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WHAT ARE COMPLETELY INTEGRABLE HAMILTON SYSTEMS

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Abstract. This paper is aimed to undergraduate students of mathematics and mechanics to draw their attention to a modern and exciting field of mathematics with applications to mechanics and astronomy. We cared to keep our exposition not to go beyond the supposed knowledge of these students.

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1. Introduction

Solving concrete mechanical and astronomical problems was one of the main mathematical tasks until the beginning of XX century (among others, see the work of Euler, Lagrange, Hamilton, Abel, Jacobi, Kovalevskaya, Chaplygin, Poincare). The majority of problems are unsolvable. Therefore, finding of solvable systems and their analysis is of a great importance.

In the second half of XX century, there was a breakthrough in the research that gave a basis of a modern theory of integrable systems. Many great mathematicians such as P. Laks, S. Novikov, V. Arnold, J. Mozer, B. Dubrovin, V. Kozlov contributed to the development of the theory, which connects the beauty of classical mechanics and differential equations with algebraic, symplectic and differential geometry, theory of Lie groups and algebras (e.g., see [1–4] and references therein).

The aim of this article is to present the basic concepts of the theory of integrable systems to readers with a minimal prior knowledge in the graduate mathematics, so we shall not use a notion of a manifold, symplectic structure, etc.

The most of mechanical and physical systems are modelled by *Hamiltonian equations*. We shall introduce Hamiltonian systems with one of the easiest problems, the system of n independent harmonic oscillators (Section 2). The corresponding equations, as linear, are easily solvable. The solutions are expressed as a trigonometric functions of time and the dynamics is linearized on invariant tori.

It turns out that much more complex Hamiltonian problems, if they have a sufficient number of integrals of motion (conservation laws), have qualitatively a similar behavior. It is a content of the *Liouville-Arnold theorem* which we state without a proof in Section 4. In Section 3, we define the *canonical Poisson brackets*. This is a very important geometric structure that makes possible to study Hamiltonian systems using analytical, algebraical and geometrical methods.

Finally in Section 5, the system of two identical harmonic oscillators is considered. It is shown that this system provides a natural description of an important geometrical object, the *Hopf fibration*.

2. Hamiltonian equations and harmonic oscillators

2.1. We have learned already in a high school about the *harmonic oscillator*, describing the motion of a material point under the influence of a force $-aq$, $a > 0$, where q is a shift from the equilibrium position $q = 0$ (for example, a motion of a material point related to the elastic spring with the elastic coefficient a). The system is described by the equation:

$$m\ddot{q} = -aq, \tag{1}$$

where m is the mass of a material point.

By introducing of a new variable $p = m\dot{q}$ (the moment) and the function $h(q,p) = \frac{1}{2m}p^2 + \frac{a}{2}q^2$ (the sum of kinetic and potential energy of the system), the second order equation (1) takes the form of the system of two first order equations in the space $\mathbb{R}^2(q,p)$:

$$\dot{q} = \frac{p}{m} = \frac{\partial h}{\partial p}, \qquad \dot{p} = -aq = -\frac{\partial h}{\partial q}. \tag{2}$$

Fig. 1. The system of n independent harmonic oscillators

Now, let us consider the system of n independent harmonic oscillators (e.g., the system of n elastic springs). As above, we can write equations of motion in the space $\mathbb{R}^{2n}(q,p) = \mathbb{R}^{2n}(q_1,\dots,q_n,p_1,\dots,p_n)$:

$$\dot{q}_i = \frac{p_i}{m_i} = \frac{\partial h}{\partial p_i}, \qquad \dot{p}_i = -a_i q_i = -\frac{\partial h}{\partial q_i}, \qquad i = 1,\dots,n \tag{3}$$

$(m_i, a_i > 0)$, where $h(q,p) = \sum_{i=1}^n (p_i^2/m_i + a_i q_i^2)/2$ represents the total mechanical energy of the system.

2.2. The system of equations

$$\dot{q}_i = \frac{\partial h}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial h}{\partial q_i}, \qquad i = 1,\dots,n \tag{4}$$

defined in a region U of the space $\mathbb{R}^{2n}(q_1,\dots,q_n,p_1,\dots,p_n)$ is called a *Hamiltonian system*. The function h is called *Hamiltonian*, while the region U is called the *phase space* of the system. By introducing the variable $x = (q_1,\dots,q_n,p_1,\dots,p_n)$ and a *Hamiltonian vector field*

$$X_h = \left(\frac{\partial h}{\partial p_1},\dots,\frac{\partial h}{\partial p_n},-\frac{\partial h}{\partial q_1},\dots,-\frac{\partial h}{\partial q_n}\right), \tag{5}$$

system (4) reads simply $\dot{x} = X_h(x)$.

2.3. Let us recall some basic notions from the theory of ordinary differential equations. Consider the equations

$$\dot{x} = X(x), \qquad x = (x_1, \dots, x_d) \in U \subset \mathbb{R}^n, \tag{6}$$

where $X = (X_1(x), \dots, X_d(x))$ is a smooth vector field on U. By the theorem on existence and uniqueness of solutions, through each point $x^0 \in U$ and a moment of time t_0, there is a unique solution $x(t)$, such that $x(t_0) = x^0$.

Under *integrability* of the system (6), in broadest context, we mean the finding of the solutions $x(t)$ for a general initial condition $x(t_0) = x^0$.

The most important objects in solving the equation are (first) integrals of the motion. A function F is a *first integral* of equation (6) if it is constant along solutions: $f(x(t)) = \text{const}$. It is clear that f is an integral if and only if the derivative of f in the direction vector field X identically equals to zero:

$$X(f) = \sum_{i=1}^{d} X_i \frac{\partial f}{\partial x_i} \equiv 0.$$

Geometrically, this means that if at some point of time t_0 a trajectory $x(t)$ belongs to $M_c = \{f(x) = c\}$, then it lies within M_c for every t. If the differential df is different from zero on M_c, then M_c is a smooth hyper-surface. The vectors $X(x)$ are tangent to M_c for every $x \in M_c$.

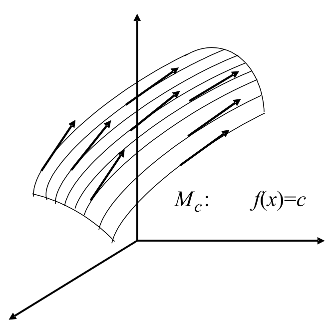

Fig. 2. Invariant surface

Now, if equation (6) have l functionally independent integrals, the problem reduces (for general values of the parameters $c = (c_1, \dots, c_l)$) to solving the restricted system on $(d - l)$-dimensional smooth invariant surface

$$M_c: \; f_1(x) = c_1, \dots, f_r(x) = c_l.$$

It is clear that the existence of $d - 1$ integrals implies solvability of (6).

2.4. As we shall see soon, in the framework of Hamiltonian mechanics, for solving equation (4) there should be "only" n integrals. Even more, the phase space of a completely integrable Hamiltonian system has a very fine structure: it is, almost everywhere, decomposed on invariant n-dimensional tori in which the dynamics is linearized.

We mentioned that we shall skip using of manifolds. However, there is one exception. The main geometrical object in the theory of integrable systems is a torus, and we hope readers will become familiar with its use.

A *torus* (n-dimensional) $\mathbf{T}^n$ is a direct product of n circles:

$$\mathbf{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n}.$$

Let $(\varphi_1, \dots, \varphi_n)$ be linear coordinates of the vector space $\mathbb{R}^n$. Then we have a natural mapping $\pi\colon \mathbb{R}^n \to \mathbf{T}^n$, $\pi(\varphi) = \varphi \mod 2\pi$, where $\varphi_i \pmod{2\pi}$ is the angular variable on the i-th circle. The mapping π is 2π-periodic in each variable. Thus, we can represent torus as an n-dimensional cube $[0, 2\pi]^n \subset \mathbb{R}^n(\varphi_1, \dots, \varphi_n)$ with identified opposite sides of the cube:

$$(\varphi_1, \dots, \varphi_{i-1}, 0, \varphi_{i+1}, \dots, \varphi_n) \equiv (\varphi_1, \dots, \varphi_{i-1}, 1, \varphi_{i+1}, \dots, \varphi_n), \quad i = 1, \dots, n.$$

A linear (quasi-periodic, conditionally periodic) motion on $\mathbf{T}^n$ is a projection of a line with the mapping π:

$$\varphi_i(t) = \varphi_i^0 + \omega_i t, \quad i = 1, \dots, n. \tag{7}$$

The numbers $\omega_1, \dots, \omega_n$ are called *frequencies*. The trajectory (7) is called a *winding* of the torus, as well.

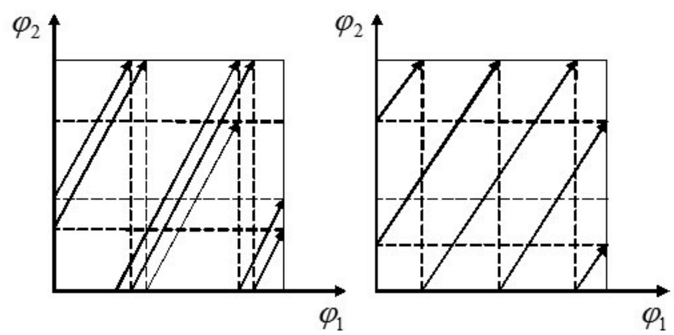

Fig. 3. Linear trajectories on a two-dimensional torus for $\omega_2/\omega_1 \notin \mathbb{Q}$, $\omega_2/\omega_1 \approx 3/2$ (non-periodic) and $\omega_2/\omega_1 = 3/2$ (periodic), respectively.

2.5. One can solve (2) easily. First note that the energy h is an integral:

$$\frac{d}{dt} h(q, p) = \frac{1}{2m} 2p\dot{p} + \frac{a}{2} 2q\dot{q} = \frac{1}{m} apq - \frac{1}{m} apq = 0.$$

The equation $h(q,p) = E$, for $E > 0$ defines the ellipse M_E in $\mathbb{R}^2(q,p)$. Let us introduce an angular variable φ (mod 2π) on M_E (see Fig. 2):

$$q = \sqrt{2E/a}\cos(\varphi), \quad p = -\sqrt{2Em}\sin(\varphi). \tag{8}$$

From the equations of motion (2) we get $\dot{\varphi} = \sqrt{a/m}$. Hence, by integration, we obtain the well known expressions

$$q(t) = \sqrt{2E/a}\cos\left(\sqrt{a/m}t + \varphi^0\right), \quad p(t) = -\sqrt{2Em}\sin\left(\sqrt{a/m}t + \varphi^0\right).$$

The constants $E \geq 0$ (energy) and $\varphi^0 \in [0, 2\pi)$ are determined from the initial condition $(q(t_0), p(t_0))$. The initial condition $(q(t_0), p(t_0)) = (0,0)$ gives the *equilibrium position*, namely the solution is $(q(t), p(t)) \equiv (0,0)$.

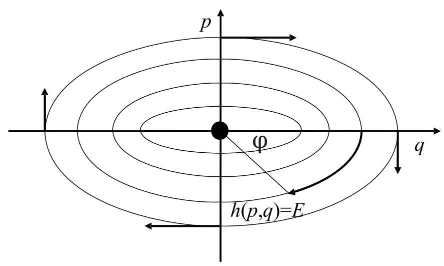

Fig. 4. The phase space of a harmonic oscillator

2.6. System (3) has n independent integrals:

$$f_i(q,p) = f_i(q_i,p_i) = \frac{1}{2m_i}p_i^2 + \frac{a_i}{2}q_i^2, \qquad i = 1,\dots,n \tag{9}$$

(the energy of i-th harmonic oscillator) and the phase space is foliated on invariant surfaces

$$M_c = \{(q_1,\dots,q_n,p_1,\dots,p_n) \in \mathbb{R}^{2n} \mid f_1 = c_1,\dots,f_n = c_n\}. \tag{10}$$

In general, if all constants c_i are greater then zero, M_c is a product of ellipses: $M_c = \mathcal{E}_{c_1} \times \cdots \times \mathcal{E}_{c_n}$, $\mathcal{E}_{c_i} = \{(q_i,p_i) \mid f_i = c_i\}$ and represents an n-dimensional torus $\mathbf{T}^n$. If one of the constants c_i is zero, the invariant manifold M_c is less dimensional torus.

The energy of the system, for the motions on M_c, is given by $E = c_1 + \cdots + c_n$. On the other side, the iso-energetic hyper-surface $h = E > 0$ is a $(2n-1)$-sphere S^{2n-1}. It is the union of all possible tori M_c with $E = c_1 + \cdots + c_n$ (we shall use this observation in the last section).

Equations (3) on M_c, in angular variables $(\varphi_1,\dots,\varphi_n)$ defined by

$$q_i = \sqrt{2c_i/a_i}\cos(\varphi_i), \quad p_i = -\sqrt{2c_i m_i}\sin(\varphi_i), \quad i = 1,\dots,n,$$

are quasi-periodic:

$$\dot{\varphi}_i = \omega_i = \sqrt{a_i/m_i}, \qquad i = 1, \dots, n. \tag{11}$$

Therefore, a general solution of the system is:

(12)

$$q_i(t) = \sqrt{2c_i/a_i}\cos\left(\omega_i t + \varphi_i^0\right), \; p_i(t) = -\sqrt{2c_i m_i}\sin\left(\omega_i t + \varphi_i^0\right), \; i = 1, \dots, n,$$

where constants $c_i \geq 0$ and $\varphi_i^0 \in [0, 2\pi)$ are determined from the initial conditions. In particular, the equilibrium position of the system is $(q_1, \dots, q_n, p_1, \dots, p_n)|_{t_0} = 0$ (zero-energy level).

3. Poisson brackets

3.1. Let us consider a Hamiltonain system (4) defined in a region U of the space $\mathbb{R}^{2n}(q,p)$ and let $f\colon U \to \mathbb{R}$ be an arbitrary smooth function. A direct computation shows that the derivative of f along a trajectory $(q(t), p(t))$ of (4) is

$$\frac{d}{dt}(f(q(t),p(t))) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}\right)\Big|_{(q,p)=(q(t),p(t))}. \tag{13}$$

Therefore, in order that f is the integral of a motion, the expression on the right-hand side of (13) must be zero. From (13) we can see that the Hamiltonian h itself is the integral of a motion (the conservation of energy for the natural mechanical systems).

Definition 1. A *canonical Poisson bracket* within a region $U \subset \mathbb{R}^{2n}(q,p)$ is the mapping $\{\cdot,\cdot\}\colon C^\infty(U) \times C^\infty(U) \to C^\infty(U)$, defined via:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{14}$$

In particular, for the coordinate functions q_i, p_j we have *canonical relations*:

$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}, \qquad i, j = 1, \dots, n.$$

As a reformulation of the definition we get the following algebraic characterization of the first integrals:

Corollary 1. *A function f is the first integral of the Hamiltonian system (4) if and only if it commutes with h: $\{h, f\} = 0$.*

Proposition 1. *The Poisson bracket is a bilinear, skew-symmetric mapping (15) that satisfies Leibniz's rule (16) and the Jacobi identity (17):*

(15)

$$\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}, \qquad \{f,g\} = -\{g,f\},$$

(16)

$$\{fg, h\} = f\{g,h\} + g\{f,h\},$$

(17)

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0, \quad \alpha, \beta \in \mathbb{R}, \quad f, g, h \in C^\infty(U).$$

Exercise 1. Prove Proposition 1.

Corollary 2. *The Poisson bracket of two first integrals f_1, f_2 of a system with Hamiltonian h is again the first integral.*

Proof. By the Jacobi identity

$$\{\{f_1, f_2\}, h\} = \{f_1, \{f_2, h\}\} + \{f_2, \{h, f_1\}\} = 0 + 0,$$

as was to be shown. ■

Remark 1. In this way, by knowing two first integrals, we can find a third, fourth, etc, by a simple computation. Of course, not all the integrals we get will be essentially new, since there cannot be more than $2n$ independent functions on $\mathbb{R}^{2n}$. On the other side, if first integrals f_1 and f_2 commute, we do not get a new integral, but we have the following geometrical property: the functions h, f_1, f_2 are all constant along the Hamiltonian vector fields X_h, X_{f_1} and X_{f_2}, that is, the vector fields X_h, X_{f_1} and X_{f_2} are all tangent to the invariant surfaces $h = c_0$, $f_1 = c_1$, $f_2 = c_2$.

3.2. As it is usual in mathematics, having a class of objects with a certain structure, we consider appropriate mappings that preserve the given structure. Let U and V be regions within $\mathbb{R}^{2n}(q, p)$ endowed with the canonical Poisson brackets. A diffeomorphism $\Psi\colon U \to V$[1] is a *Poisson isomorphism (canonical transformation)* if it preserves the Poisson bracket:

$$\{f \circ \Psi, g \circ \Psi\} = \{f, g\} \circ \Psi, \quad f, g \in C^\infty(V).$$

If $\Psi\colon U \to V$ is a canonical transformation, so it is $\Psi^{-1} : V \to U$. Also, a composition of two canonical transformations is a canonical transformation.

Proposition 2. *A diffeomorphism $\Psi\colon U \to V$ given by the functions $Q_i = Q_i(q, p)$, $P_i = P_i(q, p)$, $i = 1, \dots, n$ is a canonical transformation if and only if the functions Q_i, P_j satisfy canonical relations:*

$$\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij}, \quad i, j = 1, \dots, n.$$

Corollary 3. *Let $\Psi\colon U \to V$ be a canonical transformation, given by functions $Q_i = Q_i(q, p)$, $P_i = P_i(q, p)$, $i = 1, \dots, n$. The Hamiltonian equation in new coordinates (Q, P) have the same form:*

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H}{\partial Q_i}, \qquad i = 1, \dots, n,$$

where $H(Q, P) = h(q(Q, P), p(Q, P))$. The functions $q_i = q_i(Q, P)$, $p_i = p_i(Q, P)$ represent the inverse canonical transformation Ψ^{-1}.

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[1] Recall that a differentiable mapping $\Psi\colon U \to V$ $(U, V \subset \mathbb{R}^d)$

$$y_i = y_i(x_1, \dots, x_d), \qquad i = 1, \dots, d$$

is a *diffeomorphism* if it is a bijection and the Jacobian $\det(\partial y_i/\partial x_j)$ is different from zero on U. One can consider $(y_1, \dots, y_d)$ as a new coordinate system within U as well.

4. Complete integrability

4.1. Notice, the integrals (9) of the system of n independent harmonic oscillators commute between themself.

Definition 2. A Hamiltonian system (4) is called *completely integrable* if there are n Poisson-commuting smooth integrals $f_1, \dots, f_n$:

$$\{f_i, f_j\} = 0, \qquad i, j = 1, \dots, n,$$

whose differentials are independent in an open dense subset of U.

The global regularity of dynamics in the case of complete integrability follows from the following classical Liouville-Arnold theorem:

Theorem 1. (Liouville-Arnold [1]) *Suppose that the equations* (4) *have* n *Poisson-commuting smooth integrals* $f_1, \dots, f_n$ *and let*

$$M_c = \{f_1 = c_1, \dots, f_n = c_n\}$$

be a common invariant level set.

(i) *If* M_c *is regular (the differentials of* $f_1, \dots, f_n$ *are independent on it), compact and connected, then it is diffeomorphic to the* n*-dimensional torus.*

(ii) *In a neighborhood of* M_c *there exist canonical variables* $I, \varphi \mod 2\pi$*, called action-angle variables:*

$$\{\varphi_i, I_j\} = \delta_{ij}, \quad \{\varphi_i, \varphi_j\} = 0, \quad \{I_i, I_j\} = 0, \quad i, j = 1, \dots, n.$$

such that the level sets of the actions $I_1, \dots, I_n$ *are invariant tori and* $h = h(I_1, \dots, I_n)$*. Thus, the Hamiltonian equations are linearized:*

$$\dot\varphi_1 = \omega_1(I) = \frac{\partial h}{\partial I_1}, \dots, \dot\varphi_n = \omega_r(I) = \frac{\partial h}{\partial I_n}, \quad \dot I_1 = 0, \dots, \dot I_n = 0.$$

Remark 2. Let us comment on the fact that the invariant manifold M_c is a torus. As in Remark 1, we get that the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_n}$ are tangent to M_c. Further, from the independency of differentials $df_1, \dots, df_n$ on M_c and the definition (5), the vector fields $X_{f_1}, \dots, X_{f_n}$ are also independent at M_c. For the dimensional reason, they span the tangent spaces $T_x M_x$ for all $x \in M_c$. An important property of the Poisson bracket is that the Poisson-commutativity $\{f_i, f_j\} = 0$ implies the commutativity of vector fields $[X_{f_i}, X_{f_j}] = 0$ (see [1]). Since M_c is compact, vector fields X_{f_i} are complete. Now, the complete commuting vector fields $X_{f_1}, \dots, X_{f_n}$ determine a locally free, transitive action of an Abelian group $\mathbb{R}^n$ on M_c with a discrete isotropy group Γ.[2] Therefore M_c is diffeomorphic to $\mathbb{R}^n/\Gamma$. Again, since M_c is compact, the rank of Γ equals n ($\Gamma \cong \mathbb{Z}^n$) and $\mathbb{R}^n/\Gamma$ is

[2] The readers not familiar with the above notions can skip this remark.

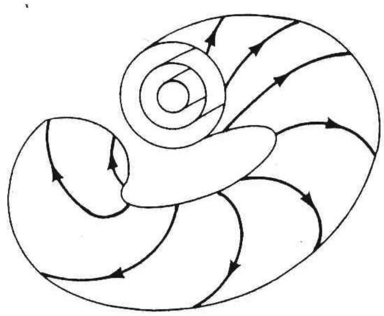

Fig. 5. Fibration of the phase space on invariant tori

a torus. The Hamiltonian vector field X_h is a linear combination of vector fields X_{f_i}. Thus, the Hamiltonian equations medelled on a torus $\mathbb{R}^n/\Gamma$ are linear.

We see that the study of integrability of the Hamiltonian systems consist of two basic nontrivial problems: the finding of enough commuting first integrals and the explicit description of the solutions (e.g., see [2, 3, 4]).

Completely integrable Hamiltonian systems are very rare: if we choose a "random" Hamiltonian, the associated Hamilton equations will be non-integrable. However, the Kolmogorov-Arnold-Mozer (KAM) theorem states that if we have a completely integrable system with Hamiltonian $h_0(x)$ and perform the perturbation $h(x) = h_0(x) + \epsilon h_1(x)$, then the system with Hamiltonian $h(x)$ keeps certain properties of the original integrable system. Namely, some invariant tori "survive" the perturbation (see [1]).

For example, consider our solar system and imagine that it consists only of the Sun which is stationary and the eight planets moving in the same plane. If we neglect the interaction among the planets, and take into account only the gravitational attraction of the sun and each planet individually, Kepler's laws give us that the bounded motion of planets, take place in ellipses where Sun is in one of the focuses. It is a completely integrable system. In reality, the planets attract each other, which corresponds to a perturbation of the system. The real problem is non-integrable. We can approximate the motion of planets by ellipses for a certain period of time, but we can not predict a long-term dynamics (when time goes to infinity). In some sense, the regularity of dynamics of the planets we observe is a "shadow" of the underlying integrable model.

Remark 3. Action variables can be found by the integration of the form $pdq = p_1dq_1 + \cdots + p_ndq_n$ along the basic cycles $\gamma_1, \ldots, \gamma_n$ of the torus M_c:[3]

$$I_i|_{M_c} = \frac{1}{2\pi} \oint_{\gamma_i} p\, dq$$

(see [1]). In the case of the harmonic oscillator (1), applying the Green formula,

[3] Ibid.

we get

$$I = \frac{1}{2\pi}\oint_{M_E} p\,dq = \frac{1}{2\pi}\int_{\Pi} dp\,dq = E\sqrt{\frac{m}{a}} = h\sqrt{\frac{m}{a}} = \frac{h}{\omega}, \tag{18}$$

where Π is a region within the ellipse $M_E : h = \frac{1}{2m}p^2 + \frac{a}{2}q^2 = E$.

EXERCISE 2. Show that the functions (8) and (18) satisfy the canonical relation $\{\varphi, I\} = 1$, i.e., (I, φ) are action-angle coordinates for the harmonic oscillator. What are action-angle coordinates for the system of n independent harmonic oscillators (3)?

4.2. A quasi-periodic motion (7) is *nonresonant* if the frequencies ω_i are independent over the field of rational numbers: if $\omega_1 k_1 + \cdots + \omega_n k_n = 0$, $k_i \in \mathbb{Z}$ then $k_1 = \cdots = k_n = 0$. A nonresonant winding (7) is uniformly distributed on the torus $\mathbf{T}^n$. More precisely, the following averaging theorem holds (see [1]).

THEOREM 2. *Let $f\colon \mathbf{T}^n \to \mathbb{R}$ be a Riemann-integrable function and let $\omega_1, \dots, \omega_n$ be independent over the field of rational numbers. Then for every point $\varphi^0 \in \mathbf{T}^n$, there exist the limit*

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s f(\omega t + \varphi^0)\,dt$$

(the time average of f) and it is equal to

$$\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\varphi)\,d\varphi_1 \dots d\varphi_n$$

(the space average of f).

In particular, let f be a characteristic function of a Zhordan measurable region $D \subset \mathbf{T}^n$ ($f(\varphi) = 1$, $\varphi \in D$; $f(\varphi) = 0$, $\varphi \notin D$) of the measure $\mu(D)$. Let $\tau_D(s)$ be the amount of time that in the interval $[0, s]$ of time, the trajectory $\varphi(t)$ is inside of D. Then

$$\lim_{s\to\infty} \frac{\tau_D(s)}{s} = \frac{\mu(D)}{(2\pi)^n}.$$

This means that the time the trajectory (7) spends in D is proportional to the measure of D. In particular, a nonresonant winding is dense on the torus.

The theorem on averages may be found implicitly in the work of Laplace, Lagrange and Gauss on celestial mechanics. A rigorous proof was given in 1909 by P. Bohl, V. Sierpinski and H. Weyl. As an example, consider the torus (10), where parameters c_i are greater then zero. The projection of M_c to the space $\mathbb{R}^n(q_1, \dots, q_n)$ is a cube $K_c = \{(q_1, \dots, q_n) \in \mathbb{R}^n \,|\, 0 \le q_i \le c_i,\ i = 1, \dots, n\}$. If the frequencies are nonresonant, the projection of a solution (12):

$$q(t) = \left(\sqrt{2c_1/a_1}\cos\left(\omega_1 t + \varphi_1^0\right), \dots, \sqrt{2c_n/a_n}\cos\left(\omega_n t + \varphi_n^0\right)\right), \qquad t \in \mathbb{R}$$

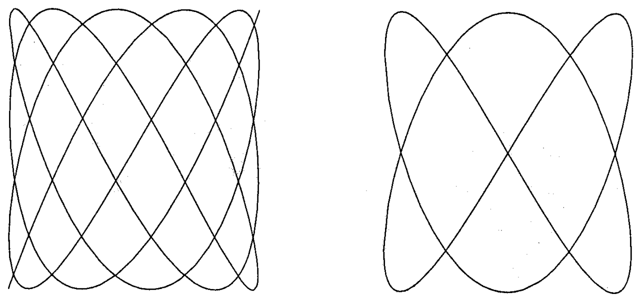

Fig. 6. Lissajous figures for $\omega_2/\omega_1 \notin \mathbb{Q}$, $\omega_2/\omega_1 \approx 3/2$ and $\omega_2/\omega_1 = 3/2$.

is everywhere dense in K_c. In the case $n = 2$, curves $(q_1(t), q_2(t))$ are called *Lissajous figures*. They are closed curves within rectangular K_E for ω_1/ω_2 being a rational number, while they fill the rectangle densely for ω_1/ω_2 being an irrational number.

5. Hopf fibration

The Hopf fibration of a sphere S^3 is one of the basic examples of a nontrivial fibration in geometry and topology. It appears that the Hopf fibration has a natural description by the use of a system of two identical independent harmonic oscillators ($a_1 = a_2, m_1 = m_2$). Note that in this case $\omega_1/\omega_2 = \sqrt{a_1/m_1}/\sqrt{a_2/m_2} = 1$ and Lissajous figures are ellipses, segment of lines and a point $(0, 0)$ (equilibrium position, $c_1 = c_2 = 0$). We note that an interesting description of the system of two harmonic oscillators is given in [5].

For simplicity, let us consider the harmonic oscillators with all parameters equal to one: $a_1 = a_2 = m_1 = m_2 = 1$:

$$
\begin{aligned}
&\dot q_1 = p_1 = \frac{\partial h}{\partial p_1}, \quad \dot q_2 = p_2 = \frac{\partial h}{\partial p_2},\\
&\dot p_1 = -q_1 = \frac{\partial h}{\partial q_1}, \quad \dot p_2 = -q_2 = \frac{\partial h}{\partial q_2}, \quad h = \frac{1}{2}(q_1^2 + p_1^2 + q_2^2 + p_2^2).
\end{aligned} \tag{19}
$$

The angular variables (φ_1, φ_2) on invariant tori

$$
M_{c_1,c_2}: \quad f_1 = \frac{1}{2}(q_1^2 + p_1^2) = c_1, \quad f_2 = \frac{1}{2}(q_2^2 + p_2^2) = c_2 \tag{20}
$$

are defined by

$$
q_1 = \sqrt{2c_1}\cos(\varphi_1),\ p_1 = -\sqrt{2c_1}\sin(\varphi_1),\ q_2 = \sqrt{2c_2}\cos(\varphi_2),\ p_2 = -\sqrt{2c_2}\sin(\varphi_2). \tag{21}
$$

The corresponding equations of motion read

$$
\dot\varphi_1 = 1, \qquad \dot\varphi_2 = 1.
$$

Therefore, all trajectories are 2π-periodic and an arbitrary 2π-periodic function of $\varphi_1 - \varphi_2$ is an additional first integral of the system. As an example, from the trigonometric identity $\cos(\varphi_1-\varphi_2) = \cos(\varphi_1)\cos(\varphi_2)+\sin(\varphi_1)\sin(\varphi_2)$ and (21) we get the integral $(q_1q_2+p_1p_2)/\sqrt{f_1f_2}$. Since $\sqrt{f_1f_2}$ is also an integral, we obtain an additional integral in the form:

$$m_1 = \frac{1}{2}(q_1q_2 + p_1p_2).$$

Exercise 3. Let $m_2 = \frac{1}{2}(q_2p_1 - p_2q_1)$, $m_3 = \frac{1}{2}(f_2 - f_1)$. Prove the following relations:

$$\{m_1, f_1\} = \{f_2, m_1\} = m_2, \quad \{f_1, m_2\} = \{m_2, f_2\} = m_1, \quad \{f_1, f_2\} = 0,$$
$$\{m_1, m_2\} = m_3, \quad \{m_2, m_3\} = m_1, \quad \{m_3, m_1\} = m_2.$$

Note that the integral m_2 corresponds to the functions $\sin(\varphi_1 - \varphi_2)$. Also, since the phase space $\mathbb{R}^4(q,p)$ is four-dimensional, integrals f_1, f_2, m_1, m_2 are not all independent. Indeed,

$$f_1f_2 = m_1^2 + m_2^2.$$

Let us consider the energy level set $h = E > 0$. It is a sphere of radius $\sqrt{2E}$:

$$S^3_E: \quad q_1^2 + q_2^2 + p_1^2 + p_2^2 = 2E. \tag{22}$$

Trajectories of the system lying on S^3_E defines the fibration of the sphere on circles, which is exactly the *Hopf fibration*. The usual definition is as follows. Consider a sphere S^3 realized in a two-dimensional complex space $\mathbb{C}^2(z_1, z_2)$ by the equation $|z_1|^2 + |z_2|^2 = 1$. Then the complex projective line $\mathbb{P}^1 \cong S^2$ is defined as a quotient of S^3 with respect to the relation $(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$, $\lambda \in S^1 = \{e^{i\theta}\}$. The classes of equivalences are circles that defines the Hopf fibration.

There is a mapping to a two-sphere

$$\pi\colon S^3_E \longrightarrow S^2_E,$$

such that the inverse images $\pi^{-1}(x)$, $x \in S^2_E$ are trajectories of the system. By using the integrals of the system, we can explicitly describe the mapping π:

$$\pi(q_1, q_2, p_1, p_2) = (m_1, m_2, m_3) = \frac{1}{2}\Big(q_1q_2 + p_1p_2, q_2p_1 - p_2q_1, \frac{1}{2}(q_2^2 + p_2^2 - q_1^2 - p_1^2)\Big).$$

Namely, we have

$$m_1^2 + m_2^2 + m_3^2 = \frac{1}{4}(4f_1f_2 + f_1^2 - 2f_1f_2 + f_2^2) = \frac{1}{4}(f_1 + f_2)^2 = \frac{1}{4}h^2,$$

so the point $\pi(q_1, q_2, p_1, p_2)$ belongs to the sphere

$$S^2_E = \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = \frac{1}{4}E^2\}, \tag{23}$$

while from the independency of m_1, m_2, m_3 we get that the inverse images $\pi^{-1}(x)$ are trajectories of the system (19). Further, we also have the fibration of the sphere (22) on invariant tori (20), where $c_1 + c_2 = E$. Note that instead of (20), in the definition of M_{c_1,c_2}, we can use the equations $h = f_1 + f_2 = E$ and $m_3 = \frac{1}{2}(f_2 - f_1) = \frac{1}{2}(c_2 - c_1)$.

Remark 4. A fibration on invariant tori for the system (19) is not unique. For example, the invariant tori defined by h and m_1 are different from (20).

Consider a decomposition of the sphere (23) on the antipodal points (the "south" and "north pole") and parallels:

$$S^2_E = \{(0,0,-E/2)\} \bigcup_{-\frac{E}{2}<c<\frac{E}{2}} S^1_c \cup \{(0,0,E/2)\}, \quad S^1_c = S^2_E \cap \{x_3 = c\}.$$

With the above notation, it is clear that the inverse image of the circle S^1_c is a torus:

$$(24)\quad M_{c_1,c_2} = \pi^{-1}(S^1_c), \quad c_1 = \frac{1}{2}(E-2c), \quad c_2 = \frac{1}{2}(E+2c), \quad (c \in (-E/2, E/2)),$$

while the inverse images of the antipodal points $(0,0,-E/2)$ and $(0,0,E/2)$ are circles

$$\gamma_1 = \pi^{-1}((0,0,-E/2)) : \quad f_1 = E,\ f_2 = 0,$$
$$\gamma_2 = \pi^{-1}((0,0,E/2)) : \quad f_1 = 0,\ f_2 = E.$$

Therefore, we have a decomposition of the sphere S^3_E on circles γ_1, γ_2 and a family of tori (24) This picture helps us to visualize the Hopf fibration, which can be modelled as follows.

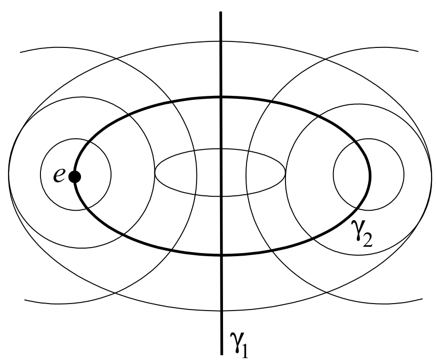

Fig. 7. The Hopf fibration

Let us realize the sphere S^3_E as a 3-dimensional Euclidean space $\mathbb{R}^3$ with the additional infinity point ∞. Let γ_1 be a union of a line and the infinity point ∞ and let $e \in \mathbb{R}^3$ be an arbitrary point which does not belong to γ_1. Consider a fibration by circles $\mathcal{F}$ of the open half plane $(\gamma_1, e$, without the point e, such that all circles contain e in their interiors (see Figure 7). By the rotation of the point e and the circles $\mathcal{F}$ around the line γ_1 we obtain the circle γ_2 and a family of rotational surfaces $\mathcal{T}$, respectively.

Rotational surfaces $\mathcal{T}$ are tori that corresponds to (24) and the Hopf fibration is an additional fibration of tori, given by cycles that wind once along the meridians and once along the parallels.

EXERCISE 4. It is obvious that the circles γ_1 and γ_2 are linked in S^3_E, i.e., one of the circles intersect an arbitrary disc bounded by the other circle. It is also obvious that the circle γ_1 (or γ_2) is linked with on arbitrary closed trajectory on invariant tori (24). Make sure yourselves that the closed trajectories laying on invariant tori are linked between themselves as well.

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