field at the singular point a and denoted by $Ind(a)$.

EXERCISES:

E_5 : Compute the index of the singular point for each of the vector fields depicted in Figure 2. For any integer $k \in \mathbb{Z}$ construct a vector field with an isolated singular point a such that $Ind(a) = k$.

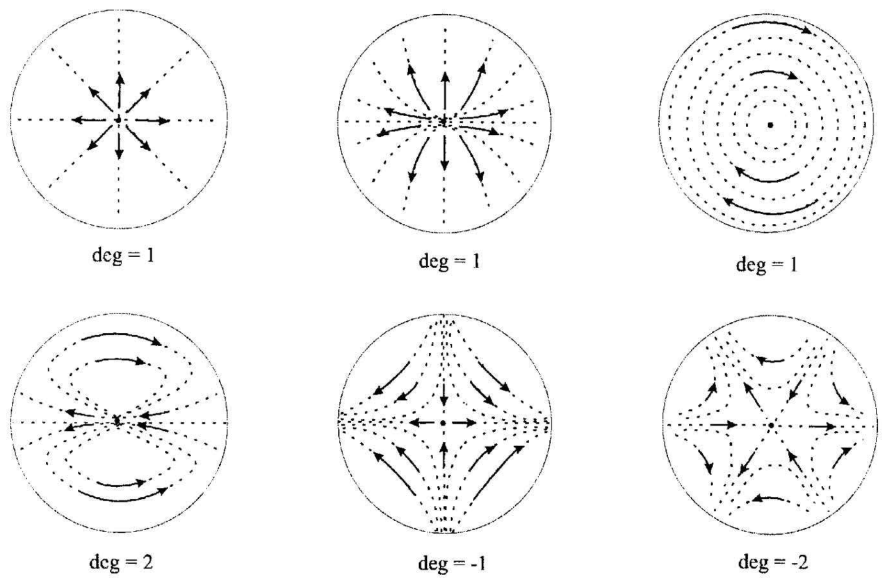

Fig. 2. Vector fields in the vicinity of an isolated singular point.

Hint: Don't forget that the needle of the compass is always tangent to the flow lines of the magnetic field.

E_6 : Let X be a vector field on a sphere S^2 which has only isolated singular points. Show by examples that the total sum of the indices of all singular points is 2. Generalize this to other surfaces.

3. Mapping degree in higher dimensions

Essentially all the definitions and formulas for the calculation of the mapping degree listed in Section 2 can be extended to higher dimensions. Here we list only three of the most important versions.

In all these examples $\phi : M \to N$ is a continuous mapping between oriented manifolds of the same dimension, $\dim(M) = \dim(N) = n$.

(1) Suppose that a is a *regular value* of a smooth map $\phi : M \to N$, i.e. $a \in N$ has the property that for each point in the pre-image $x \in \phi^{-1}(a)$ the differential $d\phi_x : T_x(M) \to T_a(N)$ is a regular linear map. The sign $\mathrm{sgn}(x)$ of the point $x \in \phi^{-1}(a)$ is by definition $+1$ (alternatively -1) if the differential $d\phi_x$ is an orientation preserving (orientation reversing) map of tangent spaces $T_x(M)$ and $T_a(N)$. Then,

$$\deg(\phi) = \sum_{x \in \phi^{-1}(a)} (-1)^{\mathrm{sgn}(x)}.$$

(2) If $\phi : M \to N$ is a smooth map and $\omega \in \Omega^n(N)$ a smooth n-form, then

$$\int_M f^*(\omega) = \deg(f) \int_N \omega.$$

(3) The degree $\deg(\phi)$ is the unique integer $k \in \mathbb{Z}$ such that $\phi_*(x) = kx$ where

$$\phi_* : H_n(M;\mathbb{Z}) \to H_n(N;\mathbb{Z})$$

is the induced map of n-dimensional homology groups with integer coefficients. Recall that by assumptions $H_n(M;\mathbb{Z}) \cong \mathbb{Z} \cong H_n(N;\mathbb{Z})$.

4. Argument principle and Sperner's lemma

One of the origins of the mapping degree is complex analysis, notably its *Argument principle* and *Rouche's theorem* which we already met in Section 2. The argument principle is essentially the equation

$$W = Z - P \tag{3}$$

where Z and P are respectively the number of zeros and the number of poles of a function meromorphic in a given region while W is the winding number of an associated curve. Here is a more precise formulation of this principle modelled on the presentation in a classical analysis text.[2]

- ... We consider a closed, continuous, oriented curve L in the z-plane that avoids the origin. If, starting from an arbitrary point, z describes the entire curve in the given direction (returning to its starting point) the argument of z changes continuously and its total variation is a multiple, $2\pi W$, of 2π. The integer W is called the winding number of the curve.

  ... L denotes a closed continuous curve without double points and D the closed interior of L. The function $f(z)$ is assumed to be regular in D, except possibly at finitely many poles, finite and non-zero on L. As z moves along L in the positive sense the point $w = f(z)$ describes a certain closed continuous curve the winding number of which is equal to the number of zeros inside L minus the number of poles inside L.

As emphasized in the Introduction, one of our objectives is to explore the underlying ideas which reveal the deeper nature of the principles associated to the mapping degree. For this reason let us compare the argument principle with a result equivalent to the 2-dimensional instance of the classical Sperner's lemma. Recall that the Sperner's lemma is a combinatorial statement about labelled triangulations of simplices which was invented for a combinatorial proof of the Brouwer fixed point theorem.

- Suppose that L is the boundary of a triangulated region D in the plane. Suppose that each vertex of the triangulation is labelled by $0, 1$ or 2. This labelling

[2] G. Pólya, G. Szegö, Problems and Theorems in Analysis I, Springer 1998.

defines a simplicial map (simplicial = affine on triangles) $f : D \to \Delta$ where Δ is a triangle with vertices A_0, A_1, A_2. A triangle T that appears in the triangulation of the region D is called a "zero of the function f" if its vertices are labelled by all three numbers $0, 1, 2$ provided their order on T is counterclockwise. Similarly T is a "pole of the function f" if labels $0, 1, 2$ appear on T in the clockwise order. Let W be the winding number of the restriction map $f' : L \to \partial\Delta$. Then $W = N - P$, i.e. the degree of f' counts the difference between the number of zeros and poles of the function f.

From a view point of a topologist, both the Argument principle and our modified version of Sperner's lemma can be established by the essentially the same method. More accurately both statements are special cases of a general principle about mapping degrees which is naturally formulated and in full generality established in *homology theory*. Here is an outline of the main geometric idea behind this proof.

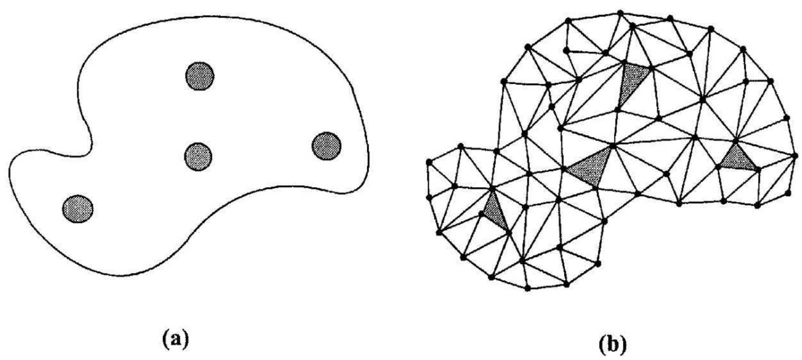

Fig. 3. The domain of the function with small neighborhoods of zeros and poles removed.

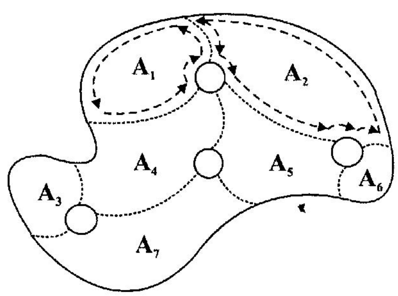

Fig. 4. Homology of cycles implies the equality of degrees!

Let us cut out from D small neighborhoods of zeroes and poles of the function f. Figure 3(a) depicts these neighborhoods in the case of a meromorphic function f as small (open) circular discs, while in Figure 3(b), associated to Sperner's lemma, these neighborhoods are interiors of triangles labelled by all three labels

$0, 1, 2$. It remains to be shown that the degree of the map f restricted on the boundary curve $L = \partial D$ is equal to the total sum of degrees of f restricted on the boundaries of small neighborhoods. A geometric explanation why this equality of degrees is true is presented in Figure 4.

We finish this section with an interesting (albeit typical) application of the winding degree, argument principle and related results. The "challenge exercise" appears to be sufficiently complicated for a straightforward attack by numerical methods so it may serve as an illustration of the power of more "qualitative methods".

Challenge exercise. Prove that the equation

$$z \cos z = 1 \tag{4}$$

has precisely two solutions in the set $\mathbb{C} \setminus \mathbb{R}$ of purely complex numbers.

Solution. Let us consider the region

$$D = [-2K\pi, +2K\pi] \times [-Ni, Ni]$$

in the complex plane where K and N are very big natural numbers. We shall demonstrate, applying the *Argument principle* on the set D and the meromorphic function $\phi : D \to \mathbb{C}$ defined by $\phi(z) = -\frac{1}{z} + \cos(z)$, that the equation $z\cos(z) = 1$ has in this region precisely two strictly complex solutions.

We begin with the observation that the equation $\phi(z) = 0$ has precisely $4K-1$ real solutions in the interval $[-2K\pi, +2K\pi]$. Indeed, since $y = \cos(x)$ is an even function, the number of these solutions is equal to the number of intersections of the curve $y = \cos(x)$ and the graph of the relation $|y| = \frac{1}{x}$ for $x \in [0, 2K\pi]$ (Figure 5).

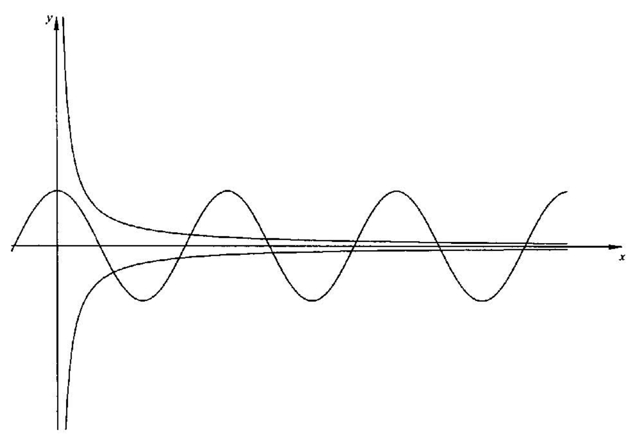

Fig. 5. Real zeros of the equation $z\cos(z) = 1$.

The function $\phi(z) = -\frac{1}{z} + \cos(z)$ has precisely one pole of degree 1 at $z = 0$. Let

$$L = \{\pm 2K\pi\} \times [-Ni, +Ni] \cup [-2K\pi, +2K\pi] \times \{\pm Ni\}$$

be the boundary of the region D. Our objective is to determine the winding number of the map $\phi : L \to \mathbb{C} \setminus \{0\}$. If $z = x + iy$ then

$$\phi(z) = -\frac{1}{z} + \cos(z) = -\frac{1}{z} + \frac{1}{2}[e^{-y+ix} + e^{+y-ix}]. \tag{5}$$

Since for all $z \in L$ the modulus $|1/z|$ of the complex number $1/z$ is very small (by assumption K and N are very big natural numbers) we conclude that $\phi(z)$ has the same winding number around $0 \in \mathbb{C}$ (when z moves along L in the positive direction), as the curve defined by the function

$$\psi : L \to \mathbb{C} \setminus \{0\}, \quad \psi(z) = \cos(z).$$

This statement is an instance of *Rouche's theorem* (the dog-on-leash theorem from Section 2.1). Our next step is to determine the degree $\deg(\psi)$. If $x = \pm 2K\pi$ then (in light of (5)) $\psi(z) = (1/2)(e^{-y} + e^{+y}) \in \mathbb{R}$. Consequently for the evaluation of the degree $\deg(\psi)$ we can focus on the horizontal parts of the curve L determined by the conditions $y = \pm N$.

By Rouche's theorem the function $\psi(z)$ has the same winding number on the fragment of the curve L determined by the equation $y = -N$ as the function $\alpha(z) = \exp(-y + ix)$. This function winds around zero $2K$ times while x moves along the interval $[-2K\pi, +2K\pi]$. Similarly, on the top boundary of the region D, the winding number of the function $\psi(z)$ (as x is gradually decreasing from $+2K\pi$ to $-2K\pi$) is again $2K$, since it is equal to the winding number of the function $\beta(z) = \exp(+y - ix)$.

Finally, according to the Argument principle, the number of zeros of the function $\phi(z)$ in the region $D \setminus \mathbb{R}$ is equal to $2K + 2K - (4K - 1) + 1 = 2$. ∎

5. Polynomials

5.1. Configuration spaces and polynomials

A point $X = (a, b, c)$ in the 3-dimensional Euclidean space is conveniently described as an ordered triple of real numbers a, b, c. Similarly a point $A = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ is identified as an ordered set (ensemble, configuration) of n points on the real line. This innocent change of perspective is actually a far reaching idea which allows us to "visualize" the geometry in higher dimensions by replacing the points in $\mathbb{R}^n$ by ordered point configurations on the real line (and vice versa).

Similarly, the geometry of the n-dimensional torus $T^n = (S^1)^n$ is recovered as the geometry of ordered, n-element configurations of points on the circle S^1.

What if the points are not ordered so for example we do not distinguish (a, b, c) from (b, a, c) or (a, b, a) from (b, a, a). More precisely what higher dimensional geometry is detected by configurations (multisets) $\{a_1, a_2, \dots, a_n\}$ where the order is no longer important. Recall that in *multisets* elements are allowed to appear more than once so for this reason $\{a, a, a, b, c\}$ and $\{a, b, a, c, a\}$ are equal unordered configurations (multisets) while $\{a, b, a\} \neq \{b, a, b\}$ in spite of the fact that they are equal as sets![3]

A reader who is used to sets of elements and feels somewhat uncomfortable with multisets should recall that natural numbers are nothing but multisets of prime numbers, for example $28 = 2^2 \cdot 7 = \{2, 2, 7\}$ and $135 = 3^3 \cdot 5 = \{3, 3, 3, 5\}$. This example makes it obvious why the order of factors is of no importance while the multiplicity, i.e. the number of occurrences of individual elements, is important.

Let us focus now on multisets of complex numbers and, for the sake of example, let us examine the geometry of 4-element multisets $\{\alpha, \beta, \gamma, \delta\} \subset \mathbb{C}$. Guided by the analogy with natural numbers which admitted a description as multisets of primes, let us ask ourselves if something similar is possible with multisets of complex numbers. Naively, since 28 is obtained from its multiset representation $\{2, 2, 7\}$ by multiplication $28 = 2 \cdot 2 \cdot 7$ why don't we try something similar with the multiset $M = \{\alpha, \beta, \gamma, \delta\}$!? The idea actually works if we replace M with the associated multiset $M' = \{x - \alpha, x - \beta, x - \gamma, x - \delta\}$ of prime polynomials! In turn we observe that, very much in analogy with natural numbers, each monic polynomial $p(x)$ with complex coefficients

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n) \tag{6}$$

is essentially nothing but the multiset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of its roots.

Summarizing we conclude that a point $(a_1, \ldots, a_n) \in \mathbb{C}^n$ is associated via (6) to the multiset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and the geometry of monic polynomials of degree n can be visualized as the geometry of planar unordered configurations (multisets).

Let us illustrate this point of view by a brief analysis of the set $\mathcal{M}_2$ of all monic, quadratic polynomials $q(x) = a_0 + a_1 x + x^2$ with real coefficients. Clearly $\mathcal{M}_2 \cong \mathbb{R}^2$ via the correspondence $q(x) \longleftrightarrow (a_0, a_1)$. The set $\mathbb{D}$ of all monic polynomials $q(x) \in \mathcal{M}_2$ with a root of multiplicity 2 is the parabola with the equation $a_1^2 - 4a_0 = 0$. For obvious reasons the set $\mathbb{D}$ is called the discriminant and the complement $\mathbb{R}^2 \setminus \mathbb{D}$, being the complement of a parabola, has two connected components. It is not difficult to see that polynomials $q_1(x) = x^2 + 1 = (x - i)(x + i)$ and $q_2(x) = x^2 - 1 = (x - 1)(x + 1)$ belong to different connected component.

We finish this subsection with a sequence of exercises designed to describe the structure of the set $\mathbb{R}^3 \setminus \mathbb{D}$ where $\mathbb{R}^3 \cong \mathcal{M}_4 = \{q(x) = a_0 + a_1 x + a_2 x^2 + x^4 \mid a_i \in \mathbb{R}^3\}$ and $\mathbb{D} \subset \mathcal{M}_4$ is the set of all polynomials with zeros of multiplicity at least 2.

Exercises:

E_7 : Show that the discriminant $\mathbb{D} \subset \mathbb{R}^3$ is a hypersurface $\mathbb{D} = \{a \in \mathbb{R}^3 \mid f(a_0, a_1, a_2) = 0\}$ where f is a polynomial in variables a_0, a_1, a_2 with integer coefficients.

E_8 : Show that $Q = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is the multiset representation of a polynomial $q(x) \in \mathcal{M}_4$ if and only if

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$$

[3]The geometry of configurations spaces of sets is equally rich and attractive, see [2, 10] and the references in these papers.

and if $\alpha \in Q$ then $\overline{\alpha} \in Q$ (with equal multiplicities).

E_9 : Show that the complement of the discriminant $\mathbb{R}^3 \setminus \mathbb{D}$ has three connected components.

Hint: Show that the representatives of the components are

$$q_1(x) = (x^2-1)(x^2-4) \qquad q_2(x) = (x^2-1)(x^2+1) \qquad q_3(x) = (x^2+1)(x^2+4)$$

E_{10} : Draw the picture of the hypersurface $\mathbb{D}$ indicating the position of the polynomials

$$q(x) = x^4 \qquad x^2(x^2-1) \qquad x^2(x^2+1) \qquad (x-1)^2(x+1)^2 \qquad (x-1)^3(x+3)$$

as well as the polynomials listed in E_9.

Hint: This is the so called "Swallow-tail surface" important in the singularity theory, see [1, page 47] or [6, page 381].

5.2. The degree of the multiplication of polynomials

Let $\mathcal{P}_m := \{p(x) = a_0 + a_1x + \ldots + a_{m-1}x^{m-1} + a_mx^m \mid a_i \in \mathbb{R}\}$ be the vector space of polynomials of degree at most m with coefficients in the field of real numbers. Let

$$\mu_{m,n} : \mathcal{P}_m \times \mathcal{P}_n \to \mathcal{P}_{m+n}$$

be the multiplication of polynomials. We focus our attention on the (affine) space $\mathcal{P}^0_m := \{p(x) = a_0 + a_1x + \ldots + a_{m-1}x^{m-1} + x^m \mid a_i \in \mathbb{R}\}$ of monic polynomials of degree m and the associated multiplication map

$$\mu^0_{m,n} : \mathcal{P}^0_m \times \mathcal{P}^0_n \to \mathcal{P}^0_{m+n} \tag{7}$$

Problem: Evaluate the mapping degree of the map $\mu^0_{m,n}$.

An alert reader will notice that here we talk about the mapping degree of a map $f : M \to N$ between noncompact manifolds! The degree $\deg(f)$ is defined as before as the algebraic count of points in the pre-image $f^{-1}(z)$ of a regular point $z \in N$. However, we need an extra condition on the function f which will guarantee the finiteness of the set $f^{-1}(z)$. Moreover, we still insist on the homotopy invariance of the degree and its independence of the regular point $z \in N$. Both conditions are satisfied if we assume that f is a proper map in the sense of the following definition.

Definition 3. A continuous map $f : X \to Y$ is *proper* if $f^{-1}(Z)$ is a compact subset of X whenever Z is a compact subset of Y.

Let us illustrate the meaning of Definition 3 by proving the *properness* of the map (7). The space $\mathcal{P}_m$ of monic polynomials is naturally parameterized by the Euclidean space $\mathbb{R}^m$ of real vectors $(a_0, a_1, \ldots, a_{m-1})$ of length m. Hence, $\mathcal{P}^0_m \times \mathcal{P}^0_n \cong \mathbb{R}^{m+n}$ and $\mathcal{P}^0_{m+n} \cong \mathbb{R}^{m+n}$ and we want to know what is the meaning of *properness* for a continuous map between Euclidean spaces of the same dimension.

The answer is provided by the following criterion which we leave as an exercise to the reader.

EXERCISE 4. (*Criterion for proper maps*) Prove that a continuous map $g : \mathbb{R}^d \to \mathbb{R}^d$ is proper if and only if $g^{-1}(A)$ is bounded whenever A is a bounded subset of $\mathbb{R}^d$. Equivalently g is proper if for each sequence $(u_n)_{n\in\mathbb{N}}$ of vectors in $\mathbb{R}^n$ such that $\lim_{n\to\infty} \|u_n\| = +\infty$, the sequence $(g(u_n))_{n\in\mathbb{N}}$ is unbounded in sense that for some subsequence (u_{n_k}) of (u_n), $\lim_{k\to\infty} \|g(u_{n_k})\| = +\infty$.

PROPOSITION 5. *The multiplication* (7) *of monic polynomials is a proper map of manifolds.*

Proof. In order to apply the criterion given in Exercise 4 assume that $A \subset \mathcal{P}^0_m$ and $B \subset \mathcal{P}^0_n$ are sets of polynomials such that $A \cdot B := \{p \cdot q \mid p \in A, q \in B\}$ is bounded as a set of polynomials in $\mathcal{P}^0_{m+n} \cong \mathbb{R}^{m+n}$. We want to conclude that both A and B individually a bounded sets of polynomials. This is easily deduced from the following claim.

Claim: If $A \subset \mathcal{P}^0_n$ is bounded set of polynomials then the set $Root(A) := \{z \in \mathbb{C} \mid p(z) = 0 \text{ for some } p \in A\}$ is also bounded. Conversely, if $Root(A)$ is a bounded, A is also a bounded set of polynomials.

Proof of the Claim: The implication $\Leftarrow$ follows from Viète's formulas, while the opposite implication $\Rightarrow$ follows from the inequality $|\lambda| \leq \max\{1, \sum_{j=0}^{n-1} |a_j|\}$, where λ is a root of a polynomial with coefficients a_j. ■

The next step needed for computation of the mapping degree of the map (7) is the evaluation of the differential $d\mu_{m,n}$. The tangent space $T_p(\mathcal{P}^0_m)$ at the monic polynomial $p \in \mathcal{P}^0_m$ is naturally identified with the space $\mathcal{P}_{m-1}$ of all polynomials of degree at most $m-1$.

LEMMA 6. *Given monic polynomials $p \in \mathcal{P}^0_m$ and $q \in \mathcal{P}^0_n$ and the polynomials $u \in \mathcal{P}_{m-1}, v \in \mathcal{P}^0_{n-1}$, playing the role of the associated tangent vectors, the differential $d\mu_{m,n} = d\mu$ is evaluated by the formula*

$$d\mu_{(p,q)}(u,v) = \frac{d}{dt}(p+tu)(q+tv)_{|t=0} = pv + uq. \tag{8}$$

Let us determine the matrix of the map $d\mu_{(p,q)}$ in suitable bases of the associated tangent spaces $T_{(p,q)}(\mathcal{P}^0_m \times \mathcal{P}^0_n) \cong \mathcal{P}_{m-1} \times \mathcal{P}_{n-1}$ and $T_{pq}(\mathcal{P}^0_{m+n}) \cong \mathcal{P}_{m+n-1}$. A canonical choice of basis for $\mathcal{P}^0_{m-1}$ is $u_0 = 1, u_1 = x, \dots, u_{m-1} = x^{m-1}$ with similar choices $v_0 = 1, v_1 = x, \dots, v_{n-1} = x^{n-1}$ and $w_0 = 1, w_1 = x, \dots, w_{p+q-1} = x^{m+n-1}$ for $\mathcal{P}^0_{n-1}$ and $\mathcal{P}^0_{m+n-1}$ respectively. Formula (8) applied to this basis gives

$$d\mu_{(p,q)}(0, v^j) = d\mu_{(p,q)}(0, x^j) = x^j p(x) \qquad d\mu_{(p,q)}(u_i, 0) = d\mu_{(p,q)}(x^i, 0) = x^i q(x).$$

Possibly as a pleasant surprise, we conclude from here that the determinant of this matrix is equal to the classical *resultant* (8) of two polynomials!

$$\mathcal{R}(p,q) = \operatorname{Det}\begin{bmatrix} a_0 & a_1 & \dots & a_{m-1} & 0 & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{m-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{m-1} \\ b_0 & b_1 & \dots & b_{n-1} & 0 & 0 & \dots & 0 \\ 0 & b_0 & b_1 & \dots & b_{n-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & b_0 & b_1 & \dots & b_{n-1} \end{bmatrix} \tag{9}$$

In particular we can use classical formulas for $\mathcal{R}(p,q)$, see [6, Chapter 12], among them the formula

$$\mathcal{R}(p,q) = \prod_{i,j}(\alpha_i - \beta_j) \tag{10}$$

where α_i are roots of p and β_j are roots of q respectively, counted with the appropriate multiplicities.

This observation makes the computation much easier and the following lemma is a first example.

LEMMA 7. *Suppose that* $p(x) = a_0 + a_1x + \ldots + a_{m-1}x^{m-1} + x^m$ *and* $p(x) = b_0 + b_1x + \ldots + b_{n-1}x^{n-1} + x^n$ *are two polynomials with real coefficients such that the corresponding roots* $\alpha_1, \ldots, \alpha_m$ *and* $\beta_1, \ldots, \beta_n$ *are all distinct and non-real,* $\{\alpha_i\}_{i=1}^m \cup \{\beta_j\}_{j=1}^n \subset \mathbb{C} \setminus \mathbb{R}$. *Then the resultant of polynomials* p *and* q *is real and positive,* $\mathcal{R}(p,q) > 0$.

Proof. By assumption all roots of $p(x)$ (respectively $q(x)$) can be divided in conjugate pairs $\alpha, \overline{\alpha}$ (respectively $\beta, \overline{\beta}$). These two pairs contribute to the product (10) the factor

$$(\alpha - \beta)(\overline{\alpha} - \overline{\beta})(\alpha - \overline{\beta})(\overline{\alpha} - \beta) = A\overline{A}B\overline{B} > 0. \quad \blacksquare$$

PROPOSITION 8. *Suppose that* $m = 2k$ *and* $n = 2l$ *are even integers. The degree of the map* $\mu_{m,n} : \mathcal{P}_m^0 \times \mathcal{P}_n^0 \to \mathcal{P}_{m+n}^0$ *is*

$$\deg(\mu_{m,n}) = \binom{k+l}{k}. \tag{11}$$

Proof. We compute the degree $\deg(\mu_{m,n})$ by an algebraic count of the number of points in the pre-image $(\mu_{m,n})^{-1}(\rho)$ of a carefully chosen generic polynomial $\rho \in \mathcal{P}_{m+n}^0$.

Assume that $\rho = \rho_1\rho_2\ldots\rho_{k+l}$ is a product of pairwise distinct, irreducible, quadratic (monic) polynomials ρ_i. Equivalently ρ does not have real roots and all its roots are pairwise distinct. Note that such a polynomial can be easily constructed by prescribing its roots, for example it can be found in any neighborhood of the polynomial x^{m+n}. The inverse image $(\mu_{m,n})^{-1}(\rho)$ is,

$$(\mu_{m,n})^{-1}(\rho) = \{(p,q) \in \mathcal{P}_m^0 \times \mathcal{P}_n^0 \mid p \cdot q = \rho\}.$$

It follows from Lemma 7 that $\mathcal{R}(p,q) > 0$ for each pair of polynomials in the inverse image $(\mu_{m,n})^{-1}(\rho)$. In particular ρ is a regular value of the map $\mu_{m,n}$ and each pair (p,q) such that $p \cdot q = \rho$ contributes $+1$ to the degree. From here we deduce that

$$\deg(\mu_{m,n}) = \binom{k+l}{k}$$

which implies the formula (11). ■

6. Suggestions for further reading

Here are a few more examples from different mathematical disciplines where the mapping degree plays a central role. The exposition is intentionally sketchy (with some hints and suggestions) so the reader should have some fun independently exploring the literature and the world wide web.

6.1. Quaternionic "Fundamental theorem of algebra"

The classical "fundamental theorem of algebra" says that each non-constant complex polynomial must have a root. This result easily follows from the basic properties of the mapping degree for the functions $f : S^1 \to S^1$, see e.g. [11]. The reader familiar with this proof may try to solve the following exercises which are based on the article [5].

Exercises:

E_{11} : Let $f(x)$ be a quaternionic polynomial such that

$$f(x) = a_0 x a_1 x \ldots x a_n + \phi(x), \quad a_i \neq 0,$$

and $\phi(x)$ is a sum of similar monomials $b_0 x b_1 x \ldots x b_k$, where $k < n$. Prove that there exists a quaternion $q \in \mathbb{Q}$ such that $f(q) = 0$.

E_{12} : Is it true that each quaternionic polynomial of degree n has at most n distinct roots in the algebra $\mathbb{Q}$ of quaternions?!

6.2. Gauss-Bonnet theorem

Gauss-Bonnet theorem [7] is both one of the most fundamental and the most beautiful results of Geometry and Topology. The following exercise indicates its connection with the mapping degree.

E_{13} : Let M be a closed, orientable surface embedded in $\mathbb{R}^3$ (Figure 6) and $\gamma : M \to S^2$ the associated *normal* map. Show that

$$\deg(\gamma) = \frac{1}{2}\chi(M).$$

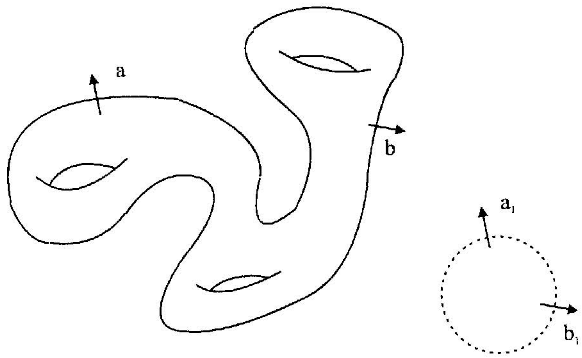

Fig. 6. The Gauss (normal) map.

6.3. Milnor number and Milnor's Problem 3

At the time of writing this article John Milnor is the most recent recipient of the *Abel Prize* (2011). Knowing that he is also a Fields medalist (1962) and a recipient of the Wolf Prize (1989), it is not a surprise that Milnor is one of the most renowned and authoritative living mathematicians.

In his influential book [9] on isolated singularities of complex hypersurfaces, he introduced a positive integer μ (Milnor number) measuring the amount of degeneracy of a critical point of a holomorphic map $f : \mathbb{C}^m \to \mathbb{C}$. Milnor number μ is defined as the mapping degree of the map $z \mapsto g(z)/\|g(z)\|$ (from a small sphere around the singular point to $S^{2m-1} \subset \mathbb{C}^m$) where $g_j := \partial f/\partial z_j$.

Milnor concludes his book [9] (Appendix B) with three problems for the reader, the first two easier and the third "more difficult".

- Problem 3. The ring $\mathbb{C}[[z]]$ of formal power series in the variables z_j can be considered as a module over the subring $\mathbb{C}[[g_1, \dots, g_m]]$. This module is free of rank μ. Hence, if I denotes the ideal spanned by $g_l, \dots, g_m$ in $\mathbb{C}[[z]]$, then the quotient ring $\mathbb{C}[[z]]/I$ has dimension μ over $\mathbb{C}$. (I am told that these statements can be proved by showing first that the map $g : \mathbb{C}^m \to \mathbb{C}^m$ induces a proper and flat map from a small neighborhood U of the origin to a small neighborhood V of the origin; and then that the direct image under g of the sheaf $\mathcal{O}_U$ of germs of holomorphic functions on U is locally free over the corresponding sheaf $\mathcal{O}_V$.)

E_{14} : Milnor's book [9] was published in 1968, opening a new era of *Singularity Theory.* Try to find out what happened afterwards and what is the contemporary point of view on Milnor's Problem 3. Hint. The books [1, Chaper 5] and [4, Chapter 4] are excellent sources of information.

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