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SOME CLASSICAL INEQUALITIES AND THEIR APPLICATION TO OLYMPIAD PROBLEMS

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Dedicated to Professor Milosav Marjanović on the occasion of his 80th birthday

Abstract. The technique introduced by M. Marjanović in [10] is used to prove several classical inequalities. Examples of application are given which can be used for preparing students for mathematical competitions.

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1. Introduction

M. Marjanović deduced in [10] some refinements of classical inequalities of Karamata [7] and Steffensen [16], making also shorter proofs (see also [8]). Similar technique was used in [12] to make a short proof of Chebyshev's inequality [4]. These proofs are reproduced in Section 2 of this paper.

It is well known that these classical results can be used in solving some of the most difficult problems in mathematical olympiads (in fact, the authors of these problems probably used the mentioned results, and made elementary solutions afterwards). We shall present in Section 3 some examples of this kind, giving solutions based on classical inequalities.

2. Majorization of sequences and functions and its application

Recall that, for decreasing finite sequences $a = (a_i)_{i=1}^n$ and $b = (b_i)_{i=1}^n$ of real numbers, a is said to *majorize* b, what is denoted by $a \succeq b$, or $b \preceq a$, if the terms of these sequences satisfy the following two conditions:

$$1^\circ \quad \sum_{i=1}^{k} a_i \geqslant \sum_{i=1}^{k} b_i, \text{ for each } k \in \{1, 2, \dots, n-1\}; \qquad 2^\circ \quad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

Analogously, for two integrable functions $\psi_1, \psi_2 \colon [\alpha, \beta] \to \mathbf{R}$, ψ_1 is said to *majorize* ψ_2, what is denoted by $\psi_1 \succeq \psi_2$, or $\psi_2 \preceq \psi_1$, if the following two conditions are satisfied:

$$1^\circ \quad \int_\alpha^x \psi_1\,dt \geqslant \int_\alpha^x \psi_2\,dt \text{ for } x \in [\alpha, \beta); \qquad 2^\circ \quad \int_\alpha^\beta \psi_1\,dt = \int_\alpha^\beta \psi_2\,dt.$$

Example 1. (a) If $a=(a_i)_{i=1}^n$ is an arbitrary decreasing sequence of nonnegative numbers, having the sum equal to 1, then

$$(1,0,\dots,0)\succeq(a_1,a_2,\dots,a_n)\succeq\Big(\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}\Big)$$

(for details see Lemma 2 in [12]).

(b) If $\psi_1\colon[\alpha,\beta]\to\mathbf{R}$ is a decreasing integrable function, and

$$\psi_2(x)=\frac{1}{\beta-\alpha}\int_\alpha^x\psi_1\,dt\quad\text{for}\quad x\in[\alpha,\beta],$$

then $\psi_1\succeq\psi_2$.

(c) The sequences $(4,4,1)$ and $(5,2,2)$ are incomparable in the sense of the relation $\succeq$, i.e., none of the two majorizes the other one. △

Lemma 1. [15] *Let $\psi_1,\psi_2\colon[\alpha,\beta]\to\mathbf{R}$ be two integrable functions, such that $\psi_1\succeq\psi_2$, and let $\varphi\colon[\alpha,\beta]\to\mathbf{R}$ be an increasing (integrable) function. Then*

$$\int_\alpha^\beta\varphi\psi_1\,dx\leqslant\int_\alpha^\beta\varphi\psi_2\,dx.$$

Proof. [8] Put $\psi(x)=\psi_1(x)-\psi_2(x)$ and $g(x)=\int_\alpha^x\psi(t)\,dt$. Then, by the hypothesis, $g(x)\geqslant0$ for $x\in[\alpha,\beta]$ and $g(\alpha)=g(\beta)=0$. Using integration by parts in the Stieltjes integral, we get

$$\begin{aligned}\int_\alpha^\beta\varphi(t)\psi(t)\,dt&=\int_\alpha^\beta\varphi(t)\,dg(t)=\varphi(t)g(t)\Big|_\alpha^\beta-\int_\alpha^\beta g(t)\,d\varphi(t)\\&=-\int_\alpha^\beta g(t)\,d\varphi(t)\leqslant0.\quad\blacksquare\end{aligned}$$

Following [10], one can use this lemma to deduce the following classical inequality, which is connected with various names—I. Schur [14], G. H. Hardy, J. I. Littlewood, G. Polya [3], H. Weyl [18], and J. Karamata [9]. Following articles [5], [10] and [13], we shall call it *Karamata's inequality.*

Theorem 1. *Let $a=(a_i)_{i=1}^n$ and $b=(b_i)_{i=1}^n$ be two (finite) decreasing sequences of real numbers from an interval (α,β). If $a\succeq b$, and if $f\colon(\alpha,\beta)\to\mathbf{R}$ is a convex function, then the following inequality holds*

$$\sum_{i=1}^n f(a_i)\geqslant\sum_{i=1}^n f(b_i).\tag{1}$$

Proof. [10] The given function f, being convex, is continuous and it can be represented in the form $f(x)=\int_\alpha^x\varphi\,dt$ for an increasing function φ. Introduce functions $A,B\colon[\alpha,\beta]\to\mathbf{R}$ by

$$A(x)=\sum_{i=1}^n(\min\{x,a_i\}-\alpha),\qquad B(x)=\sum_{i=1}^n(\min\{x,b_i\}-\alpha).$$

It is easy to see that $A(x) \leqslant B(x)$, for $x \in [\alpha, \beta]$ and $A(a_1) = B(a_1)$. Moreover, $A'(x)$ and $B'(x)$ exist everywhere except in a finite set of points. Applying Lemma 1, we conclude that

$$\int_\alpha^{a_1} \varphi \, dA(x) \geqslant \int_\alpha^{a_1} \varphi \, dB(x). \tag{2}$$

But,

$$\begin{aligned} \int_\alpha^{a_1} \varphi \, dA(x) &= n \int_\alpha^{a_n} \varphi \, dx + (n-1) \int_{a_n}^{a_{n-1}} \varphi \, dx + \cdots + \int_{a_2}^{a_1} \varphi \, dx \\ &= f(a_1) + f(a_2) + \cdots + f(a_n), \end{aligned}$$

and the similar relation holds for the integral on the right-hand side of (2). This proves Karamata's inequality. ■

If the function f is strictly convex, it can be easily checked that the equality in (1) is obtained if and only if the sequences (a_i) and (b_i) coincide.

By the standard technique (passing from natural to rational and then to real weights) one can deduce the weighted form of Karamata's inequality (sometimes called Fuchs' inequality, see [2]):

$$\sum_{i=1}^n \lambda_i f(a_i) \geqslant \sum_{i=1}^n \lambda_i f(b_i)$$

if $\lambda_i \in \mathbf{R}^+$ and (a_i) and (b_i) are decreasing sequences, $\sum\limits_{i=1}^k \lambda_i a_i \geqslant \sum\limits_{i=1}^k \lambda_i b_i$ for $k \in \{1, 2, \ldots, n-1\}$ and $\sum\limits_{i=1}^n \lambda_i a_i = \sum\limits_{i=1}^n \lambda_i b_i$. An immediate consequence is the classical Jensen's inequality:

THEOREM 2. [6] *Let $f \colon [\alpha, \beta] \to \mathbf{R}$ be a convex function, let $x_i \in [\alpha, \beta]$, $i \in \{1, 2, \ldots, n\}$ and let $\lambda_i \in [0, 1]$ be such that $\sum\limits_{i=1}^n \lambda_i = 1$. Then*

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leqslant \sum_{i=1}^n \lambda_i f(x_i). \tag{3}$$

If f is strictly convex, equality in (3) holds if and only if $x_1 = x_2 = \cdots = x_n$ or all but one λ_i's are equal to 0.

Lemma 1 can also be used in proving Steffensen's inequality [16]:

THEOREM 3. *Let $f, g \colon [0, a] \to \mathbf{R}$, $0 \leqslant g(x) \leqslant 1$, f be decreasing on $[0, a]$, and let $F(x) = \int_0^x f \, dt$. Then*

$$\int_0^a fg \, dx \leqslant F\left(\int_0^a g \, dx\right).$$

Proof. [10] If we denote $c = \int_0^a g\,dx$, then $0 < c \leqslant a$. Let

$$h(x) = \begin{cases} 1, & x \in [0, c], \\ 0, & x \in (c, a]. \end{cases}$$

Then, it is easy to check that $h \succeq g$, and so, applying Lemma 1, we obtain Steffensen's inequality in the form

$$\int_0^c f\,dx = \int_0^a fh\,dx \geqslant \int_0^a fg\,dx. \quad \blacksquare$$

In order to deduce Chebyshev's inequality, we first prove the following lemma [12].

LEMMA 2. *Let $a = (a_i)_{i=1}^n$, $b = (b_i)_{i=1}^n$ and $c = (c_i)_{i=1}^n$ be three decreasing sequences of real numbers, such that $a \succeq b$. Then the following inequality holds:*

$$\sum_{i=1}^n a_i c_i \geqslant \sum_{i=1}^n b_i c_i.$$

Proof. Denote $A_i = \sum\limits_{j=1}^i a_j$, $B_i = \sum\limits_{j=1}^i b_j$, for $i \in \{1, 2, \dots, n\}$, and put $A_0 = B_0 = 0$. Then we have

$$\begin{aligned} \sum_{i=1}^n a_i c_i - \sum_{i=1}^n b_i c_i &= \sum_{i=1}^n (a_i - b_i) c_i = \sum_{i=1}^n (A_i - A_{i-1} - B_i + B_{i-1}) c_i \\ &= \sum_{i=1}^n (A_i - B_i) c_i - \sum_{i=1}^n (A_{i-1} - B_{i-1}) c_i \\ &= \sum_{i=1}^{n-1} (A_i - B_i) c_i - \sum_{i=0}^{n-1} (A_i - B_i) c_{i+1} \\ &= \sum_{i=1}^{n-1} (A_i - B_i)(c_i - c_{i+1}) \geqslant 0, \end{aligned}$$

being $A_i - B_i \geqslant 0$ and $c_i - c_{i+1} \geqslant 0$ for each $i \in \{1, 2, \dots, n-1\}$. $\blacksquare$

In particular, when a and b are decreasing, $a \succeq b$ and $\sum\limits_{i=1}^n a_i = \sum\limits_{i=1}^n b_i = 1$, the above inequality holds for convex combinations of points c_1, c_2, ... , c_n.

THEOREM 4. *If $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ are decreasing sequences of real numbers, then the following inequality holds:*

$$\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right) \leqslant n \sum_{i=1}^n x_i y_i. \tag{4}$$

Equality in (4) *holds if and only if $x_1 = x_2 = \dots = x_n$ or $y_1 = y_2 = \dots = y_n$.*

Proof. [12] Without loss of generality, we can assume that the terms x_i and y_i of the given sequences are nonnegative (if, for example, some of x_i's or y_i's were negative, we would apply the procedure that follows to the terms $x_i' = x_i - x_n \geqslant 0$ and $y_i' = y_i - y_n \geqslant 0$).

Denote $X = \sum\limits_{i=1}^{n} x_i$, $a_i = \dfrac{x_i}{X}$ and $b_i = \dfrac{1}{n}$. Then the sequences (a_i) i (b_i) are decreasing and, by Example 1, $(a_i) \succeq (b_i)$ holds true. Applying Lemma 2 to the sequences (a_i), (b_i) and taking $c_i = y_i$, we obtain

$$\sum_{i=1}^{n} \frac{x_i}{X} \cdot y_i \geqslant \sum_{i=1}^{n} \frac{1}{n} \cdot y_i,$$

i.e., $n \sum\limits_{i=1}^{n} x_i y_i \geqslant \left(\sum\limits_{i=1}^{n} x_i\right)\left(\sum\limits_{i=1}^{n} y_i\right)$. ■

We state also the following version of Chebyshev's inequality:

Theorem 4'. *Let $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ be decreasing sequences of real numbers, and let π be an arbitrary permutation of the set $\{1, 2, \dots, n\}$. Then the inequality*

$$\sum_{i=1}^{n} x_i y_{\pi(i)} \leqslant \sum_{i=1}^{n} x_i y_i \tag{5}$$

holds. If the sequence $(x_i)_{i=1}^n$ is strictly decreasing, then equality in (5) holds if and only if $y_{\pi(i)} = y_i$ for $i \in \{1, 2, \dots, n\}$.

3. Examples of olympiad problems

Problem 1. [Asian-Pacific Olympiad, 1996] Let a, b, c be the length of sides of a triangle. Prove that the inequality

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leqslant \sqrt{a} + \sqrt{b} + \sqrt{c}$$

holds.

Solution. Suppose, without loss of generality, that $a \geqslant b \geqslant c$ and apply Karamata's inequality to the concave function $f(x) = \sqrt{x}$ and sequences

$$(a+b-c, c+a-b, b+c-a) \succeq (a, b, c). \quad \triangle$$

Problem 2. [13] Prove that the inequality

$$\frac{a_1^3}{a_2} + \frac{a_2^3}{a_3} + \cdots + \frac{a_n^3}{a_1} \geqslant a_1^2 + a_2^2 + \cdots + a_n^2$$

holds for arbitrary positive numbers $a_1, a_2, \dots, a_n$.

Solution. Making the substitution $x_i = \log a_i$, $i \in \{1, 2, \dots, n\}$, we obtain an equivalent inequality

$$e^{3x_1 - x_2} + e^{3x_2 - x_3} + \cdots + e^{3x_n - x_1} \geqslant e^{2x_1} + e^{2x_2} + \cdots + e^{2x_n}.$$

This is obtained by applying Karamata's inequality to the (convex) function $f(x) = e^x$, and the sequences $a = (3x_1 - x_2, 3x_2 - x_3, \dots, 3x_n - x_1)$ and $b = (2x_1, 2x_2, \dots, 2x_n)$. It is enough to prove that these sequences, when arranged to be decreasing, satisfy $a \succeq b$.

Let indices $m_1, \dots, m_n$ and $k_1, \dots, k_n$ be chosen so that

$$\{m_1, \dots, m_n\} = \{k_1, \dots, k_n\} = \{1, \dots, n\},$$

$$3x_{m_1} - x_{m_1+1} \geqslant 3x_{m_2} - x_{m_2+1} \geqslant \cdots \geqslant 3x_{m_n} - x_{m_n+1}, \tag{6}$$

$$2x_{k_1} \geqslant 2x_{k_2} \geqslant \cdots \geqslant 2x_{k_n}. \tag{7}$$

Then

$$3x_{m_1} - x_{m_1+1} \geqslant 3x_{k_1} - x_{k_1+1} \geqslant 2x_{k_1}$$

(the first inequality holds because $3x_{m_1} - x_{m_1+1}$ is, by the choice of numbers m_i, the greatest of numbers of the form $3x_{m_i} - x_{m_i+1}$; the second one follows by the choice of numbers k_i). By similar reasons,

$$(3x_{m_1} - x_{m_1+1}) + (3x_{m_2} - x_{m_2+1}) \geqslant (3x_{k_1} - x_{k_1+1}) + (3x_{k_2} - x_{k_2+1}) \geqslant 2x_{k_1} + 2x_{k_2},$$

and, generally, the sum of the first l terms of sequence (6) is not less than the sum of the first l terms of sequence (7), for $l \in \{1, \dots, n-1\}$. For $l = n$, obviously, the equality is obtained, and so all the conditions for applying Karamata's inequality are fulfilled. $\triangle$

PROBLEM 3. [International Mathematical Olympiad 1999] Let n be a fixed integer, $n \geqslant 2$.

(a) Determine the minimal constant C such that the inequality

$$\sum_{1 \leqslant i < j \leqslant n} x_i x_j (x_i^2 + x_j^2) \leqslant C \left(\sum_{1 \leqslant i \leqslant n} x_i \right)^4$$

is valid for all real numbers $x_1, x_2, \dots, x_n \geqslant 0$.

(b) For the constant C found in (a) determine when the equality is obtained.

Solution. As far as the given inequality is homogeneous, we can assume that $x_1 + x_2 + \cdots + x_n = 1$. In this case the inequality can be written as

$$x_1^3(1 - x_1) + x_2^3(1 - x_2) + \cdots + x_n^3(1 - x_n) \leqslant C.$$

The function $f(x) = x^3(1-x)$ is increasing and convex on the segment $[0, 1/2]$. Let x_1 be the greatest of the given numbers. Then the numbers $x_2, x_3, \dots, x_n$ are not greater than $1/2$. If $x_1 \in [0, 1/2]$ as well, then from $(x_1, x_2, \dots, x_n) \preceq (\frac{1}{2}, \frac{1}{2}, 0, \dots)$ using Theorem 1, we obtain that

$$f(x_1) + f(x_2) \cdots + f(x_n) \leqslant f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + (n-2) f(0) = \frac{1}{8}.$$

If, to the contrary, $x_1 > 1/2$, then it is $1 - x_1 < 1/2$ and we have that $(x_2, x_3, \dots, x_n) \preceq (1 - x_1, 0, \dots, 0)$. Applying Karamata's inequality once more, we obtain that

$$f(x_1) + f(x_2) \cdots + f(x_n) \leqslant f(x_1) + f(1 - x_1) + (n-2) f(0) = f(x_1) + f(1 - x_1).$$

It is easy to prove that the function $g(x) = f(x) + f(1-x)$ has the maximum on the segment $[0, 1]$ equal to $g(1/2) = 1/8$. Thus, in this case also, $f(x_1) + f(x_2) + \dots + f(x_n) \leqslant 1/8$ follows.

Equality holds, e.g., for $x_1 = x_2 = 1/2$, which proves that $C = 1/8$. △

Problem 4. [G. Szegö, [17]] Let $f\colon [0, a_1] \to \mathbf{R}$ be a convex function and $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{2n+1} \geqslant 0$. Then the inequality

$$f(a_1 - a_2 + a_3 - \dots + a_{2n+1}) \leqslant f(a_1) - f(a_2) + f(a_3) - \dots + f(a_{2n+1})$$

holds.

Solution. [5] Put $a = a_1 - a_2 + a_3 - \dots + a_{2n+1}$. Then the given inequality can be rewritten as

$$f(a_1) + f(a_3) + \dots + f(a_{2n+1}) \geqslant f(a) + f(a_2) + \dots + f(a_{2n}).$$

To apply Karamata's inequality it is enough to check that $(a_1, a_3, \dots, a_{2n+1}) \succeq (a_2, \dots, a_{2n}, a)$. But this follows directly because $a_{2k-1} \geqslant a_{2k}$ for all k.

For another proof of Szegö's inequality see [11] or [8]. △

Problem 5. Let a, b and c be the length of sides of a triangle and let s be its semiperimeter. Prove that for a positive integer n, the inequality

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geqslant \left(\frac{2}{3}\right)^{n-2} s^{n-1}$$

holds.

Solution. [7] Without loss of generality, we can suppose that $a \leqslant b \leqslant c$; then also $\dfrac{1}{b+c} \leqslant \dfrac{1}{c+a} \leqslant \dfrac{1}{a+b}$. Chebyshev's inequality, applied to the sequences (a^n, b^n, c^n) and $\left(\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}\right)$, implies that

$$\frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} \geqslant \frac{a^n + b^n + c^n}{3} \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right).$$

By Cauchy-Schwarz inequality, we have

$$2(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \geqslant 9,$$

and by the mean inequality of order n,

$$\frac{a^n + b^n + c^n}{3} \geqslant \left(\frac{a+b+c}{3}\right)^n.$$

Now,

$$\begin{aligned} \frac{a^n}{b+c} + \frac{b^n}{c+a} + \frac{c^n}{a+b} &\geqslant \left(\frac{a+b+c}{3}\right)^n \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \\ &\geqslant \frac{1}{3} \cdot \frac{1}{2} \cdot \left(\frac{2}{3}s\right)^{n-1} \cdot 9 = \left(\frac{2}{3}\right)^{n-2} s^{n-1}. \quad \triangle \end{aligned}$$

Problem 6. [University student's competition, Ostrawa 2002] Let $0 < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ $(n \geqslant 2)$ and

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} = 1.$$

Prove that

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \geqslant (n-1)\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right).$$

Solution. [7] It is enough to prove that

$$\left(\sqrt{x_1} + \frac{1}{\sqrt{x_1}}\right) + \left(\sqrt{x_2} + \frac{1}{\sqrt{x_2}}\right) + \cdots + \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}\right) \geqslant n\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right),$$

or, equivalently,

$$\left(\frac{1+x_1}{\sqrt{x_1}} + \cdots + \frac{1+x_n}{\sqrt{x_n}}\right)\left(\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n}\right) \geqslant n\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right). \tag{8}$$

Take the function $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}} = \dfrac{x+1}{\sqrt{x}}$, $x \in (0, +\infty)$. It is easy to check that f is increasing on $(1, +\infty)$ and that $f(x) = f\left(\dfrac{1}{x}\right)$ for each $x > 0$.

By the assumptions, it follows that only x_1 could be less than 1 and $\dfrac{1}{1+x_2} \leqslant 1 - \dfrac{1}{1+x_1} = \dfrac{x_1}{1+x_1}$. Hence, $x_2 \geqslant \dfrac{1}{x_1}$. Now it is clear that (both in the case $x_1 \geqslant 1$ and in the case $x_1 < 1$)

$$f(x_1) = f\left(\frac{1}{x_1}\right) \leqslant f(x_2) \leqslant \cdots \leqslant f(x_n).$$

This means that the sequence $\left(\dfrac{1+x_k}{\sqrt{x_k}}\right)_{k=1}^{n}$ is increasing. Applying Chebyshev's inequality, we obtain that inequality (8) holds.

We leave it to the reader to check that for $n > 2$ equality holds if and only if $x_1 = x_2 = \cdots = x_n = n - 1$. If $n = 2$, then equality holds for $(x_1, x_2) \in \left\{ \left(\dfrac{1}{t}, t\right) \;\middle|\; t \geqslant 1 \right\}$. $\triangle$

Problem 7. [Serbian Mathematical Olympiad 2007] Let k be a positive integer. Prove that for positive real numbers x, y, z, having the sum equal to 1, the following inequality holds

$$\frac{x^{k+2}}{x^{k+1} + y^k + z^k} + \frac{y^{k+2}}{y^{k+1} + z^k + x^k} + \frac{z^{k+2}}{z^{k+1} + x^k + y^k} \geqslant \frac{1}{7}.$$

Solution. The inequality will be proved by several applications of Chebyshev's inequality. Note first, that the expression on left-hand side is symmetric, so we can suppose that $x \geqslant y \geqslant z$. Let us prove now that

$$x^{k+1} + y^k + z^k \leqslant y^{k+1} + z^k + x^k \leqslant z^{k+1} + x^k + y^k.$$

It is enough to prove the first inequality, which is equivalent to $x^{k+1} + y^k \leqslant y^{k+1} + x^k$, i.e., to $\left(\frac{y}{x}\right)^k \leqslant \frac{1-x}{1-y}$ (numbers x and y are less than 1). Since $y \leqslant x$, it is enough to prove that $\frac{y}{x} \leqslant \frac{1-x}{1-y}$, which is equivalent to $0 \leqslant x - x^2 - y + y^2 = (x-y)(1-x-y) = (x-y)z$, and this inequality obviously holds.

Applying Chebyshev's inequality to the triples

$$(x^{k+2}, y^{k+2}, z^{k+2}) \quad \text{and} \quad \left(\frac{1}{x^{k+1}+y^k+z^k}, \frac{1}{y^{k+1}+z^k+x^k}, \frac{1}{z^{k+1}+x^k+y^k}\right),$$

we obtain that the left-hand side of the given inequality is not less than

$$A = \frac{1}{3}(x^{k+2}+y^{k+2}+z^{k+2})\left(\frac{1}{x^{k+1}+y^k+z^k} + \frac{1}{y^{k+1}+z^k+x^k} + \frac{1}{z^{k+1}+x^k+y^k}\right).$$

A new application of Chebyshev's inequality, this time to the triples (x, y, z) and $(x^{k+1}, y^{k+1}, z^{k+1})$, gives the inequality

$$\begin{aligned} x^{k+2} + y^{k+2} + z^{k+2} &= x^{k+1} \cdot x + y^{k+1} \cdot y + z^{k+1} \cdot z \\ &\geqslant \frac{1}{3}(x^{k+1} + y^{k+1} + z^{k+1})(x+y+z) = \frac{1}{3}(x^{k+1} + y^{k+1} + z^{k+1}). \end{aligned}$$

Thus,

$$\begin{aligned} A \geqslant \frac{1}{3} \cdot \frac{1}{3}&(x^{k+1} + y^{k+1} + z^{k+1}) \times \\ &\times \left(\frac{1}{x^{k+1}+y^k+z^k} + \frac{1}{y^{k+1}+z^k+x^k} + \frac{1}{z^{k+1}+x^k+y^k}\right) =: B. \end{aligned}$$

But, Cauchy-Schwarz inequality implies that

$$\begin{aligned} &\left(\frac{1}{x^{k+1}+y^k+z^k} + \frac{1}{y^{k+1}+z^k+x^k} + \frac{1}{z^{k+1}+x^k+y^k}\right) \times \\ &\qquad \times \left((x^{k+1}+y^k+z^k) + (y^{k+1}+z^k+x^k) + (z^{k+1}+x^k+y^k)\right) \geqslant 9, \end{aligned}$$

and hence $B \geqslant \dfrac{x^{k+1}+y^{k+1}+z^{k+1}}{x^{k+1}+y^{k+1}+z^{k+1}+2(x^k+y^k+z^k)}$. It remains only to prove that

$$3(x^{k+1} + y^{k+1} + z^{k+1}) \geqslant x^k + y^k + z^k.$$

This follows directly by another application of Chebyshev's inequality.

Equality holds if and only if $x = y = z = 1/3$. △

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