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A PROBLEM FROM THE PISA ASSESSMENT RELEVANT TO CALCULUS

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Abstract. In this paper, one of the multiple choice tasks from the PISA test is considered. This particular task deals with the relation between a water tank, in which water is poured at a constant rate, and the given graphs of functions showing how the height of water in that tank changes in time. For the purpose of better understanding the types of tasks that resemble this one, a few similar tasks have been considered. These tasks could be used on various levels of education. We also hope that our paper could exemplify the case of a research project to be assigned to the best students and that the paper can also be useful for all those who teach introductory calculus.

ZDM Subject Classification: I24; *AMS Subject Classification*: 97I20.

Key words and phrases: PISA tests; the height of water surface in a tank.

1. Introduction

PISA[1] tests attract the professional mathematical community precisely because the given problems do not serve only to examine formal knowledge, but also intuitive and logical ingenuity. The degree of applicability of the acquired knowledge in daily life is also tested.

Among the other things, we are motivated to consider this matter because of the existing interest for PISA tests and their possible influence on the teaching of mathematics. We add that M. Marjanović's paper [8] and the communication between him and the first of these authors, have provoked our interest to research as it was also in the case of papers [10, 11].

In this paper, which is continuation of the papers [10, 11], we discuss one of the PISA problems where students are assumed to understand the given functional dependence as well as to select the intervals of convexity and concavity of the drawn graph. This paper includes the proper methodical responses in both cases of elementary and secondary school. Let us also note that P. Eisenmann [5], has considered similar questions related to functional thinking.

2. The analysis of a task from PISA testing

In this section, we will first present one of the tasks from PISA assessment [14]. This task has been an initial point of our interest in this matter. In particular, we have become interested in the students' (at various levels of education) interpretation of convexity and concavity of functions in a practical task. In Section 4, we

[1]Programme for International Student Assessment

will present one of the ways in which students can reach the correct answer to this task, by applying proper reasoning.

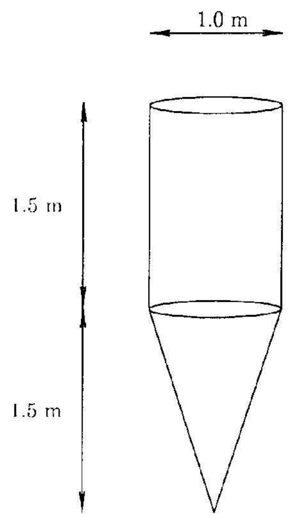


Fig. 1

EXAMPLE 1. *A water tank has a shape and dimensions as depicted in Figure 1. At the beginning, the tank is empty. Then it is filled with water at the rate of one liter per second.*

Which of the graphs in Figure 2 shows how the height of the water surface changes over time?

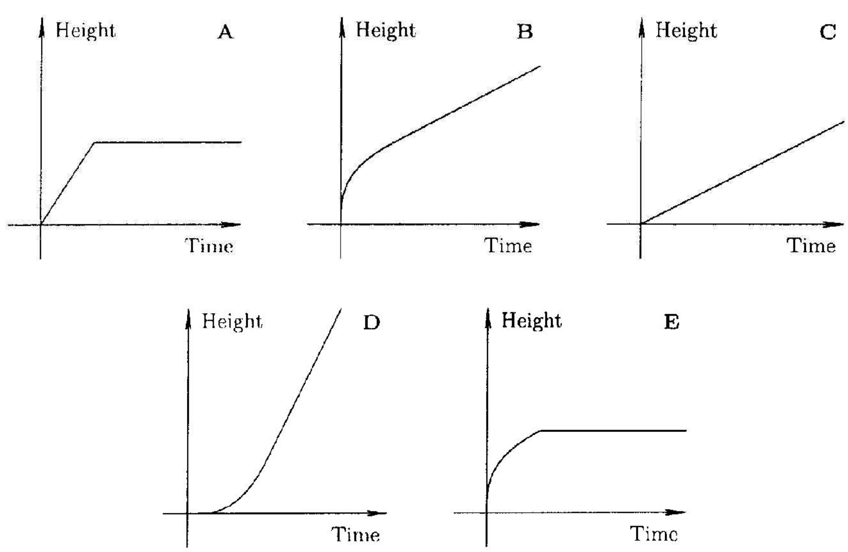


Fig. 2

In a similar task given in PISA test 2003 (which has not been published yet, and therefore cannot be shown here), approximately 20% of students gave the correct answer. Nearly the same percentage of students did not try to solve a problem, whilst the other answers were incorrect. Although isolated from the frame of the entire test, this result shows that multiple-choice tasks are not the best solution for testing in mathematics, especially because the examiners should consider the probability for a student to pick a correct answer randomly.

With the aim of a better understanding of these types of testing, and detecting required mathematical knowledge in mind, the second named author tested 238 students (from the fourth to the eighth grade of elementary school) with the task which is presented in this paper as Example 2. It is important to mention that our students learn about the basic concept of a function in the seventh grade, in the eighth they are graphing linear functions, whilst derivatives and drawing function graphs is a topic in the fourth grade of secondary school.

Example 2. [Questionnaire for the students from 4th to 8th grade of elementary school (an additional task was to explain how they got the answer, especially if they chose it randomly)]

Four cups are shown in Figure 3. One of them is being filled up with coffee at a constant rate. The graph shows how the height of the coffee surface changes over time. Which cup is observed?

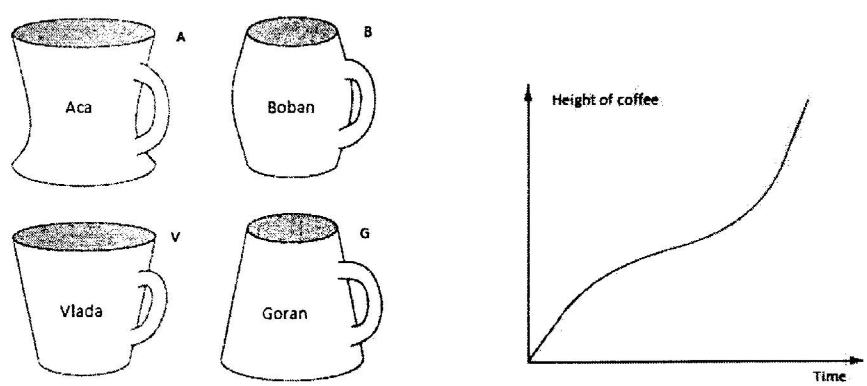


Fig. 3

The research results are presented in Table 1 and Table 2. Here we can see that the percentage of correct answers is not in a proportion to mathematical knowledge at the particular age. Those students who answered that the graph is from the cup B, no matter how old they were, based their conclusion on the shape of the cup, whilst the only difference in answers is in terminology. The most frequent answer was that the graph depends on the shape of the cup. In the majority of cases students used this explanation: “Coffee moves faster where the cup is narrow, and where it is wide, coffee moves slowly”. Some of the students gave both correct answers and then drew a function for the other three cups. The most common incorrect answer was that coffee gains the shape of a cup, so the graph is more oval shaped like cup B. In 70% of cases students said that they picked the answer randomly, or they didn't give any explanations.

Sampling		Type of answer	CUPS							
			Aca		Boban		Vlada		Goran	
Grade	Freq.		fr.	%	fr.	%	fr.	%	fr.	%
Forth	48	I think the answer is correct	0	0.00	6	12.50	0	0.00	0	0.00
		Randomly chosen	11	22.92	0	0.00	7	14.58	1	2.08
		No explanation	9	18.75	2	4.17	8	16.67	4	8.33
Fifth	50	I think the answer is correct	0	0.00	14	28.00	0	0.00	0	0.00
		Randomly chosen	13	26.00	5	10.00	4	8.00	2	4.00
		No explanation	3	6.00	4	8.00	2	4.00	3	6.00
Sixth	46	I think the answer is correct	0	0.00	6	13.04	0	0.00	0	0.00
		Randomly chosen	20	43.48	2	4.35	9	19.57	2	4.35
		No explanation	4	8.70	1	2.17	1	2.17	1	2.17
Seventh	52	I think the answer is correct	0	0.00	8	15.38	0	0.00	0	0.00
		Randomly chosen	12	23.08	4	7.69	8	15.38	4	7.69
		No explanation	12	23.08	2	3.85	0	0.00	2	3.85
Eighth	42	I think the answer is correct	0	0.00	6	14.29	0	0.00	0	0.00
		Randomly chosen	8	19.05	1	2.38	0	0.00	0	0.00
		No explanation	16	38.10	5	11.90	4	9.52	2	4.76
Total	238		108	45.38	66	27.73	43	18.07	21	8.82

Table 1. Results of survey in an elementary school

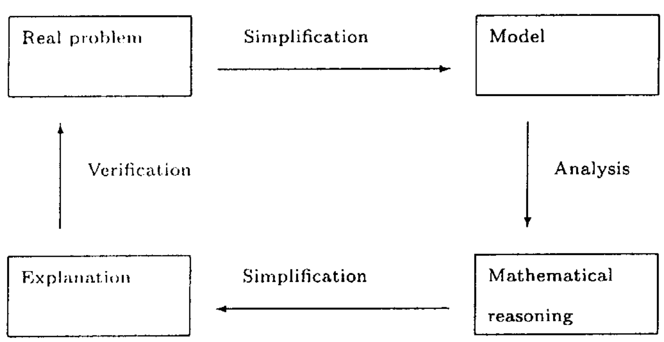
The answers were homogenous and not dependent on the age of students, so it can be concluded that they were not using any knowledge previously gained in mathematics or physics. This means that the testing was reduced to the examination of mathematical intuition and intelligence. The data related to incorrect answers are also interesting. In every grade the percentage of choosing cup A was about 45%, while the sum of the percentages for V and G was 27%. This shows that

Achievements by grades				
Grade	Correct answer	%	Other answers	%
Forth	6	12.5	42	87.5
Fifth	14	28	36	72
Sixth	6	13.04	40	86.96
Seventh	8	15.38	44	84.62
Eighth	6	14.28	36	85.71

Table 2. Student's achievements by grades

the students intuitively eliminated answers V and G, where the cup is only spreading or shrinking. On the other hand, the students who picked cups A and B were giving exactly the same explanations for their choice, so we can conclude that they understood that the graph depends on the shape of the cup. Difficulties appeared when the shape of a cup had to be related to the graph (i.e. in identifying the convexity interval). By further testing some students who gave the correct answer, we came to the conclusion that they did not really understand the relation between the height of coffee in a cup and time. There were only few of them who could identify the right graph when a simpler, cylindrical container was given. Bearing in mind that elementary school students do not have the appropriate mathematical knowledge to consider convexity or concavity of functions, the solution was reduced to a selection by elimination and to the conclusion based on the changes of coffee level in the cup over time. This type of task would be more appropriate for secondary school students in the fourth grade (within a topic—Functions), and to university students.

However, these tasks could be brought closer to elementary school students, with the respect of getting to know the process of modelling real life problems. Schematically it looks like this:



This process can be reviewed on one example, starting with cognitive level of an elementary school student.

3. Methodical approach in Elementary school

In this section we consider two examples. We believe that these examples may be presented to students of elementary school.

Example 3. *In an empty cylindrically shaped tank of a* $1.5\,\mathrm{m}$ *height and a base radius of* $0.5\,\mathrm{m}$*, the water is filled at a constant rate of* $1\,\mathrm{dm}^3/\,\mathrm{s}$*, i.e.* $0.06\,\mathrm{m}^3/\,\mathrm{min}$*. Use a table to show a relation between the volume and height of the water surface in the tank over time. Explain changes in volume and height from one minute to another. Draw a graph that shows how the height of the water surface changes over time.*

Let us use letters H and R to denote height and base radius of the tank, respectively. We also denote by v the rate of water pouring into the tank. Suppose that the process of pouring water into the tank begins at the moment $t = 0$ and ends at the moment $t = T$. The volume of water in the tank at the moment $t \in [0, T]$, which we denote by V, on the one hand equals

$$V = vt, \tag{1}$$

while the other is equal to

$$V = R^2\pi h, \tag{2}$$

where with h the height of the water surface in the tank at the moment t is denoted. Note that the total volume of the tank is equal to $0.375\pi\,\mathrm{m}^3 \approx 1.18\,\mathrm{m}^3$, whilst filling the whole tank takes $6.25\pi\,\mathrm{min} \approx 19.63\,\mathrm{min}$. From (1) and (2) we get

$$h = \frac{v}{R^2\pi}t. \tag{3}$$

From formula (3) we directly get the results given in Table 3. When we display data from the Table 3 on the graph (see the left part of Figure 4), we can make a conclusion about a look of the graph that shows change of height of water surface in the tank over time (see the right part of Figure 4). We can also make a formal conclusion, from formula (3) that the right part of Figure 4 represents how the height of the water surface in the tank changes over time.

Note that the volume and the height of water in the tank both increase over time and that the change of volume and height of water in the tank during the unit time interval is constant.

time (min) t_i	volume (m^3) V_i	change of volume (m^3) $V_{i+1} - V_i$	height (m) h_i	change of height (m) $h_{i+1} - h_i$
1	0.06		0.076	
2	0.12	0.06	0.153	0.077
3	0.18	0.06	0.229	0.077
4	0.24	0.06	0.301	0.077
5	0.30	0.06	0.382	0.077
6	0.36	0.06	0.458	0.077
7	0.42	0.06	0.535	0.077
8	0.48	0.06	0.611	0.077
9	0.54	0.06	0.688	0.077
10	0.60	0.06	0.764	0.077
11	0.66	0.06	0.840	0.077
12	0.72	0.06	0.917	0.077
13	0.78	0.06	0.933	0.077
14	0.84	0.06	1.070	0.077
15	0.90	0.06	1.146	0.077
16	0.96	0.06	1.222	0.077
17	1.02	0.06	1.299	0.077
18	1.08	0.06	1.375	0.077
19	1.14	0.06	1.452	0.077

Table 3

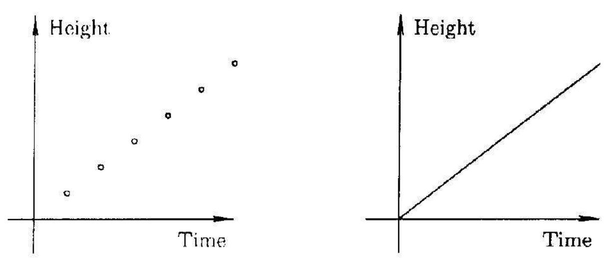


Fig. 4

EXAMPLE 4. *In an empty conically shaped tank of a* 1.5 m *height and a base radius of* 0.5 m*, the water is filled at a constant rate of* $1\,\mathrm{dm}^3/\,\mathrm{s}$*, i.e.* $0.06\,\mathrm{m}^3/\,\mathrm{min}$. *Use a table to show a relation between the volume and height of the water surface in the tank over time. Explain changes in volume and height from one minute to another. Draw a graph that shows how the height of the water surface changes over time.*

Let us use letters H and R to denote height and base radius of the tank, respectively. We also denote by v the rate of pouring water into the tank. Suppose that the process of pouring water into the tank begins at the moment $t = 0$ and ends at the moment $t = T$. Water that is in the tank, at the moment $t \in [0, T]$, fills a conical part of the tank with a height h and base radius r (see Figure 5). Using similarity of triangles we get

$$\frac{r}{h} = \frac{R}{H}.$$

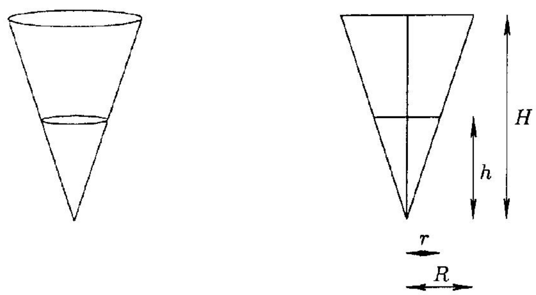


Fig. 5

The volume of water in the tank at the moment $t \in [0, T]$, which we denote by V, on one hand equals

$$V = vt, \tag{4}$$

while the other is equal to

$$V = \frac{1}{3} r^2 \pi h = \frac{1}{3} \frac{R^2}{H^2} \pi h^3 \tag{5}$$

Note that the total volume of the tank is equal to $0.125\pi\,\mathrm{m}^3 \approx 0.39\,\mathrm{m}^3$, whilst filling the whole tank takes $\frac{25\pi}{12}$ min ≈ 6.55 min. From (4) and (5) we get,

$$h = \sqrt[3]{\frac{3H^2 vt}{R^2 \pi}}. \tag{6}$$

and from formula (6) we directly get the results given in the Table 4. When we display data from the Table 4 on a graph (see the left part of Figure 6), we can make a conclusion about a look of the graph that shows change of height of water surface in the tank over time (see the right part of Figure 6). We can also make a formal conclusion, from formula (3) that the right part of Figure 6 represents how the height of the water surface in the tank, changes over time.

Note that the volume and the height of water in the tank both increase over time. Also, the change of volume of water in the tank during the unit time period is a constant value, and a change of height of water in the tank decreases during that period.

time (min) t_i	volume (m^3) V_i	change of volume (m^3) $V_{i+1} - V_i$	height (m) h_i	change of height (m) $h_{i+1} - h_i$
1	0.06		0.802	
2	0.12	0.06	1.010	0.208
3	0.18	0.06	1.156	0.146
4	0.24	0.06	1.273	0.116
5	0.30	0.06	1.371	0.098
6	0.36	0.06	1.457	0.086

Table 4

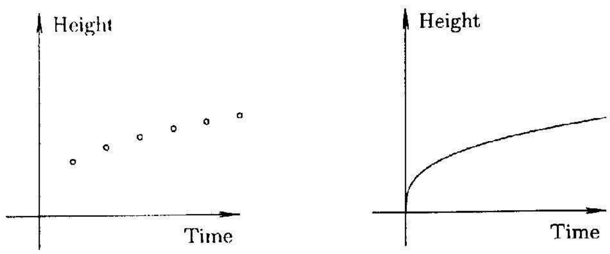


Fig. 6

4. A solution of Example 1

Let us now consider one of the ways that the student reaches the exact solution in Example 1.

It is intuitively clear that while the water poured into the cone part of the tank, the height of the water surface is increasing but more and more slowly (because the water spreads horizontally). Therefore answers A and C are certainly not accurate. It is also intuitively clear that when the poured water starts to occupy a part of the tank which is cylindrically shaped, the height of water starts to change uniformly over time, i.e. during the equal time intervals there are equal changes (linear dependence). It follows that answer E is certainly not accurate. Now we need to determine which of the answers B and D is the correct one. In order to do that, let us observe two moments (t_1 and $t_2 = 2t_1$) and the potential values of water heights in these moments ($h(t_1)$ and $h(t_2)$), as shown in Figure 7 and Figure 8.

Let us note that in the case B we have $h(t_2) - h(t_1) < h(t_1) - h(0)$, and in the case D we have $h(t_2) - h(t_1) > h(t_1) - h(0)$. But as we have already concluded—until the water pours in the conical part of the tank the water height increases, but increasingly slowly, so it follows that the correct answer is B.

We leave it to the reader to explain in similar way the solution in the Example 2.

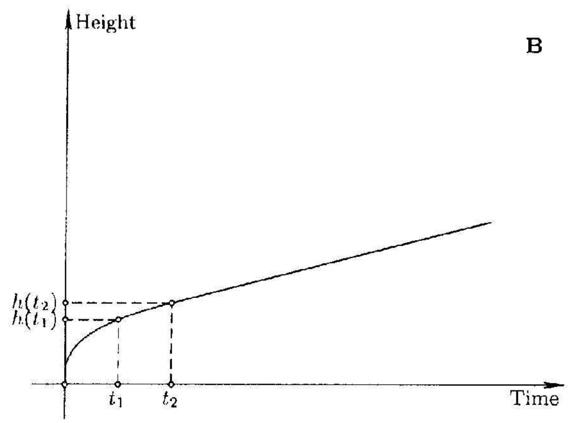


Fig. 7

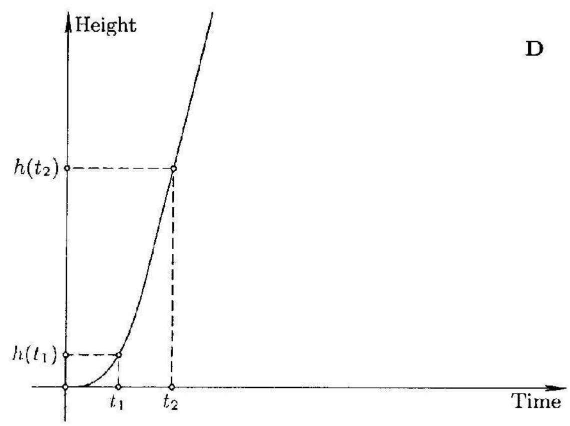


Fig. 8

5. Methodical approach in secondary school

If we leave labels as in Example 3 and Example 4 and if we denote by $h(t)$ the height of water surface contained in the tank at the moment $t \in [0, T]$, we will have a function $h\colon [0, T] \to \mathbb{R}$. Fix a moment $t_0 \in [0, T]$ and denote by Δt the increment of time. Furthermore, quotient

$$u_{avg} = u_{avg}(t_0, \Delta t) = \frac{h(t_0 + \Delta t) - h(t_0)}{\Delta t}$$

represents the average rate of rising of the height of water surface in the tank during the time interval $(t_0, t_0 + \Delta t)$ (or $(t_0 + \Delta t, t_0)$). If Δt approaches zero then the quotient u_{avg} approaches the rate of rising of the height of water surface at the moment t_0. More precisely, here the limit occurs, and for rate to be defined it is necessary that this limit exists. It follows that the rate of rising of the height of water surface in that moment t_0, i.e. $u(t_0)$, is equal to $h'(t_0)$.

Students know from Calculus classes that, if the first derivative of function h exists on some interval and if on that interval holds $h' \leqslant 0$ ($h' \geqslant 0$), then the function h decreases (increases) on this interval.

Using this knowledge, students can conclude that in Example 3 it is valid that

$$h'(t) = \frac{v}{R^2\pi} > 0,$$

and in Example 4 it is valid that

$$h'(t) = \sqrt[3]{\frac{3H^2v}{R^2\pi}}\frac{1}{3}\frac{1}{\sqrt[3]{t^2}} > 0.$$

So in both examples, we get that the function h increases, which means that the height of water surface in the tank rises. Thus, we only stated what was intuitively clear from the nature of this problem. Also because of the nature of the problem, h' exists.

Consider now the quotient

$$a_{avg} = a_{avg}(t_0, \Delta t) = \frac{h'(t_0 + \Delta t) - h'(t_0)}{\Delta t}$$

which represents the average acceleration of rising of the height of water surface, in the tank during the time interval $(t_0, t_0 + \Delta t)$ (or $(t_0 + \Delta t, t_0)$). If Δt approaches zero then the quotient a_{avg} approaches the acceleration of rising of the height of water surface at the moment t_0. More precisely, here the limit occurs, and for acceleration to be defined it is necessary that this limit exists. It then follows that the acceleration of rising of the height of water surface in that moment t_0, the $a(t_0)$, is equal to $h''(t_0)$.

Students know from Calculus classes that, if the second derivative of function h exists on some interval and if on that interval holds $h'' \leqslant 0$ ($h'' \geqslant 0$), then the function h is concave (convex) on this interval.

Using this knowledge, students can conclude that in Example 3 it is valid that

$$h''(t) = 0,$$

and in Example 4

$$h''(t) = \sqrt[3]{\frac{3H^2v}{R^2\pi}}\frac{(-2)}{9}\frac{1}{\sqrt[3]{t^5}} < 0$$

is valid. Thus, the function in Example 3 which represents how the height of the water surface in the tank dependence of time, is at the same time convex and concave, which means that the acceleration of rising of the height of water surface in the tank is equal to 0, i.e. the rate of rising of the height of water surface in the tank is constant. In Example 4 the situation is a bit different. The function which represents the dependence between the height of water surface in the tank and time is concave, whilst the acceleration of rising of the height of water surface in the tank is negative, i.e. the rate of rising of the height of water surface in the tank is decreasing.

After these considerations students could be given the next task.

EXAMPLE 5. *In an empty conically shaped tank of a* 1.5 m *height and a base radius of* 0.5 m*, the water is filled at a constant rate of* $1\,\mathrm{dm}^3/\,\mathrm{s}$*, i.e.* $0.06\,\mathrm{m}^3/\,\mathrm{min}$. *After what period of time will the height of the water surface in the tank reach* 1.2 m*, and what is the rate of rising of the height of water surface at that moment? Is the rate change of the height rising of the water surface in the tank at that moment positive or negative?*

Let us use labels from the previous example, except that we will now, more precisely, mark with $V(t)$ the volume of water that is poured in the tank up to the moment t, and with $h(t)$ the height of water surface at the moment t, and with $r(t)$ the radius of the cone base, which the water in the tank formed at the moment t. Based on the similarity of triangles we get

$$\frac{r(t)}{h(t)} = \frac{R}{H} = \frac{1/2}{3/2} = \frac{1}{3}$$

and, knowing that

$$V(t) = \frac{1}{3}\pi(r(t))^2 h(t),$$

we get

$$V(t) = \frac{1}{27}\pi(h(t))^3. \tag{7}$$

On the other hand $V(t) = vt$, so, at the moment when the water surface reached a height $h(t)$ it is true that

$$t = \frac{\pi}{27v}(h(t))^3.$$

By putting into the table values given in the task, we will get the time that water surface takes to reach the height of 1.2 m,

$$t = \frac{16\pi}{15}\,\mathrm{min} \approx 3.35\,\mathrm{min}\,.$$

To calculate the rate of rising of the height of water surface at the moment when it reaches 1.2 m, we should differentiate both sides of the equality (7) in respect to t. This gives us

$$V'(t) = \frac{\pi}{9}(h(t))^2 h'(t),$$

i.e.

$$v = \frac{\pi}{9}(h(t))^2 h'(t), \tag{8}$$

wherefrom we find that the rate of rising of the height of water surface in the tank at the moment t is equal to

$$h'(t) = \frac{9v}{\pi(h(t))^2}.$$

By substituting the specific values in the last formula we get that the rate of rising of the height of water surface in the tank at the moment when that height is 1.2 m is equal to $\frac{3}{8\pi}$ m/ min ≈ 0.12 m/ min. On Figure 9 it represents the gradient of a tangent of graph of function h at the point $(16\pi/15, 1.2)$.

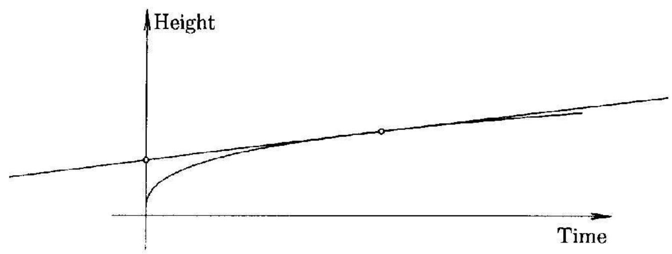


Fig. 9

To calculate the acceleration of rising of the height of water surface at the moment when it reaches 1.2 m, we should differentiate both sides of the equality (8) in respect to t. This gives us

$$0 = \frac{\pi}{9}(2h(t)(h'(t))^2 + (h(t))^2 h''(t)),$$

from which we find that the acceleration of rising of the height of water surface in the tank at the moment t equals

$$h''(t) = -\frac{2\pi(h'(t))^2}{9h(t)}.$$

Substituting the specific values in the last formula we get that the acceleration of rising of the height of water surface in the tank at the moment when that height reaches 1.2 m is equal to $-\frac{5}{72}$ m/ min$^2 \approx -0.07$ m/ min^2. Thus, the rate of change of rate of the height of water surface in the tank at the moment $t = \frac{16\pi}{15}$ is negative.

6. Conclusion

The PISA mathematics literacy test asks students to apply their mathematical knowledge to solve problems set in various real-world contexts. To solve the problems students must utilize a number of mathematical concepts as well as a broad range of mathematical content knowledge. It seems if we desire better results in PISA tests, we need to improve educational policies and outcomes. Tests in our country, on the other hand, measure more traditional classroom content (such as an understanding of fractions and decimals and the relationship between them).

In many tasks in PISA tests, students are not expected to mathematically prove the solution, and only the results were evaluated, without the explanation and the way that they were gained. In that way we cannot get a clear gauge of a student's mathematical knowledge, and in that sense it would be wrong to characterize this assessment as a complete picture of mathematical literacy of one nation. The truth is, of course, somewhere in the middle. It is evident that our students are missing applications of mathematics, so for that aim we could analyze tasks from assessment and gradually implement them in curricula so that they become closer to students, both with formal mathematics, which should not be neglected.

It is possible to conclude that many useful lessons can be derived from the PISA testing, and they can improve our approach in teaching of mathematics. Mathematics lessons can be animated with real life problems, especially at the elementary school level.

In this way, they will be brought closer to more students and will motivate them when they realize the purpose of some of the calculations.

It is interesting that, at the beginning of the 21st century, we are considering the applicability of mathematics, whilst it evolved as a science from humanity's need to solve practical problems. It has abstracted, detached from us, but we will always need it to explain every natural and social phenomenon and even art. Teachers have the obligation to expose to their students both abstract and practical problems and issues.

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