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LOCAL RATE OF CHANGE: A SOCRATIC EXPERIENCE IN VAN HIELE'S MODEL FRAMEWORK

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Abstract. Our aim is to provide an educative experience concerning the study of rate of change, to be implemented prior to its formal treatment in the classroom. By means of a Socratic dialog and the manipulation of a computer generated visualization tool, we shall provoke answers to questions such as How much?, How steep? and How fast? by identifying all key ingredients necessary to provide a sound concept image of the concept of derivative of a function at a point. Our study is framed in van Hiele's educational model methodology. This first of two articles deals with the interview and the list of descriptors which allow the detection of the levels of reasoning the model postulates. In a forthcoming article we shall concern ourselves with the analysis of all students responses, cognitive obstacles encountered and tutorial actions needed to overcome them.

ZDM Subject Classification: C34, U54; *AMS Subject Classification*: 97C30, 97U50.

Key words and phrases: Model; change; rate of change; visualization; approximation; local rate of change.

1. Objective

Calculus is the subject that treats the concept of local rate of change of one variable respect to another (the derivative) and the various applications of this concept. Newton's Second Law of motion is a statement about a rate of change: it says that the force acting on a body equals the mass of the body multiplied by the acceleration of the bodys motion: when the force is known, the law becomes a statement about acceleration, rate of change of speed compared to time. The difficulty on providing a sound definition of what local rate of change really was, made Calculus to be thought as unsound but giving correct results, perhaps because errors committed were offsetting each other. It took about a hundred and fifty years of work to produce a logical presentation of Calculus. The modern concept of local (instantaneous, if time is concerned) rate of change requires not only the usual algebraic operations (subtraction and division) but also an essentially different algebraic operation whose task is to put a dynamic process to rest and its formulation requires a certain logical-algebraic maturity which is not always available in our students.

Our purpose is to introduce the topic of the derivative via a Socratic dialog in which active participation is required by either side to students in High School who have not study differentiation yet, but are close to doing it. We are not interested in what students can do naturally, but in what they can do accompanied by instruction. We want to guide them in a journey of discovery and inquiry, not random but purposeful, testing at every significant stage of the experience the foundations of their beliefs or their ways of reasoning, very much in the Socratic spirit, and to explore how far progress can be made in providing meaningful information by using colloquial language admitting a certain degree of ambiguity present, which hopefully allow students to invest meaning in the problem studied, while manipulating a visualization tool. Although it is unclear how those manipulative aids affect cognitive functioning, its use somehow provokes less disappointment if the machine proves them wrong, favours the appearance of conjectures and frees students thought processes. This approach will prove encouraging but will show its limitations too, mainly when asked to leave the safety of conjectural reasoning and jump to exact reasoning.

The cognitive structure associated to a mathematical concept which includes all mental images, visual representations, experiences and impressions as well as associated properties and processes is called concept-image, [3]. By means of a semi-structured clinical interview, we shall provide the means for the construction of a solid concept-image of local rate of change (derivative) which incorporates visual, numerical and algebraic connotations. For this purpose and with the help of a mathematical assistant, we have designed a tool covering all those aspects. Our aim is not to develop a substitute for the concept-definition (the conventional linguistic statement precisely delimiting the frontiers of application of the concept, [3]), but rather a somewhat narrower objective: to describe a battery of actions that should be implemented prior to formal mathematical instruction in the classroom with the purpose of, on one hand, constructing a suitable concept-image which does not distort the desired concept-definition allowing an easy transition to it and, on the other, provoke the need for such a concept-definition. The natural place to implement such a strategy is High School where the problem they shall attack is, given a curve, motivate and define derivative and study its behaviour along it (our objective). In a sequel we shall provide the appropriate clues to prepare students to approach the more interesting problem of, knowing where a curve starts and how its derivative behaves at every point, find the curve.

2. Van Hiele's Educational Model

We shall anchor our experience in Van Hiele's educational model [4] which provides a description of the learning process, by postulating the existence of levels of reasoning (not identified with computational skills) classified as Level 0 (Pre-descriptive), Level 1 (Visual Recognition), Level 2 (Analysis), Level 3 (Classification and Relation) and Level 4 (Formal Deduction).

In Level 1 students are guided by a series of visual characteristics and lead by their intuition. In Level 2, individuals notice the existence of a network of relationships. This is the first level of reasoning that can be called "mathematical" because students are able to describe and generalize through observation and manipulation properties that they still do not know. Reasoning in Level 3 is related to the structure of the second level and conclusions are no longer based on the existence or non-existence of links in the network of relationships of the second level, but rather on existing connections between those links. Level 4 speaks for itself. Level 3 is our last port of call since we are interested mainly in the construction of a solid concept-image of a mathematical concept which makes easy the mental transition to its concept-definition, once the necessary logical-algebraic maturity is available.

Its success in dealing with concepts outside of the realm of Geometry (see [1], [2]) is explained because the model is more concerned on how students think about a specific topic than with the topic itself and also because the model mimics the genesis of some mathematical concepts: first, the discovery of isolated phenomena; second, the acknowledgement of certain characteristics common to all of them; third, the search for new objects, their study and classification and, fourth, through consideration of examples and counterexamples to proposed definitions, the emergence of definitive formulations. The application of this model to a specific subject requires the establishment of a series of descriptors for each level to enable their detection. To be considered within van Hiele's model: (i) levels must be hierarchical, recursive, and sequential (ii) levels must be formulated so that they include a progression in the level of reasoning as a result of a gradual process, resulting from learning experiences (iii) tests designed for the detection of levels should take into account the existing relationships among levels and the language used by apprentices and (iv) the fundamental objective of the design must be the detection of levels of reasoning, without confusing them with levels of computational skill or previous knowledge.

3. Our approach

According to the constructivist perspective which focuses in individual thinking, all that can be accomplished via an appropriate Socratic interview design that, within the context of the model and paying attention to many of the recurrent themes described in the research literature, allows the detection of students' levels of thinking with respect to the specific mathematical concept we are dealing with. This interview is not a wild-goose chase: it has to be carefully planned with a strategy in mind and orderly implemented, avoiding inappropriate tactics at the wrong times or places and allowing for feedback to planning and assessment.

Why an interview as a Socratic dialog? Most students are very comfortable functioning without formal definitions in most cases and hence common language is a perfectly reasonable tool of communication to use: it is rich in implicit mathematical rules, meanings and conventions and its proper use entails a special competence. There is no such thing as a neutral communication system: ours allows the interviewer a sense of direction proposing those questions which concentrate in conceptual understanding and avoiding those more concerned in assessing performance in procedural items, which are not clearly related. We shall stress the importance of mathematical concepts and pattern matching. The dialog provides motivation and tests the solidity of those ideas and arguments which the student brings and that are based in prior educative experiences when challenged to dig out and verbalize his own beliefs; it also allows the student to form his own connections, many of which are unexpected by the interviewer and most of which are rational and adequate, throwing light in his way of understanding things mathematical. It shows that many students do not seem to approach learning Mathematics the way mathematicians or mathematics instructors think they do.

Since students have been well trained to be passive in the classroom, the interview allows us to break this pattern and turn them in active learners: the downside is that it causes discomfort, but it can be usually ameliorated by frank and open discussion. The most delicate aspect of the dialog is how to fight the attitude of most students which are comfortable with inconsistencies, contradictions and competing meanings which are the main cause of failure in completing the interview as our questions grow in sophistication. Our only weapon is to be persuasive and gently reveal their ignorance and misinformation along with valid knowledge so that we can help replace one with the other, but only in the context of what he already knows, since otherwise a long period of incubation is needed and we are not entitled to the luxury of time. Special care has been attached to check if the projection of our understanding on the students interviewed is different from their actual understanding.

Around twenty interviews were carried out. The participants agreed to the audio recording of the interviews and also to the use of their corresponding transcriptions in our analysis. Their previous formation in Mathematics was the habitual one of students sailing through the Baccalaureate in Spain. Of course not all interviewees were capable of completing the interview; as a matter of fact, only one third of them finished it satisfactorily, that is, reached Level 3 in van Hiele's terminology. To show how the experience went for those students, we have assembled a single interview with a fictitious student, whose answers come from a variety of students who were able to reach the interview's conclusion and were heavily edited to make it readable, avoiding false starts, silences, clumsiness and hesitations, although what is given below is a fair representation of their clarity of thought and adaptability. Overcoming difficulties contribute to conceptual learning but in order to keep this article within reasonable bounds, dysfunctions do not show in the following transcription of our interview, since the responses belong to the more gifted students; once our building is completed, the scaffolding should no longer be visible. However, in a forthcoming study, we shall study them, point out the causes of failure and propose remedial or tutorial actions for those students which were not able to reach the interview's conclusion as well as making an incursion in Level 4.

We shall start by questioning on previous knowledge concerning space, time and the notion of function. Since questions are presented verbally, not algebraically or graphically or numerically, all the answers are expected to be to some extent verbal. At some stage of the interview and in order to increase his participation in the experience we shall present a tool, based on a mathematical assistant, designed to introduce ideas and methods and to let students explore them computing and visualizing all pertinent ingredients of our study.

4. The tool

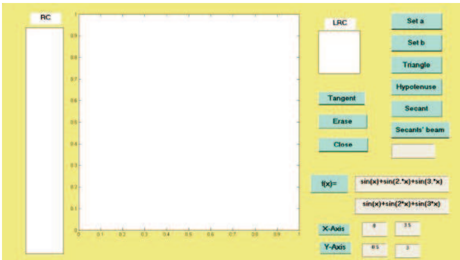

Fig. 1

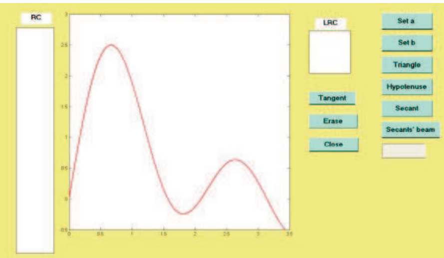

Fig. 2

The screen (Figure 1) shows several graphic, computational and textual windows; those last ones allow the entering the algebraic expression of functions as well as the portions of abscissas and ordinates axes to be seen when plotting the function, a way of choosing pertinent scales. Click the mouse on the button "$f(x) =$" and the function is plotted in the graphic window; simultaneously, the textual windows disappear (Figure 2) (we do not want the student to bother with algebraic expressions or the syntax of the program MATLAB 6.0). From now on the tool can be manipulated with the mouse just by clicking the appropriate button: you may fix abscissas a y b on the range of the independent variables, draw the corresponding right triangle (Figure 3), isolate the hypotenuse (Figure 4) and prolong it to the secant line (Figure 5). Simultaneously, the slope of this secant line will appear in the computational screen "RC". The tool allows to draw a tangent line to the graph at the point $(a, f(a))$ by clicking on the corresponding button and its slope will appear in the computational window "LRC" (Figure 6).

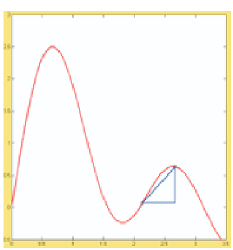

Fig. 3

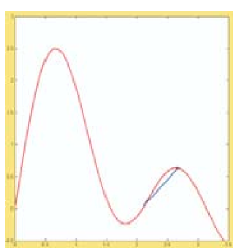

Fig. 4

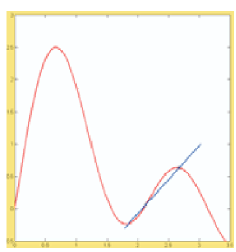

Fig. 5

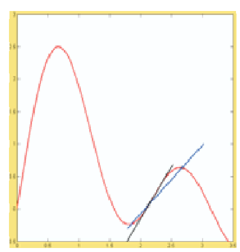

Fig. 6

We can specify how many secants through $(a, f(a))$ we want to see and the button "Secants' beam" allows us to plot the beam. Simultaneously and in the corresponding windows, we will have access to all slopes concerned as well as the slope of the tangent line, whose graph is produced automatically by the tool (Figure 7).

The button "Erase" places us where we started and a different function can be selected or, for the same function, we may use a different choice of abscissas.

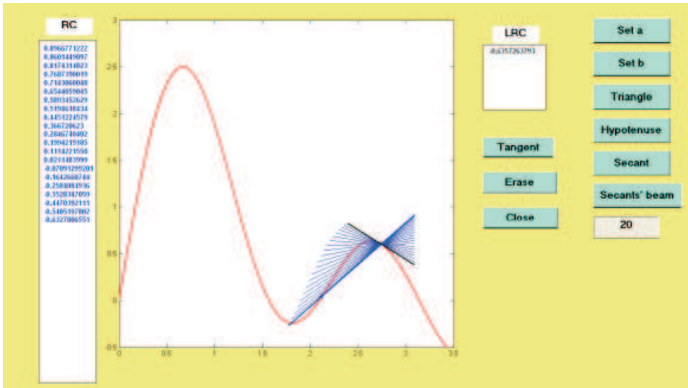

Fig. 7

5. Experience

Our first step is to deal with all those notions which are relevant to our study of rate of change and covers previously known material of his/her mathematical background. Students surviving Level 0 are well positioned to complete the interview.

5.1 Level 0 (*prerequisites*)

The notion of infinity comes neither from observation nor from physical experience. Our brain is a finite object and it cannot contain anything infinite, although it is able to produce notions of the infinite. Since Logic does not force the incorporation of the infinite in the Mathematics (geometricians always consider finite segments, not infinite straight lines). Thus certain preventions on this notion are natural in our students.

Pr.: What is a number, what types of numbers do you know and how many numbers are available?

St.: Numbers answer to the question How much? They have magnitude and are something to count with. That is, integers are available for counting purposes and fractions for estimating parts of whole things. Well, integers can be considered fractions as well.

Pr.: Let us consider the word number as a term referring to fraction. If you take the average of two fractions, is the result a fraction again?

St.: Average?

Pr.: Meaning the arithmetical mean.

St.: (checks with paper and pencil) Yes, indeed.

Pr.: Do you have any problem accepting that you can proceed to take averages indefinitely?

St.: No, at least theoretically, even if I can't perform this operation indefinitely in reality.

Pr.: Hence, under this supposition, between two numbers there is an infinity of them.

St.: An unending number of them.

Pr.: There is no such thing as an unending number.

St.: Well, you know what I mean, not a finite amount of them; OK, an infinity of numbers.

Now we engage in a dialog concerning the possibility of visualizing the number system and if this visualization may serve as a model to deal with space and time. Self-inflicted trouble is awaiting us when we'll try to identify points and numbers.

Pr.: So, if you were looking for a visual representation of the number system, what would you choose as a model or simulation?

St.: A straight line, as usual.

Pr.: Meaning that you are identifying numbers with points of a straight line, in which you select an arbitrary point as the number 0 and an arbitrary length to set the number 1 in the picture and then you can go on picturing all the numbers you want. How do you do that?

St.: Yes, that's the idea. If I want to know where 1/2 stands, take the middle point between 0 and 1 and so on. On the other hand, there is another way of representing fractions: perform the division and you get its decimal expression. It is easy to understand a decimal expression like 0.5 as a position in that straight line.

Pr.: Due to the infinity of numbers involved, is it possible that you may run out of points to be associated to numbers?

St.: I don't think so, if there is an infinity of points in a line or/and in a segment. Well, it is pretty obvious that there is an infinity of points in a straight line, it has no end and all integers are accounted for. Concerning the fractions, it depends on . . .

Pr.: How many points are to be found in a segment?

St.: Right. There are many points there.

Pr.: What is a point for you?

St.: (Staring puzzled takes pencil and draws a point) That is a point.

Pr.: For mathematicians a point it's an undefined term, and it's undefined on purpose. Let us say that it is something which has position but no magnitude or "size" as you said. What you've drawn is a representation of a point, which has magnitude, which for practical reasons is OK, but not really what it is understood as a point, meaning just location.

St.: I know where you are heading to: if each point had magnitude, there should be place only for a finite number of them.

Pr.: Right. Thinking geometrically, the operation of taking averages in the numerical system corresponds in the model to

St.: Two numbers determine a segment in the model and then we take its middle point; geometrically, we bisect the segment.

Pr.: Something which can be done by ruler and compass (shows how to do it). Under the assumption of a finite number of points in a segment, is it possible to bisect it using ruler and compass?

St.: (Hesitating) Well, the center has to belong to either half . . . The center is a point also, isn't it? Hmm, no problem if there is an odd number of points . . . but then the center doesn't belong to either part; if there is an even number of points, you halve the segment, but what happens to the center point, it belongs to both halves?

Pr.: Even if the center point question is resolved, admitting a finite number (even or odd) of points in the segment does not allow you to proceed bisecting and bisecting . . .

St.: Mm . . . , yes. Well, then we better assume an infinity of points in each segment

Pr.: Averaging fractions and its equivalent bisecting segments allow you to see fractions as points.

St.: Yes, I understand the purpose in thinking that way, but is it true in reality that there are a finite number of points in a segment?

Pr.: We are not dealing with reality, whatever that is, but we are trying to model space to be allowed to do things like bisecting indefinitely to mimic our understanding of space based in the system of numbers. Let us say that the abstract ideas of number system and space share a common model: the straight line or the segment, that is, a continuum . . .

St.: A continuum?

Pr.: No gaps, unbroken as opposed to discrete. But, let us be clear about the fact that we haven't shown that fractions or rational numbers, as we call them, cover the whole line . . .

St.: Do you mean that we are not running short of points but of numbers? Are there more points than rational numbers?

Pr.: (Uncomfortable) Yes, there are "holes" in the line not covered by rational numbers. We are not going to deal with this fact now.

St.: Why not? It seems interesting.

Pr.: (Resigned) It is, indeed but it deviate us from our purposes. Suffice to say that an expression like square root of 2, which measures the length of the hypotenuse of a right triangle with sides measuring 1, should be a number, a rational one if there were no holes in the line.

St.: Yes, by the Pythagorean Theorem.

Pr.: Aha. If you rephrase this question asking to solve the equation $x^2 = 2$ for rational x, Arithmetic proves powerless to provide a solution as stated, although Geometry suggests that a numerical solution should exist, if you identify numbers with lengths, as we want. Calculus actually solves the problem and you shall deal with it in due time by enlarging the number system to cover those holes.

St.: Are there many holes?

Pr.: As a matter of fact, quite a lot. But every hole has a rational number lying as close as I want to it. Since rational numbers are densely packed and our intention is to work intuitively, they provide a practical illusion of the continuum, not exactly but close.

To summarize, the admission of the infinite adapts well to our intuitions of space suggesting that any length, as small as it may be, can be subdivided. The mathematical formulation of space takes in consideration this property: (i) any segment can be bisected by means of constructions by rule and compass (ii) any length consists of points, each one of which does not have length and (iii) those points are related to each other in the same way as the numbers in the numerical system (between two numbers there is an infinity of them). From this point of view, a mathematical segment is infinitely divisible, whereas a material cable may be not.

Pr.: Somehow we have stated that the number system can be used to measure length. Measuring time, for instance, comes to mind as other possible use of the number system.

St.: Well any magnitude which can be measured uses the number system.

Pr.: Hence share the same model. Concerning time, an instant is conceived as a point, a temporal interval is a finite segment.

St.: And that is what time is?

Pr.: Again, let me stress that we are talking about a model not reality: we are not concerned in questions such as "what is time", which we better leave to philosophers, but in "how to measure time". That there is a distinction is shown by the fact our model of time contravenes one of the basic characteristic of it: its irreversibility.

St: I do not see any problem with that, my watch does not go backwards. But I see other problem: if time is conceived as a continuum, how you reconcile that with the fact that my watch measures time step by step, that is, discretely as you said?

Pr.: Our model is designed to deal with all possible measure instruments used in the evaluation of time. As you know there are atomic clocks which work discretely but increasing at very, very small amounts.

St.: (Impersonating the professor) I see. Points, lines, planes, instants and infinity cannot exist in this universe ...

Pr.: (smiling) They are only abstractions of our minds, a form of metaphors to explain our world.

St.: (Defiant) Well, but it still bothers me that, if we ignore what things really are, talking instead of abstractions that are not even closely related, how can be we sure of the effectiveness of such a model?

Pr.: Experience. Mathematics is not the language of the universe but a language used by people who are trying to describe the universe. It's important to emphasize that the way math works is that we create idealized concepts, and then look for situations where we can set up analogies between these concepts and things we observe in the world. Modelling space and time as equals and modelling physical phenomena with the help of relationships between sets of numbers has allowed Humanity to study physical reality with notable success recognizing structural similarities in apparently different situations and applying successful reasoning methods to new problems.

St.: Physical phenomena such as ...

Pr.: Change, for instance.

St.: And what shall we do from now on?

Pr.: In what follows, we shall concentrate on those relationships between sets of numbers.

Now we exchange considerations about change such as that every natural phenomenon is a manifestation of change and that changes in the surrounding world have always been a source of anxiety and the perception of relationships between them has been a tool to cope with the changes. We insist in that we must become sensitive to the patterns of change, learning to represent changes and their patterns in a comprehensible form and understand the fundamental types of change in order to recognize them.

Pr.: As you have seen I attach importance to deal with models.

St.: Yes, you have been very clear about it. Do we have a model for change?

Pr.: Change can be very complex but, at least in the simplest situations, we have it and you know it already. Let us see if you can guess it. Our model involves at least two sets of numbers: suppose you are dealing with measuring the position of a moving object as time flows. What changes?

St.: As time changes, the distance covered by the object changes too.

Pr.: Let us call **variable** any measurable physical magnitude whose value changes continuously.

St.: Such as space or time.

Pr.: Yes, but not exclusively. Imagine a statement such as “Pressure in the atmosphere varies with height above the surface of Earth”. What are variables here?

St: Pressure and height.

Pr.: Right. We routinely use letters to name variables, but the notion of variable is not simply a letter standing for a number but a measurable quantity which changes continually as the situations in which they occur change.

St.: But all variables change in the same way, that is, continuously as if moving along a segment.

Pr.: Yes, because we use the same model for each of them.

St.: Change is then the study of how a variable moves?

Pr: Depends on the type of change we are talking about. Suppose that we dealing with a flow of information and that this information can be measured somehow.

St.: So the variable is information.

Pr.: Yes, but there is a variable hidden. Information is tied to the passage of time, time must flow for what is hidden to be revealed.

St.: I see: information and time are the variables, although time was not mentioned. Or the types of change we talked before, where pressure and height were mentioned and so on.

Pr.: Sure. Concerning all types of change we mentioned before, observe that variables are not usually significant by themselves, but only in relation to other variables and ...

St.: And we need a way of finding one changing magnitude with the help of another?

Pr.: Yes, the conceptual heart of the matter for our desired model of change is trying to understand relations among several variables. Do you know the most useful algebraic idea for thinking about relations of this sort?

St.: The concept of function?

Now we remind him of what a function is. He has been already introduced to this notion and hence is expected to manipulate it with a certain confidence or at least have a sense of function, although we are conscious of the variety of traps hidden in this notion due to competing meanings (a graph, a formula, an action, a process or an object) as well as existing misconceptions and the little connection existing between formal and informal and diffuse views of a function.

We do so as a fine tuning on what he already knows in an abstract setting not related to the changes mentioned before: verbally, as a rule which assigns to each of certain numbers, some other number; algebraically, we follow the usual drill with no apparent opposition.

Rather hopelessly and since we are not going to deal with formulas, we point out that a formula relating both sets of numbers—viewed as a sequence of commands from the algorithm behind it allowing substitution and evaluation—is one way of doing it but is not essential: if x stands for any number in the first set, any rule which, when x is decided on, fixes corresponding y's can be said to determine a function f. It is not required that y should actually be computable, only that the rule fixes it.

Following the sometimes referred to as the Bourbaki approach, we point out that the word rule is itself vague and not necessary; what is essential to a function is the idea of pair of numbers, one y associated with the other x. We might write them as a pair in order (x, y). When this is the case, we write $y = f(x)$, read "f of x" (Euler's notation).

As said, it is hopeless and time consuming to insist in differences between rule (an object) and formula (an operational process); he will always object to the absence of a well-defined rule; for him, a function f needs that a variation in x has to be systematically reflected in the y and that manipulation carried out on x produces y, which is not surprising having in mind the historical development of the notion of function (which reproduces itself in learning processes), the lack of motivation on the student to think of functions otherwise and fear to abandon algebra, since he is helpless to replace it with something else.

The idea of a formula associated to a function keeps intruding at all stages of our interview. As long as he does not forget a general idea of what a rule may be and the idea of ordered pair linking both numbers allowing a bi-dimensional visualization, he can do without finer distinctions and so can we: our tool will deal with the computational aspect of the notion identifying functions with computer programs (although we are not clear if the analogy is properly understood) and the need of operations to be performed on functions (translation or composition) is absent of our study and hence we will not run in trouble due to his inability to deal formally with them.

Now we turn to our considerations on change. We commonly call x and y variables since they stand for any numbers in their respective classes. Since in observing changes very often students have difficulty in identifying what is changing, we point out that there is no symmetry in the treatment of both variables. It is always the second variable that is uniquely determined by the first and not always vice-versa, x is called the independent variable and y the dependent one. In applications y is often the numerical measure of a physical situation which varies with the time x or more generally a measure of some physical quantity naturally dependent on some other which may vary freely.

5.2 Level 1 (*from verbal to visual*)

Our first purpose is to discuss change and rate of change and provide visualizations for both concepts so that qualitative properties can be inferred.

Pr.: Now we shall investigate the type of change symbolized by $y = f(x)$, meaning that a (dependent) variable y is changing continuously as the (independent) variable x moves forward continuously also. What different types of change of that sort can you think of? No formulas, please. Give a verbal description.

St.: Since x always increases, y may grow or decay ... or stay the same.

Pr.: That is, the dependent variable increases or decreases as ...

St.: As the independent variable x increases. In our previous situation of distance covered versus time, we have the first case. If the change concerns the evolution of temperature with time of an ill person who is recovering, we are in the second case.

Pr.: Good. Can you imagine of a change where both cases occur?

St.: If a ball is thrown up into the air and we measure its height above the surface or if we juggle balls and measure again its height as time goes by.

Pr.: What about measuring the angle in a pendulum with regard of time elapsed as we displace it from its stable position having in mind the resistance to motion provided by air? Make a drawing of a pendulum displaced from the vertical and consider positive the angles at the right side and negative at the left.

St.: The angle decreases during the first swing to the left, increases again to reach almost its starting position and the process repeats itself several times with decreasing amplitude until it eventually rests.

Pr.: Due to friction. Is it the same situation as in the juggling you mentioned?

St.: No, in the juggling the same actions are performed over and over again by hand and could continue indefinitely.

Pr.: What about the stock market index along the day?

St.: The change combines sudden increases and decreases and no change during certain periods of time, but all of them in an unexpected manner, at least to the naked eye, I think it is ... unpredictable, chaotic ...

We summarize for him the types of change we have witnessed so far: (i) as one variable increases, another also increases or decreases (ii) as one variable increases without limit, another approaches some limiting value (iii) as one variable increases uniformly, the other increases and decreases in some repeating cycle (stable periodic motion) (iv) as one variable increases, another changes in jumps and (v) the change is chaotic or random.

Now we turn on how to replace verbal information by graphical, that is the geometrization of change: the idea that numbers can be represented by positions of points has already been used in our interview; visualizing changing numbers by curves is the idea we shall pursue. According to Piaget, going visual is usually more difficult and harder to teach because you only see what you understand: using graphs to represent functional relationships is hardly spontaneous, even if trained to look for the equations of plane figures in implicit form where both variables have the same weight or perhaps because of it. Moreover, when presented with numerical and graphical information simultaneously, students tend to disregard visual information, attaching more “certainty” to the numerical one. Nevertheless, once the graph is produced, it is more likely that the student starts to conceive a function as an (geometrical) object while looking at its graph, albeit a limited one.

Instead of the usual approach of starting with numerical tables, ask them to picture those data in a Cartesian plane and interpolate to obtain a continuous function to provide a sense of graph, our students didn't need to go that way, since they had been exposed to graphing and our first block of questions is to check if both parts agree in how a pictorial representation of change should be.

Pr.: When we discussed the meaning of the term function, our algebraic representation of change, the ordered pair $(x, f(x))$ was mentioned. What does it suggest?

St.: (confident) I've been already through that: a point in the plane, I mean a Cartesian plane where the variable x stays in abscissas and $y = f(x)$ in ordinates.

Pr.: In different axes, then. The change of x causes the change of $f(x)$, not the other way round. When the variable x increases, what is the behavior of the point $(x, f(x))$?

St.: (takes pencil) It moves in the plane drawing a curve.

Pr.: An unbroken curve or depends on how the rule f is specified?

St.: (insecure) Apparently so, if f is given by a formula. I need a formula, isn't it? Otherwise how shall I be able to know how to proceed? I obtain a continuous curve since x increases continuously. The curve is the graph of the function in question.

Pr.: Does a graph show where a particular point has been displaced?

St.: In a way yes, but if I want to measure the displacement once the graph is finished, I have to read the displacement in both axes because on the graph itself I don't see how is it possible without reference to the axes, although the graph shows where a point is landing.

Pr.: Do you mean that there are two ways at looking at it, the graph itself (a static representation) and looking progressively how the graph is taking shape (a dynamic representation)?

St.: A fancy way of putting it.

Pr.: Somehow you lose track of the individual points facing a completed graph.

St.: Yes, in the act of producing the graph step by step, one has a better chance of appreciating the displacement of a single point but, again, checking what happens in either axis.

We ask him to produce crude graphs for all the different types of general changes we have isolated so far

Pr.: Can you draw curves corresponding to changes without a formula?

St.: Change and function being the same ... Neither specific numbers involved nor a formula? I don't know. Even if I can, the curves are ... approximate representations, to say the least.

Pr.: Think of a specific change and use only your perception of how growth or decay of f(x) as x grows happens. Try (i)-type changes.

St.: (With free hand produces something looking as concave downwards curved semi-arcs) Those kind of arcs for both changes growth and decay?

Pr.: Are those curves graphs of functions?

St.: (Hesitating) I guess that there has to be a way to find formulas whose graphs are those curves, but one has to find them.

Pr.: If you understand a function as a rule and the rule is given by the verbal description on how the change behaves ...

St.: OK, then the curves are graphs of functions, the ones describing both changes. Well, the general aspect of the curve may vary from change to change, but those graphs give the general idea.

Pr.: Hmm, let's consider the case x increases and $f(x)$ increases, that is growth for both variables and suppose water is pouring into a jar at a constant rate, the jar being a truncated cone with larger bottom than top. Suppose that x stands for time and $f(x)$ for height of the water in the jar. Am I talking of a (i)-type change? Please, sketch the graph of the heights of the water as a function of time.

St.: Yes as times flows, heights increase, thus a similar curve.

Pr.: Are you sure? The bottom being larger than the top ...

St.: Height grows slower at the beginning and faster when reaching the top

Pr.: Slow means ...

St.: It takes a big increase in time to produce a small increase in height.

Pr.: And how does this fact reflect on the curve?

St.: (Untroubled) As time runs, the height does not increase much at the start, hence a quasi-horizontal curve, but as time continues running, the curve elevates itself and hence does not correspond to the graph I pictured before (draws a parabolic semi-arch concave upwards).

Pr.: If we maintain the height of the jar but make the bottom larger and the top smaller, how does the curve change?

St.: The quasi-horizontal stretch is longer and the last path of curve is steeper.

Pr.: What if the jar is a truncated cone with smaller bottom than top? Are we still in the increasing abscissa, increasing ordinate case?

St.: (Draws a parabolic semi-arch concave downwards) Yes, but height increases quickly at the beginning but slower at the end and hence the curve would be something like.

Pr.: Quick in that curve means what?

St.: For the same time interval, the measures of height are drawn as a steep curve.

Pr.: Visually at least, we have two options of the phenomenon of increase of the dependent variable.

St.: (Drawing) Curves concave down and upward ...

Pr.: Any other possibility?

St.: A juxtaposition of both; imagine the jar looking as two equal truncated cones pasted by their smaller or larger bases ...

Pr.: With how many essentially different curves are we dealing with?

St.: Well, two different curves and two possible juxtapositions or more, you could go on and on pasting paths. Essentially different curves, you mean excluding juxtapositions? Well, two curves ... or perhaps three, is you consider a straight line.

Pr.: Which corresponds to what kind of jar?

St.: Steady change, a cylinder. As time increases uniformly, so does height.

Pr.: We may picture all possibilities as an onion (Figure 8) where the straight line is a kind of frontier between ...

St.: Different growth phenomena: slow/ quick and quick/slow. The quicker the steeper the curve, the slower the flatter the curve and hence a frontier between concavities ...

Pr.: Thus the pace at which time increases, which is steady, does not correspond to the pace at which height increases, except in the cylinder situation. So we may be talking of what we may call different **rates of change** along the whole process of filling the jar with water.

St.: Yes, different speeds of increase; graphically different rates of change manifest themselves as different degrees of steepness of the curve.

Since we are going to deal with changes where the independent variable is not necessarily temporal and since some verbal terms we used have a strong temporal connotation, we ask him to reformulate our previous conclusions: what "fast" and "slow" may mean in the general context, their translation into different degrees of steepness of their pictorial representations and their relationship with the notion of rate of change as well as our classification of essentially different paths corresponding to the increasing x, increasing $f(x)$ situation.

Pr.: The questions *How fast a change*? And *How steep its pictorial representation*? have the same answer. Can you predict all kind of essentially different curves corresponding to graphs of functions f where x increases and $f(x)$ decreases?

St.: (Drawing) Essentially different? Again, an onion (Figure 9).

Pr.: Could you say the second onion might be considered as a mirror image of the first (Figure 10)?

St.: You may see it that way.

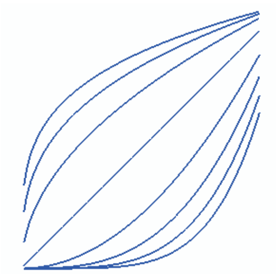

Fig. 8

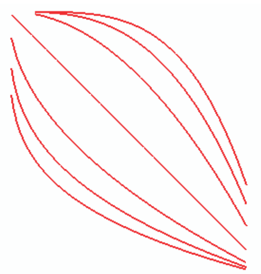

Fig. 9

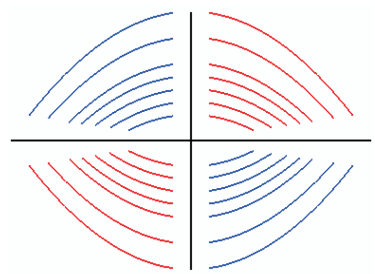

Fig. 10

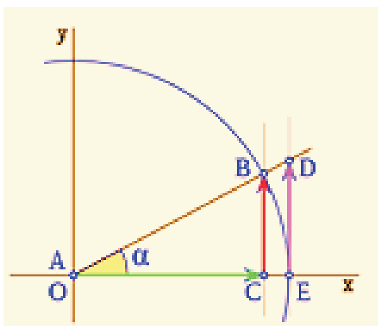

Fig. 11

We produce the following figure (Figure 11) and we remind him that, being a circumference of unit radius, the segments CB, OC and ED are identified with the trigonometric entities sinus, cosinus and tangent of the angle α, by similarity between triangles. Since all three change as the angle changes, we ask him to sketch the graphs of the sinus and tangent as functions of the angle (measured in radians).

Pr.: Sketch the graphs only where the abscissa varies from 0 to $\alpha/2$. What do you see?

St.: (Without considering the figure draws both curves) I know those curves.

Pr.: Please, replicate those graphs looking at the figure by observing the behavior of both altitudes.

St.: (Looking bored) If you insist. Both describe growth processes, but their concavity is different, which means that the sinus grows quickly at the beginning but slows down close to the angle $\alpha/2$; the tangent grows the quicker the larger the angle, hence the different concavity. In fact, the steepness of the curve is almost vertical close to $\alpha/2$.

Pr: Almost vertical accounts for the fact that small increases in the abscissa provoke …

St.: Enormous increases in the ordinate, hence the exaggerated steepness.

Pr.: Thus, speaking of rates of change (growth, in this case), we may say that in the right half of the abscissa interval the rate of change is larger than in the left half of the interval.

St.: Much larger. You can even say that, taking any abscissa interval, its rate of change is larger the more to the right is located.

Pr.: Thus, in general, the idea of rate of change is then a measure of the following observation: as x moves along an interval in abscissas, $f(x)$ moves along a certain interval in ordinates; for the same abscissas interval, the larger the rate of change, the larger this interval.

Just to check if he has grasped the importance of the notion of rate of change and how it influences the pictorial representation of change, we ask him to interpret graphically the following statement: “*the child’s temperature is still rising, but the antibiotic seems to be taking effect*”.

Pr.: Do you agree on the fact that rates of change of varying quantities are at least as significant as the fact that they are changing?

St.: Yes, rates of change have a direct bearing in how the graph or curve looks.

Pr.: Could you classify all onions leaves attaching labels to them such as: increasing at an increasing rate and so on?

St.: (Taking time, looking at the onion and gesturing) Mm . . . , increasing change and increasing rate of change are different things, as the onion clearly shows. Well, in fact there are six labels available: increasing at an increasing rate, increasing at a decreasing rate, decreasing at an increasing rate, decreasing at a decreasing rate and increasing or decreasing at a constant rate. The segments correspond to the last two ones and the “onion leaves” the other fours, an infinity of leaves sharing the same label.

Pr.: For those who share the same label, what is common to them and what is different?

St.: Common is concavity; different is steepness. To complete our pictorial representation of different types of change, we direct his attention to the aforementioned situation of a ball which is thrown up into the air and we measure its height above the surface as time goes.

St.: (Drawing) Looks like the trajectory of a bullet. The curve is composed of two paths: first height increases, reaches its highest altitude and begins to fall until it reaches the surface: the curve is the composite of two arcs one from the increasing type and the other from the decreasing type.

Pr.: I’ve noticed that your figure is somehow symmetrical in its increasing and decreasing paths.

St.: Ah, they should not look alike whether it takes longer to rise or to fall.

Pr.: The forces of gravity and friction act in concert on the way up, but oppositely on the way down, so . . .

St.: Rise time is less than fall time (redraws).

Pr: Concerning your opening remark, should we expect a too-close resemblance between the shape of a graph and the situation that the graph refers to?

St.: (Confused) What do you mean?

Pr.: Think of the bullet you mentioned or of a point moving in circles and you trying to represent displacement versus time, does a circle appear in the corresponding graph?

St.: (Pensive) That is a tricky question. The answer is no, of course.

We discuss the similar situation of juggling balls repeatedly, which provided us an example of periodic change and how this fact reflects in its graph (see (iii)). Graphs related to changes typified as (iv) and (v) pose no threat. (ii) puzzles him, but finally an appropriate graph is produced.

Pr.: Suppose we want to produce a sketch showing the impact on the population when a new fashion is introduced. Please, take pencil and paper.

St.: A graph of the number of people adopting it as a function of time?

Pr.: Yes, suppose it spreads slowly through the population at first.

St.: (Producing an arc) And then . . . ?

Pr.: It speeds up as more people become aware of it.

St.: (Continues drawing) No problem.

Pr.: Eventually, however, the pool of those willing to try new fashions begins to dry up, and while the number of people adopting the new fashion continues to increase, it does so at a decreasing rate.

St.: (Hesitating but producing the right path) What happens next?

Pr.: Later, the fashion goes out and disappears very quickly.

St.: Right.

Pr.: Although a few never give it up.

St.: (Corrects and completes the graph) That was easy. Once the curve is produced, we ask him whether, presented with such a curve but without the former information, if it would be possible to reproduce the missing information by interpreting the graph and he deals with this question without trouble.

Pr.: (Introducing him our tool, we present a graph) Confronted with a continuous graph corresponding to some change (Figure 12), can you decompose it in smaller paths in such a way that any path corresponds to one of the six labelled curves we mentioned before?

St.: (Looking at the screen) Yes, it seems reasonable (Figure 13).

Pr.: Given a graph of a change and if we want to study its rate of change, it would be useful to concentrate only in the six labelled curves, study how the rate of change behaves there and then juxtapose all curves needed to have a global idea.

St.: Again, it is reasonable, but it looks time consuming.

Pr.: No, if once studied one of the curves thoroughly, we were able to export our conclusions to the remaining ones and somehow deduce what is going on in the juncture points. First let us deal with those of constant rate.

St: Graphs being segments.

Pr.: And rate of change, as an indication of its steepness, is a measure of its inclination respect to the horizontal that is some number related with the angle the segment makes with the horizontal. Once we know what measures conveniently inclination, we are done, because it is the same everywhere.

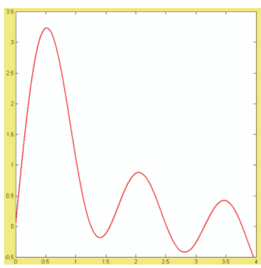

Fig. 12

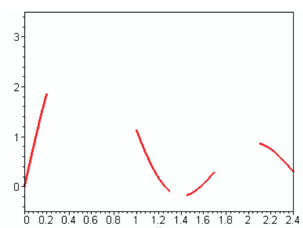

Fig. 13

Since all segments in the same line have the same inclination, we remind him of the slope of a straight line, simply as a measure of the steepness of rise or fall of the line viewed from left to right.

Pr.: Let us suppose we know already how to perform this task. If the curve we are dealing with is not a segment, is it any way we may speak of the slope of that curve?

St.: (Doubtful) A slope for the whole curve? It doesn't seem feasible since the curve varies from point to point and intuitively inclination is changing (uses his hands, implicitly associating some straight line to different points of the curve).

Pr.: Let us go from the curve to some associated line. Considering two points on the curve, is it a way of attaching a line to the curve whose slope might be measured?

St.: With two points on the curve we may speak of the inclination of the secant determined by them, although what I draw is again a segment, a larger one.

Pr.: Right. But the curve between those two points may look very different from the portion of secant joining them and hence not very representative.

St.: Sure, but not so much if the points stay close together.

Pr.: The closer they are, the better the coincidence?

St.: Yes obviously.

Pr.: In other words, every continuous curve can be viewed as locally straight.

St.: What do you mean?

Pr.: (Pointing to the screen) Imagine that you direct your attention to a point in the curve and perform a zooming process several times. What do you see?

St.: After several zooms I see a straight line, well ... a segment.

Pr.: Imagine you maintain the segment and proceed to undo the zooming process recuperating the curve. What is the relation between the curve and the segment?

St.: The segment stays tangent to the curve at this point.

Pr.: Enlarging the segment ...

St.: We have the tangent line to the curve.

Pr.: Tangent line understood as ...

St.: Line which touches the curve without cutting it ... and expected to have only one point in common with the curve.

Pr.: As the ancient Greeks understood the concept.

St.: (Surprised) Are there any other ways of understanding it?

Pr.: Yes, but we are not going to dig deeper now. Suppose we were able to measure its slope.

St.: We get a kind of measure of the steepness of the curve at this point.

Pr.: Is it possible that there are curves where you cannot perform this zooming operation to determine the tangent line?

St.: (tries the zooming process on the screen for several curves whose analytical expression he knows) I always obtain a segment.

Pr.: (Plotting the curve corresponding to modulus of x) Imagine a curve where at some point there is a vertex, a sudden change of inclination at right and left.

St.: (performing several zooms at the vertex) The vertex does not disappear. Why?

Pr.: Remember that a zoom is a simultaneous change of scale in horizontal and vertical, hence the vertex perpetuates itself.

St.: Hmm ... Thus, there is no way to obtain a tangent line there.

Pr.: Right. What does it say about a curve having a tangent line at every point?

St.: (Secure) No corners to be seen. That its path is smooth and that changes path gradually.

5.3 Level 2 (*from visual to algebraic and the other way round*)

Since we want to measure change and rate of change, we are going to attach numbers to all ingredients of our study. In order to be sure that a number sense is present in the interviewee, we start by presenting discretely defined functions given by tables of values. We are not interested in the capability of formalizing the rule in algebraic form (he hasn't been taught to recognize familiar functions from a table of values) but to know if there is an ability to scan a table to interpret verbally stated conditions or to identify changes as those typified above. We pose several questions which were answered correctly related to tables of values without any reference to where the tables came from. We ask:

Is the rate of growth constant? (we invite him to produce a rough graph):

0	1	2	3	4	5	6	7	8	9	10
111	122	134	147	161	177	195	214	235	258	283

Identify changes of decreasing behaviour at an increasing rate and at a decreasing rate:

x	0	1	2	3	4	5	6
$f(x)$	55.0	54.5	53.5	52.0	50.0	47.0	43.5

x	0	1	2	3	4	5	6
$f(x)$	100	88	77	68	60	56	53

What type of change suggests the following table?

x	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
$f(x)$	0	1	0	-1	0	1	0	1	0	-1	0	1	0

Pr.: Let us tackle now the problem of assigning a number to the inclination of a line or segment, that is, calculate the slope of the line. Since all segments of the same line share the same inclination, take one of unit length as in our former Figure 11 and ask yourself which is the most appropriate method to measure its inclination: CB or ED?

St.: Inclination and angle seem to be the same thing. What is wrong with measuring the angle α in degrees or radians? To every segment there is a different angle and the other way round.

Pr.: There is nothing wrong with measuring the angle (we mathematicians prefer to measure it in radians because it simplifies many formulas), the orientation of α being from the positive end of the x-axis counter clockwise around to the line or positive from 0 to π and negative from 0 to $-\pi$. Even if we end dealing with the angle, my question is: Is there an alternative way of doing it measuring altitudes?

St.: CB seems reasonable for angles between 0 and $\pi/2$, but not for further angles, because two segments of different inclinations will have the same number. No ... it is better to choose ED as a way of measuring inclination.

Pr.: Using similarity of triangles, ED equals ...

St.: Sinus divided by co-sinus, that is, the tangent of α. Well, here is the angle again.

Pr.: Right. Let us define the slope as $\tan(\alpha)$. If you have a look to the graph of the function $\tan(\alpha)$ with respect to α, then you realize that there are positive slopes as well as negative slopes. Thus, slope -1 is attached to ...

St.: (Drawing) The bisectrix of the second and fourth quadrant or any parallel one.

Pr.: Are there lines with undefined slope?

St.: Vertical ones?

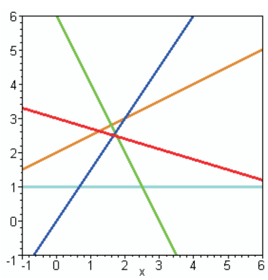

Fig. 14

Slopes

-0.2	1.5	-2	0	0.5

We present him with a picture of five lines and five numbers to be associated as slopes (Figure 14) and he performs the task flawlessly. Now we turn our attention to assign a number to the change of rate of a curve. From now on we concentrate on

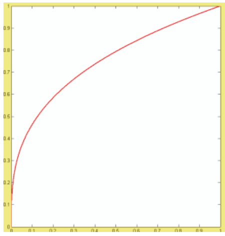

Fig. 15

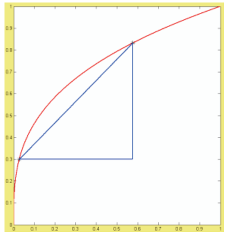

Fig. 16

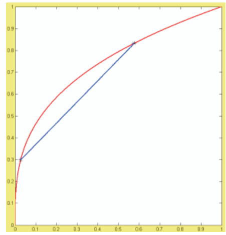

Fig. 17

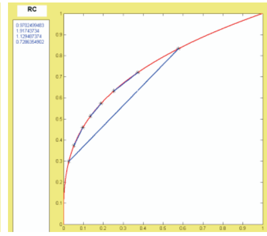

Fig. 18

the configuration “increasing at a decreasing rate” (Figure 15), first using algebraic considerations and then translating them visually:

Pr.: If a variable moves from position a to position b, which algebraic operation measures its change?

St.: Its difference, that is subtraction $b - a$.

Pr.: When x moves from a to b and you deal with a change $y = f(x)$, the dependent variable moves from $f(a)$ to $f(b)$, hence the change of the dependent variable is measured again by subtraction $f(b) - f(a)$. We need to consider a relative magnitude, a kind of comparative index, a way to measure the change of the dependent variable not by itself but with respect to the independent one. How do we express algebraically this idea? Will addition or multiplication of changes of both variables do?

St.: No, those operations do not say anything about change versus change. Since you want to make comparisons between measurements, you should put them in proportion.

Pr.: Algebraically speaking?

St.: We need to consider the ratio between both measurements; that is, we use division to calculate the quotient $(f(b) - f(a))/(b - a)$.

Pr.: Thus, this ratio is an algebraic formulation of rate of change between a and b and allows to associate a number to it.

St.: (Uneasy) Too many words to say the same thing: ratio, rate and rate of change.

Pr.: Well, let us say that a ratio is a comparison of two numbers or measurements. A rate is a special ratio in which the numbers or measurements may be expressed in different units. Our intuitive idea of rate of change is expressed algebraically as a rate, where a division may be executed. Remember that the symbol “/” refers to a fractional part of a single entity but also refers to a relationship between two quantities and last, but not least, the symbol is used to refer to an indicated algebraic operation, the division.

St.: All right.

Pr.: Thus given two different functions, fixing a and b and calculating their respective rates of change, which one is larger?

St.: The one with the bigger numerator, that is, the one which makes the jump of the dependent variable bigger.

Pr.: Now we go visual. If you draw the graph of a change f (Figure 16), what visual element corresponds to this quotient, that is, to the rate of change there?

St.: (Pointing in the screen) The tangent of the angle with the x-axis of the segment joining the points $(a, f(a))$ and $(b, f(b))$, that is, the inclination of the segment or the slope of the corresponding secant line.

Pr.: Please, identify in the figure all relevant information concerning inclination.

St.: The segment is the hypotenuse of a right triangle where $b - a$ and $f(b) - f(a)$ are sides and the quotient stands for the inclination of the hypotenuse or the slope of the secant line.

Pr.: Since we are only interested in inclination, may we dispense from the sides in the visualization as only the hypotenuse is relevant?

St.: (Figure 17) All right.

Pr.: Is this inclination representative of the steepness of the curve between abscissas a and b?

St.: In a way, yes, the better the closer both abscissas are. Concerning your question about two different changes, the bigger the quotient, the bigger the inclination, the bigger the rate of change.

Pr.: If you start drawing segments corresponding to different subintervals between a and b (Figure 18), observe that some have inclination larger than the rate of change between a and b and some smaller. How do you think the inclination corresponding to the rate of change relates to all other possible inclinations?

St.: As a kind of average?

Now we offer him our tool, where given a function, he proceeds to draw segments in the visual screen and rates of change in the computational one (Figure 19).

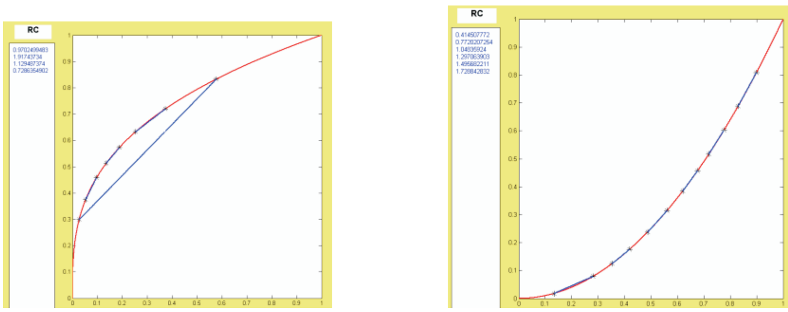

Fig. 19

Pr.: Numerically speaking, what do you think is important to know concerning the evolution of rate of change along this path?

St.: Its sign, as indicative of inclination.

Pr.: What about growth or decay of their values as we go from left to right?

St.: True, as an indicative of the degree of concavity.

Pr.: Right. What do all calculations of rate of change have in common in this path?

St.: They are all positive and their values evolve downwards from left to right.

Pr.: Now to the other configurations.

St.: Increasing at an increasing rate: positive also; the evolution is from less to more.

Using the tool, he deduces all signs and evolutions of the rest of configurations. Asking him to adjoin configurations in all sort of manners, the transition points of juncture are described such as follows "*if I paste an increasing path at decreasing rate with a decreasing path at decreasing rate, it is clear that at right and left sides of the juncture point the rates of change approach zero*" and similar comments on the other possibilities.

5.4 Level 3 (*new problem, new ideas*)

Pr.: Rate of change can be appreciated in somewhat long stretches, but we need to go local to know what happens exactly at the juncture point.

St.: You mean rate of change at a point? Everything said suggests that in the former picture this rate of change is 0 at the juncture point.

Pr.: Yes, once we understand what local rate of change means. Let us explore it: to define rate of change we need two points and, to calculate it, we need to assign a number to the inclination of the segment joining both points of the curve calculating a quotient. Unfortunately, algebraically does not makes any sense since being the points the same, the quotient is 0/0 which makes impossible to assign a concrete number to this quotient. Agreed?

St.: Yes, because division undoes what multiplication does: any number will do as 0/0.

Pr.: How would you asses this problem visually?

St.: Both points being the same, the segment collapses to a single point and it does not make sense to talk of the inclination of a point.

We propose to understand our dilemma by means of an interpretation of change and rate of change: if $f(x)$ stands for the distance from an origin, at time x, of a particle moving in a straight line through that origin, the rate of change $f(b) - f(a)/b - a$ is known as the average speed or velocity or speed of the particle between times a and b. We discuss why knowing average speed will not give any idea of what is happening at any particular instant and, by means of a simple example, we ask him to adventure why closer estimations of average speed might be better estimations of what happens at a specified instant.

Pr: I offer you the following suggestion: instantaneous speed is the number approached by average speeds as the intervals of time over which the average speeds are computed approach zero. Understood?

St.: Hmm ..., I am in the situation where I know how to compute the speed for each tiny temporal interval. Right?

Pr.: However ...

St.: The speed changes over each tiny interval and so I only get an approximation to the correct answer at the instant in question.

Pr.: Correct. But the key idea is that the smaller you make the tiny intervals used in your computation, the more accurate you will be able to compute the actual speed.

St.: (Thinking) I see, the closer I stay to the time in question, the better the approximation to the actual speed.

Pr.: How does this idea translate visually?

St.: (Uses pencil) Select points getting closer to the point in question, draw the corresponding segments ...

Pr.: Please do it using our tool.

St.: (Figure 20) Again, the segments collapse in a point ... (disappointed). There is no way out ...

Pr.: What if you substitute the segments by the corresponding secant lines?

St.: (Figure 21) Let us see: the inclinations of the segments are the same as the slope of the corresponding secants. Ah ..., if I select many points approaching the fixed one, those secants tend to glue into a line ... a kind of secant for which the two points come closer and closer together until they coincide.

Pr.: Is it a visual end to this procedure of taking more and more secants?

St.: Well, I do not know if there is an end to the procedure or not, but it looks as if the tangent line to the curve at this point is obtained.

Pr.: Visually, at least, instantaneous speed will correspond to ...

St.: The slope of the tangent line. I see, changing segments by lines is the way out. But, if this is the conclusion, why not forget about secants and simply zoom several times until a straight line is produced and measure its slope?

Pr.: Well, you mentioned it: the secants are necessary to arrive to the tangent line. But, since we have to put numbers to our figures, we have to find a way to calculate this slope. Your suggested procedure provides a very rough estimation. On the other hand, the secant's beam

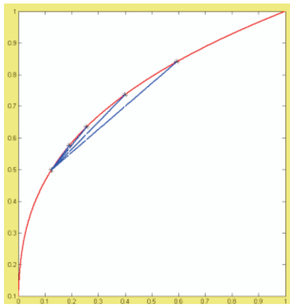

Fig. 20

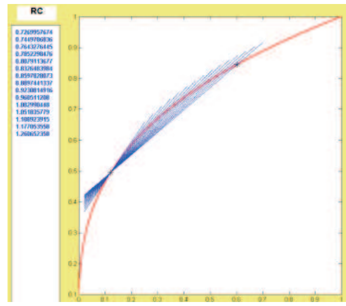

Fig. 21

approximation procedure can be converted to a numerical approximation one, which is shown in the computational screen (Figure 21). Please, make a guess for that slope.

Since we are going to deal with situations where the variables are not necessarily displacement versus time it is convenient to replace the concept of instantaneous speed by that of **local rate of change** which geometrically is the slope of the curve at any point P, understood as the slope of the tangent line to the curve at P. He understands that our former considerations do not depend of the temporal setting chosen. If f stands for the function and if a denotes a value of the independent variable, we shall ask him to use the symbol $\boldsymbol{f'(a)}$ to denote the local rate of change at the value a and we shall call the **derivative** of f in a. We point out that we have arrived to a visual realization of what the local rate of change or derivative is and envisaged a raw method to calculate it by a process of approximation.

We put the tool to his disposal, ask him to type familiar algebraic expressions, select points, draw corresponding secant beams and we encourage him to experiment numerically providing conjectures for the local rate of change at that point to be compared with those provided by the tool.

Concerning the visualization of local rate of change as a slope, we invite him to switch contexts and we ask him to produce by hand a pictorial representation of the growth of a population with time where verbal considerations have to be translated visually.

Pr.: Is it natural to assume that the rate of growth of the population is proportional to the existing population; in another words a larger population produces proportionately more offspring than a smaller one.

St.: Yes, hence an increasing path: small growth at the beginning and speeding up.

Pr.: Since rate of growth of the population P is proportional to P, suppose the constant of proportionality is 2, use paper and pencil and draw small tangents at forthcoming points.

St.: (Starts to draw small tangents with increasing inclinations) As P increases the slope of the tangent line to the curve increases too. Its aspect depends on the constant chosen. Hmm . . . , the curve becomes steeper and steeper, that is, P rises faster and faster, with no end in sight.

Pr.: Indeed. But isn't also reasonable to assume that physical conditions set an upper limit, let us call it L, to the population and that the rate of growth of the population is also proportional to the possibility for popular expansion, that is, to the difference between L and the existing population?

St.: (Redrawing) Mm . . . , right. Up to a point slopes have to interrupt their increase and start going down . . . the curve has to bend at some time and the slopes of tangents become smaller and smaller, that is, the curve is flattening and changes concavity: well, that has to be the curve (see Figure 22); it is an example of juxtaposition of increasing curves but with varying rates of change.

Pr.: Right. How and when it flattens depends on . . .

St.: I guess on the size of the other constant of proportionality. But not all populations increase in reality. Can it happen that the population actually decreases?

Pr.: Good question. Then our model of population growth has to altered to take into account other factors such as predators, illnesses, conflicts, natural disasters and so on.

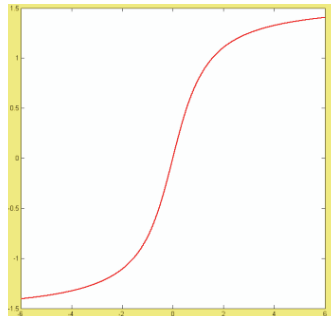

Fig. 22

Moving to the general context, now we inquire on the signs of the local rate of change.

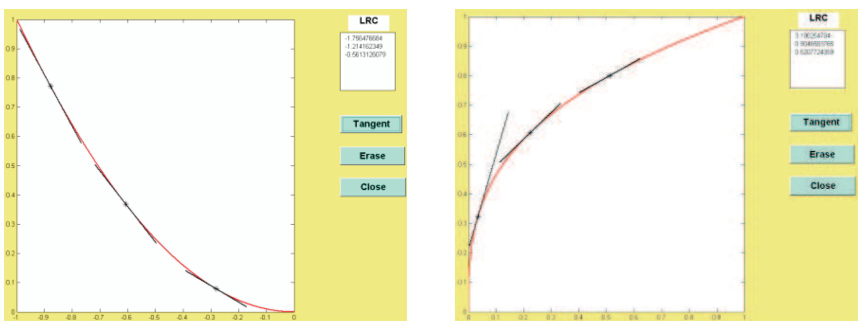

Fig. 23

Pr.: Changing from rate of change to local rate of change or derivative, do you stand by your findings regarding the sign of them in all our labelled paths? (show Figures 23).

St.: (Using his hands to depict tangent lines) Let us have a look. Derivative sign is the sign of the slope of the tangent line to the curve at the point considered, hence both increasing paths have positive derivative, both decreasing paths have negative derivative.

Pr.: Thus, if a path depicts an increasing change, derivative at every point is positive.

St.: Indeed.

Pr.: What about if you do not know the path, but somehow suppose you know that the derivative is positive at all points?

Moves his hands up and down and takes paper and pencil to selects a point in the plane where the curve should start and begins to draw at several points of the grid small segments suggesting tangent lines with positive slopes. We ask him to explore all possibilities. After several trials, all curves presented follow both "increasing" paths or juxtaposition of them (Figure 24).

St.: If the derivative is positive at every point, the curve depicts an increasing change. If the derivative is negative, we are talking of a decreasing change. Hence it goes both ways . . .

Pr.: As the independent variable x changes, so does the dependent variable $f(x)$, but also the derivative $f'(x)$.

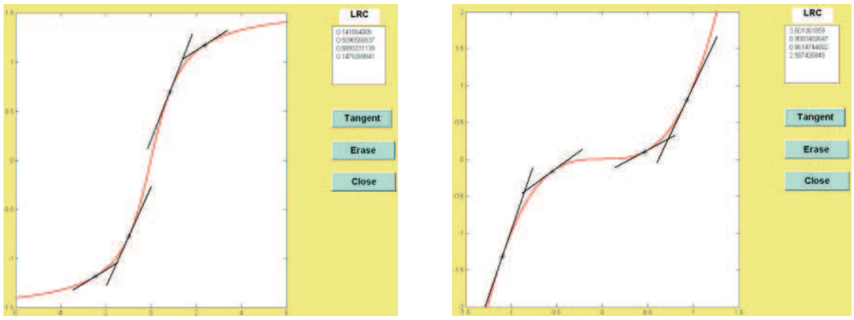

Fig. 24

St.: Yes.

Pr.: As we have seen, the sign of f' monitors the change of f (f' positive, f increasing and f' negative, f decreasing). What monitors the change of f'?

St.: The derivative of f'?

Pr: Does it make sense to talk about the derivative of the derivative?

St.: Hmm, if $f(x)$ is given by a formula and its slope keeps changing from point to point, there should be an x in the expression for the slope accounting for this change.

Pr.: Do you mean that, if the derivative has a value at all points, then it's clear that it's also a function?

St.: (Unsure) Well . . . , yes

Pr.: So, for our function f, we can talk about the function f', which is its derivative, where $f'(x)$ stands for the derivative of f at the given abscissa x . . .

St.: Thus it makes sense to talk about the derivative of the function derivative.

Pr.: Let us call it the second derivative and write f'' to denote it and $f''(x)$ if specified at any abscissa x.

St.: I understand its algebraic meaning, but a further explanation would be nice.

Pr.: Turning again to our context of displacement versus time, if you were walking along and decided to walk a little faster, the first derivative (your speed) would also increase. The second derivative measures the change in the first derivative per unit of time. The common language for the change in speed per unit of time is acceleration: if you are driving a car, when you step on the gas, you "change" your speed. How fast it is changing is the idea of a second derivative.

St.: Right.

Pr.: And if your second derivative is changing, guess what we call that?

St.: The third derivative?

Pr.: Sure. We don't have names for all the different derivatives, but you can see how each one is related to the one before it. By the way, the third derivative of position with respect to time, which is the change in acceleration with respect to time, is named for what happens when you experience a sudden change of acceleration: you feel a jerk.

St.: Visually?

Pr.: Graphically you should also try to understand that if you graph y versus the second derivative of x, then the slope of the curve at any point is the third derivative of y with respect to x.

St.: And you could go on and on.

Pr.: Yes, but it makes no much sense in this context.

Back again to the general context:

Pr.: Let's go back to the second derivative. My question is the following: since there are essentially two different ways of f increasing, is it possible that the sign of f'' may distinguish between them?

St.: (Drawing confidently) Yes. If f increasing at a decreasing rate, f' is positive and f'' negative. If f increasing at an increasing rate, f' positive and f'' positive.

Pr.: If f decreasing ...

St.: If f decreasing at decreasing rate, f' negative and f'' negative. If f decreasing at increasing rate, f' negative and f'' positive.

Pr.: Graphically f'' distinguishes between ...

St.: Concavities up and downwards.

Pr: (Showing several figures 25) If you join configurations in all manners possible, what happens at the juncture points which are called local maxima and minima and inflection points?

St.: At local maxima and minima, f' is zero. At inflexion points, f'' is zero.

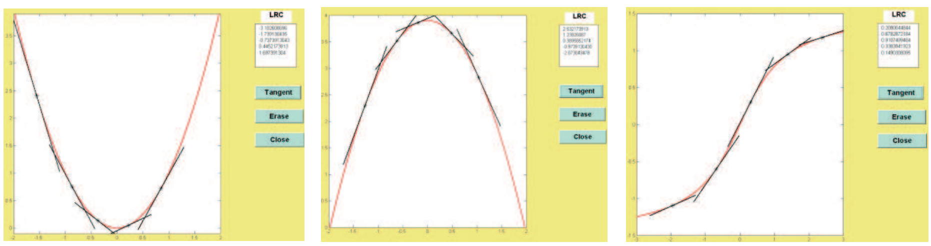

Fig. 25

We produce images showing graphs for f, f' and f'' for some algebraically unspecified function (a simple example presented separately and a more complex one presented simultaneously for the use of recently acquired knowledge, see Figures 26) and we ask him to identify which is which: no trouble with our three first graphs, but it takes some time to process the former information to come with the right answer for the last picture.

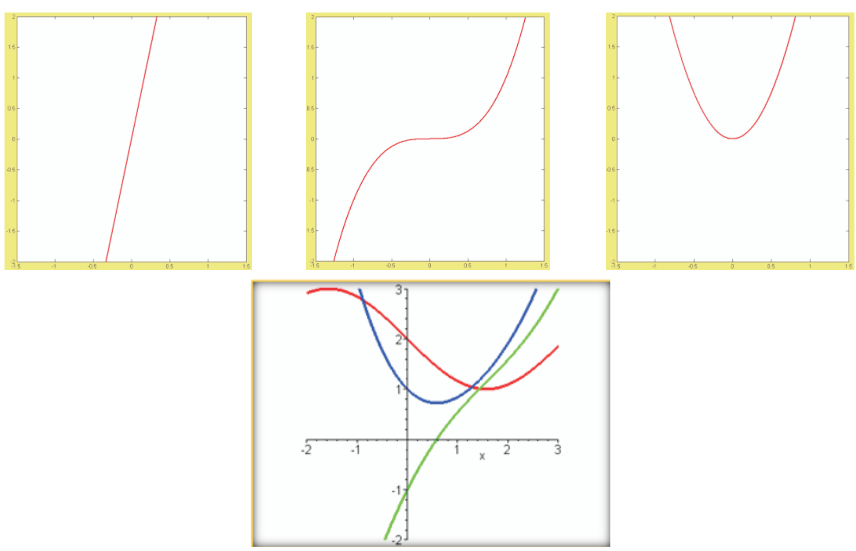

Fig. 26

We proceed with several considerations on what information should follow. First, we emphasize that it is clear that, if we have an algebraic expression for the function but no tool available, and, if we knew how to calculate derivatives at every point, the considerations above would be of great help to produce a graph of the function by determining stretches of growth and decay and maxima, minima and inflection points.

Then, we point out that we cannot escape the fact that, in absence of an algebraic method of calculating derivatives, but having an algebraic expression for f and being able to draw its graph using our tool, there are limits to what we can do calculating derivatives as approximate slopes instead of having an algebraic expression for them (showing different examples of oscillating curves (Figures 27) where it is difficult to see if there derivative at a point and which one it is and whether it contravenes his idea of what a tangent line should be). For a Socratic dialog leading to this end framed in van Hiele's model, we refer the interested reader to [1].

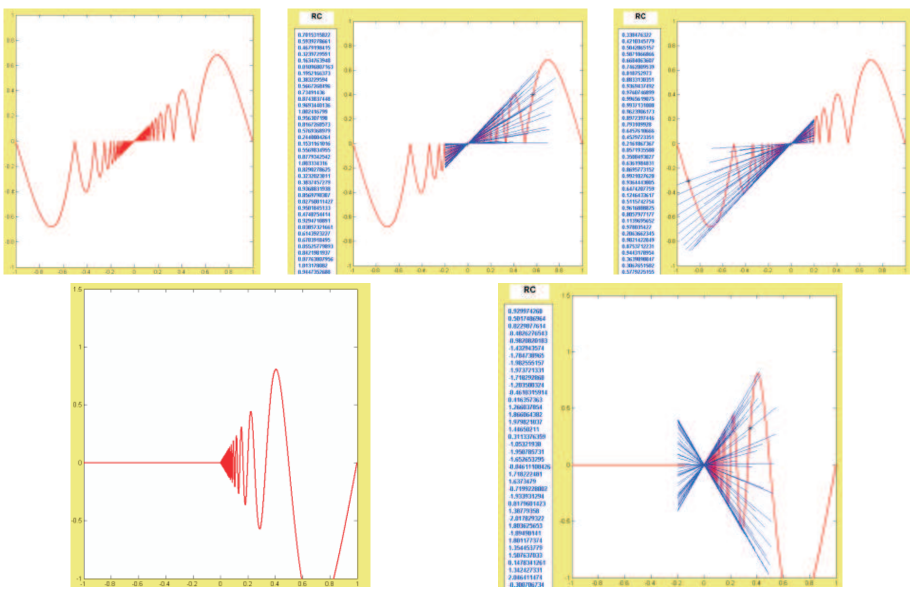

Fig. 27

He understands that this gets really tiring when you need the derivative at many points on the graph, because it is rather cumbersome to proceed to calculate the slopes of approximating secants and guessing the final value of the tangent slope, not to mention that secants can be taken in many different ways and we need some warranty that the same approximating value would be obtained independent on how we proceed. Furthermore, our approximating value is rather limited if we need better accuracy. The moral of the story is that our tool is not enough. What is the problem? Clearly, to develop algorithms to calculate derivatives efficiently a geometrical understanding of what a derivative means is not enough; we are in need of a precise algebraic definition of what a derivative is. All ingredients are present except the algebraic operation which treats an indefinite sequence of numbers (a dynamic process) and puts it to rest by showing a precise number which intuitively stands for the number the sequence approaches or gets close. In other words, we need to define algebraically what "*approach*" or "*get close*" means. For a Socratic dialog leading to put all those considerations under the optic of a visualization tool, study how those images translate in mathematical statements favoring the use of logical quantifiers and framing the whole study in van Hiele's model allowing the study the cognitive obstacles that arise, we refer the interested reader to [2].

Once the definition of derivative is understood, we tell him that in due time he will be able to proceed in the usual classroom manner to develop those algorithms which frees him of conceptual thinking, proceeding with procedural thinking converting a deep idea in a mechanical easy procedure. Once this mastery is acquired, the problems to deal with will go in three steps (i) figure out which quantities we're interested in (ii) find a relation between the quantities and (iii) calculate their derivatives to find a relation between their rates.

As a résumé and moving to the most popular context, we point out that to have a method for calculating instantaneous speeds we need working from a formula relating distance and time. We discuss that, in order to treat the concept of instantaneous speed, we have idealized space and time so that we can speak of something existing at an instant of time and at some point in space. The layman finds his imagination and intuition strained by the notions of instant, point, speed at an instant and he might prefer to speak of speed during some very small interval of time. Yet mathematics produces its idealization not merely a concept but will allow in time for, not a simple guess, but a formula for speed at an instant that is precise and more readily applied than is the notion of average speed during some sufficiently small interval. We comment on how those considerations enlighten what we may call the paradox of mathematics: that by introducing seeming difficulties it simplifies and renders easy a truly complex problem. We promise him that he shall encounter this paradox in other guises along his mathematical life.

6. Level descriptors

6.1 Level 0: The interviewee

0.1 Sees the need of a number system, identified as the rational numbers, for measuring purposes.

0.2 Accepts the possibility of operations which may be performed as many times as desired, like taking averages. Accepts the existence of infinity, as opposed to finite.

0.3 Allows the visual representation of the number system in a line, admitting the existence of enough points to do the job.

0.4 Starts to grasp what a model is as an idealization of an idea, such as space, taking some of its characteristics into account disregarding others.

0.5 Accepts an incomplete view of the continuum.

0.6 Understands that time can be treated with the same model as space.

0.7 Understands the idea of variable and accepts that change can be modelled putting variables into correspondence.

0.8 Agrees that the idea of function as a rule serves our modelling purposes and understands that the first variable conditions the behavior of the second one. Concerning its algebraic representation, the existence of a formula is primordial for him/her and resists the idea of rule without a specific computing mean.

6.2 Level 1: Words are translated into pictures and the process of recognizing visual characteristics of change starts. The interviewee

1.1 Describes changes as functions with verbal statement such as: when x grows, $f(x)$ changes either growing or decaying. Explores different types of changes.

1.2 When asked to translate pictorially what is described verbally as different change phenomena (a limited number of them such as displacement vs. time), the graph of a function is his way of picturing it, even if a formula for that function is absent. Recognizes that the pictorial representation of change can be understood as a dynamic process.

1.3 Recognizes that the growth phenomenon has a pictorial representation which is different than the decay one. Explores all possibilities of change for increasing vs. increasing variables, from where he deduces all possibilities in a change with increasing vs. decreasing variables.

1.4 Recognizes that different degrees of steepness may occur in the graph, uses temporal expressions like quick and slow to describe them and connects those pictures to verbal conditions stated in the description of the change under study. Different rates of change are for him a compact way of referring to them, understands what they mean verbally and visually, and recognizes their significance in explaining change.

1.5 Armed with newly acquired visual prowess and being confronted with verbal descriptions of different changes where rate of change is significant, is able to translate the changes into pictures, where different degrees of steepness are visible and, confronted with a graph, he is able to verbalize qualitative aspects of the change represented by it.

1.6 Having distributed all possible changes in simple paths, looks for their minimal expression (the onions) and is able to attach labels to all essentially different ones. Recognizes that, among those sharing the same label, concavity distinguishes between them and explains why in terms of rate of change.

1.7 Having been introduced to our manipulative tool, recognizes the possibility of decomposing a general graph into elementary labelled paths. Given the mirror symmetry exhibited by those elementary paths, accepts that a study of one of the elementary paths may lead to export our findings to the other ones and, hence, to the global graph.

1.8 Once confronted with steady change, when dealing with a general change, relates steepness with inclination of some segment attached to the graph, recognizing that the slope of the secant line may play a role in a future attempt to measure rate of change.

1.9 Admitting that secant and graph may look quite different, advances the idea that not so much if both points are "sufficiently close", leading to recognize that a graph might be locally straight, that is, a graph zoomed around a point looks like a line, such action being performed by our tool.

1.10 Admitting that there is no way to talk about the slope of a graph, understands that local straightness can be seen as a measure of inclination of the graph locally.

6.3 Level 2: Ability to deduce properties of functions from the analysis of the visual characteristics grasped in level 1 is achieved. The interviewee

2.1 Understands the importance in attaching numbers to the study of rate of change. For some changes described as tables, is able to deduce where growth and decay occur and at what pace.

2.2 Is able to attach numbers as slopes of different straight lines. Studying the graph of the function $\tan(\cdot)$, deduces that there are negative slopes and what they mean.

2.3 Understands that change of one variable is measured by the algebraic operation of subtraction and deduces that change of the second variable with respect to the first involves division. A suitable quotient is the realization of the concept of rate of change.

2.4 Fixing one of the elementary paths and provided with the tool, understands rate of change visually as the inclination of a segment, which is the hypotenuse of a right triangle and is able to discard visual elements which are not substantial.

2.5 Looking at both screens of the tool, is able to switch from visual to arithmetic and the way round studying different rates of change along the graph and deducing what those rates of change mean in both contexts.

2.6 Given an elementary path understands that there are two significant pieces of information: sign of rate of change and numerical evolution of it from left to right. Confirms that, on all possible paths belonging to the chosen elementary one, all rates of change share the same sign and evolve likewise. Produces mature explanations on why that happens.

2.7 By deduction, is able to export similar conclusions, albeit with different signs and evolutions, to the rest of elementary paths by using the mirror symmetry.

6.4 Level 3: Properties deduced in level 2 are used to obtain conclusions about the derivative of a function. The interviewee

3.1 Understands that, for getting information at the juncture points which appear when we paste elementary paths in the global picture, we have to go local and concludes that serious problems appear when we try to speak of local rate of change, either algebraically or visually.

3.2 By moving to the displacement vs. time context, understands what local rate of change may signify, once the key idea of approximation is apprehended.

3.3 Moving to the general context again, concludes that considerations in 3.2 indicate the route to follow to solve this problem: convincingly, in the visual setting and tentatively, in the algebraic setting.

3.4 By enlarging segments to lines, is able to obtain a visual answer by looking at local rate of change as the slope of the tangent line at this point of the graph, which can be obtained either by zooming locally or by considering this line as the result of a process of visual approximation.

3.5 Understands that he has arrived to a visual explanation, although not an algebraically sound one, but he is left with a visual approximation procedure which can be transformed in a numerical approximation one allowing the establishment of convincing guesses, apparently.

3.6 Accepts a change of nomenclature: from change/local rate of change to function/derivative.

3.7 Once established the relationship between slope of the tangent line and local rate of change or derivative, he is able to produce graphs of functions when there is a verbal description on how the derivative behaves.

3.8 Concludes that sign and numerical evolution considerations stated in 2.6 and 2.7 hold if rate of change is substituted by derivative.

3.9 Accepts that the derivative may be seen as a function and therefore concludes that the second derivative monitors the change of the first and assigns signs of the second derivative to each of the elementary paths.

3.10 Translates his conclusions to the visual setting in term of concavities and pasting elementary paths concludes on what happens at maxima, minima and inflection points.

3.11 Acknowledges the importance of being able to calculate derivatives at desired points and how tiring this procedure might be if limited to our tool.

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