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INVESTIGATION AND PROOF OF A PROPERTY OF INTERIOR ANGLE BISECTOR IN A TRIANGLE

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Abstract. The paper is focused on educational practices which may enable students to discover the property of interior angle bisector in a triangle with the help of exploration of dynamic constructions. Interactive geometry software is used for exploring relationships between geometrical objects, for transition from conjecture to verification of a statement, and for development of discovered relationships. Second part of the paper presents various methods to prove the formulated theorem based on the similarity of triangles, trigonometric law of sines, and analytic method.

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1. Introduction

The properly selected problem situations in mathematics teaching could motivate students to investigation of object properties and relationships between objects, in order to identify unclarities, formulate questions and make conjectures. Students' mathematical abilities focused on reasoning and justification of statements are developed by verification and proof of formulated conjectures, whereby logical thinking of students is stimulated. Geometry provides many interesting theorems, which can also be proved in mathematics teaching and which may offer students new perspectives that reveal the beauty of geometric relationships. In Principles and Standards for School Mathematics of NCTM [10], it is stated: "Geometry is a natural place for the development of students' reasoning and justification skills, culminating in work with proof."

To meet these objectives in mathematics teaching, we can use mathematical theorems which can be discovered and proved using different approaches. We can reach deeper understanding of discovered relationships with explanation of their interesting implications and applications. These objectives are also declared in Principles and Standards for School Mathematics of NCTM [10]: "Students who can use many types of reasoning and forms of argument will have resources for more-effective reasoning in everyday situations."

Interactive geometry software (IGS) provides advanced means of investigation of properties of geometrical figures and a new access to direct manipulations of geometrical drawings. Experimentation and investigation of dynamic constructions allow discovering invariant properties of drawings and relationships between geometrical objects. Manipulation with dynamic constructions promotes new didactic possibilities to discovery patterns and relationships between geometric objects. Diković [3] also emphasizes this property of technology: "The goal is to use technology to provide an environment for active exploration of mathematical structures through multiple representations, or to show students some aspects of mathematics that are not possible with pen and paper." Exploration of more specific examples of geometric shapes facilitates students to perceive and understand geometry in dynamic environment which vitalizes the arena of experimental mathematics and opens up mathematics classroom into scientific-like laboratory [9]. IGS provides students possibilities to obtain empirical evidence as the source for insight that can form the ground for further conjectures and generalizations. As Bruckheimer and Arcavi [2] point out: "Geometrical theorems in 'microworlds' can become much more than propositions waiting to be proven, they can become projects for investigation, which rely on the ease with which many instances of a proposition can be obtained, analyzed, measured, and compared."

A new generation of IGS has been developed in recent years. The systems like GeoGebra (`www.geogebra.org`) represent multi-platform software that combines dynamic geometry, calculus, and algebra. New geometric software Geometry Expressions (`www.geometryexpressions.com`) introduces dynamic symbolic geometry environment that provides advanced possibilities to generalize and justify geometric relationships. This geometric system can automatically generate algebraic expressions from geometric figures. As Todd [13] notes: "A Symbolic Geometry system such as Geometry Expressions is an effective bridge between the two dominant mathematics education technologies: dynamic geometry and CAS." Algebraic formulas provide ways to clear patterns more readily and use symbolic geometry in mathematical reasoning and generalization.

The paper describes educational practices that may lead students to discovering the property of interior angle bisector in a triangle using construction of the figures in paper and pencil environment and exploration of dynamic constructions. We tried to outline the possibilities of using IGS for transition from conjectures to justifying claims and for development of discovered relationships.

The investigation of dynamic constructions should not reduce the justification of statements on the basis of logical considerations in mathematics teaching. As Izen [7] claims computer-based discovery activities in a geometry course should be followed by rigorous proofs of the conjectures. This paper also uses the idea of using mathematical theorem which may be discovered by students on their own with the help of exploration of dynamic constructions:

(1) *The bisector of any angle of a triangle divides the opposite side into two segments that are proportional to the other two sides.*

This theorem is proved in the above mentioned paper by means of similarity of triangles and relationships between angles, which are determined by a line intersecting a pair of two parallel lines. We would like to extend the ideas presented in this paper and to describe other approaches to investigation of this theorem in mathematics teaching. Main objectives of this paper are to design a lesson plan for guiding students through exploration of constructions to discovery and verification of relationships between geometrical figures described in theorem (1) and to apply various methods to prove it starting with the use of similarity of triangles up to analytic proof based on vector calculus. IGS will be used in order to make patterns visible more readily and to enhance student learning in certain stages of the lesson. Discovered and proved relationships are used for derivation of an interesting implication about properties of Apollonius circle at the end of the paper.

Scheme of teaching focused on discovery, justification and verification of the property of the interior angle bisector in a triangle is divided into four parts. Lesson plan is based on a model of the interactive geometry approach described by Scher [12]. This model contains four phases that lead naturally from visual evidence to deductive argument: visualization, hands-on exploration, dynamic software investigation, deductive reasoning.

2. Initial exploration in paper and pencil environment

Due to the fact that the ratios of lengths of segments are referred to in theorem (1), it would be appropriate to repeat the construction of segments with the given ratio of their lengths derived from similarity of triangles before the beginning of the lesson. Given a triangle ABC, let points F, G be the midpoints of sides AC, BC. According to triangle similarity test SAS, triangles ABC and FGC are similar. Therefore FG and AB are parallel and we have $AC/BC = FC/GC$. Let also the length m of the segment be given. How could a segment with length x be constructed so as to have $x/m = AC/BC$? We construct the point M on ray CB so that $CM = m$. Then we draw a line $p \parallel AB$ through the point M and we construct the point $K \in CA \cap p$, for which $CK = x$. According to triangle similarity it can be easily shown that $CK/AK = CM/BM$. The final construction is shown in Fig. 1. We will use this knowledge to check the correctness of theorem (1) using IGS, and also for its proof.

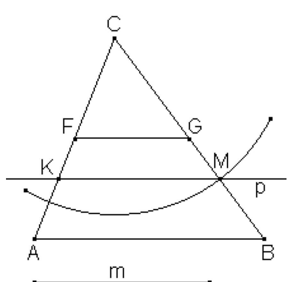

Fig. 1

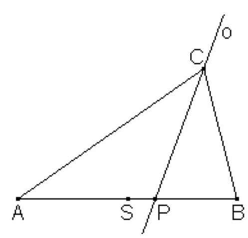

Fig. 2

The following problem is presented to students at the beginning of the lesson: *The line o is angle bisector of interior angle $\angle C$ in triangle ABC. It intersects the* *opposite side AB in the point P. Find relationships among AP, PB and lengths of sides of triangle ABC, which contain the vertex C.*

Visualization

At first, students should make a construction in paper and pencil environment for the understanding of the problem (see Fig. 2).

The point S is the midpoint of side AB. Visualization of the investigated objects can help students in the first phase of problem solving focused on the exploration of a simple special case: *If we construct the angle bisector of interior angle $\angle C$ in triangle ABC, when is the point P identical with the point S?*

Isosceles triangles with base have this property. Can this case also happen in other triangles? If $P \equiv S$ then we can take point D on angle bisector o so that $DS = SC$. Thus triangle ABC can be completed into a parallelogram $ADBC$. One can easily prove that the parallelogram $ADBC$ is a rhombus. Hence triangle ABC has to be isosceles.

Hands-on exploration

After the special case resolution students will find out where the point P lies, if the explored triangle is not isosceles. Students could construct another triangle ABX and angle bisector o' of interior angle $\angle X$ so that the point $X \in o$ (see Fig. 3). Also in this case the intersection point between o' and the side will be labelled P.

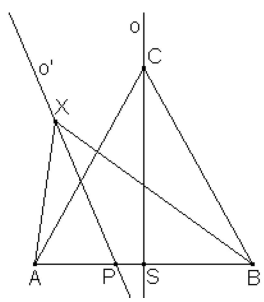

Fig. 3

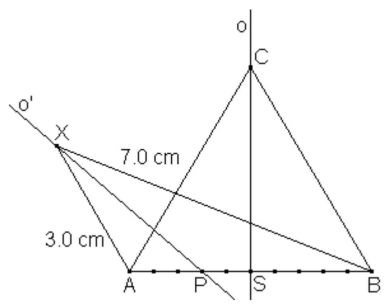

Fig. 4

Students could look for an answer to the following question: *How does the position of the point P on side AB relate to the position of the point X in one of the half-planes oA, oB?* This problem is more difficult than the first partial problem. Students would be able to notice that the points X and P lie in the same half-plane determined by perpendicular bisector o.

At this stage of the investigation students might feel intuitively that the distance between points X and A influences the position of the point P. The purpose of the third construction is to explore this dependence more precisely. This construction also leads students to focus on the ratio of the lengths of sides AX and BX. Side AB with length 5 cm is divided into 10 equal parts. Students are required to construct a not isosceles triangle ABX with integer sides, so that the following equation holds: $AX + BX = 10\,\text{cm}$. There are several pairs of integers that can be chosen for the lengths of sides AX, BX (for example $(1,9)$, $(2,8)$, $(3,7)$, $(4,6)$, $\dots$). It can happen that triangle ABX does not exist for any chosen pairs of integers (e.g. $(1,9)$, $(2,8)$). Figure 4 shows triangle ABX with $AX = 3\,\mathrm{cm}$ and $BX = 7\,\mathrm{cm}$. The point P divides side in the ratio $3:7$.

As a result of exploration of different particular cases, students could be able to make conjectures which express the following relationship: when one side is made smaller compared to the other side of the triangle, the point P approaches the vertex lying on the shorter side. At this stage of problem solving a transition from this conjecture to comparison of the ratios of the lengths of the corresponding sides of the triangle and AP, PB is natural.

The proposed lesson plan was partially tested in classroom practice for 15-year old students at a Slovak grammar school (first grade) during one class. After construction of the triangle ABC with the angle bisector of interior angle $\angle C$, the students solved the initial problem, correctly finding that the point P is identical with the point S in isosceles triangles. Most students reasoned that the angle bisector of interior angle $\angle C$ is identical with a perpendicular bisector of side AB. The majority of students found out and noted down in their own words the fact that points X and P lie in the same half-plane determined by perpendicular bisector of side AB.

Many students were not convinced that construction of one specific example was sufficient enough for the solution of the last problem of the Hands-on exploration phase. They constructed two special triangles ABX with integer sides for which the term $AX + BX = 10\,\mathrm{cm}$ applies. But even the construction of two specific examples was not sufficient for most of the students to formulate a statement corresponding to theorem (1). Only two students (the 1/8 of number of students in the class) were able to express the general relationship between the lengths of segments: $AP/PB = AX/BX$. Many students' conjectures were equivalent to the statement: numbers of parts expressing the distance of the point P from vertices A, B are equal to the lengths of sides AX, BX. Some students characterized the position of the point P more accurately by means of relationships: $AP = AX/2$ and $PB = BX/2$.

IGS could be incorporated into the exploration of this problem to test several specific examples which would help to revise errors in students' statements. Students can use the dynamic nature of the software to help them gain confidence in the conjectures they have made. Additionally, students can ask other questions by making and exploring the constructions to which finding answers is rather difficult in paper and pencil environment. For example, we can ask the following question: *Where could other vertices X of triangle ABX lie so that angle bisector o' of interior angle $\angle X$ also divides side in the ratio* $3:7$? Colette Laborde [8] refers that the role of visualization in geometry remained hidden in paper and pencil environment because of the very low level of reliability of the production of drawings and the very small number of possible experiments. IGS would allow students to quickly make accurate drawings of different triangles. Therefore we will proceed to investigation of dynamic constructions with IGS for the review of students' conjectures and for a deeper understanding of relationships between geometrical objects.

3. Investigation of triangles using dynamic constructions

Three dynamic constructions represent the basis of stimulative learning environment, in which students can explore property of an interior angle bisector in a triangle. The teacher prepares these constructions together with instructions for students before the lesson. According to Chazan and Yerushalmy [5] teachers using geometry construction program try to create experimental environments where collaborative learning and student exploration are encouraged. Subsequently, students would be led to discover and formulate theorem (1) with the help of exploration of different particular cases in dynamic constructions under teacher's guidance. The simple logical considerations will be used for the initial verification of theorem (1) in the third dynamic construction.

At first, students will explore different triangles using dynamic construction that corresponds to the construction in Fig. 3. Dragging the point C along the angle bisector o of interior angle $\angle C$, which is simultaneously the perpendicular bisector of side AB, forms isosceles triangles. The point P is identical with the point S. We draw a simple conclusion: if $AC = BC$, then $AP = PB$. If a vertex of a triangle does not lie on the line o (it is labelled as X), then the point $P \in o' \cap AB$ is always situated in the same half-plane determined by line o, in which point X also lies.

For the justification of this statement triangle ABX can be completed into a parallelogram $ADBX$ (see Fig. 5). The point S is the midpoint of side AB and it is also the intersection point of the parallelogram diagonals.

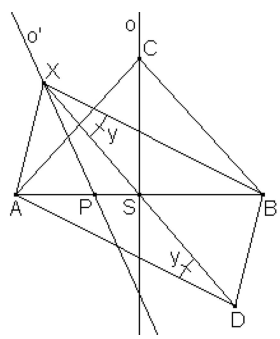

Fig. 5

Since $AD > AX$ in triangle ADX then $\angle X - \angle y > \angle y$ and thus we have $\angle y < (\angle X)/2$. Therefore the point $P \in AS$. Students can observe the position of the point P by changing the position of the point X: the shorter one length of changed side is to another changed side, the nearer the point P is to a vertex, which lies on shorter side.

The next dynamic construction is focused on the investigation of the ratio of the lengths of sides AX and BX. This construction is derived from the construction in Fig. 4. Students can explore the position of the point P in different triangles ABX with integer sides. Side AB with length 3 cm is divided into 6 equal parts (see Fig. 6).

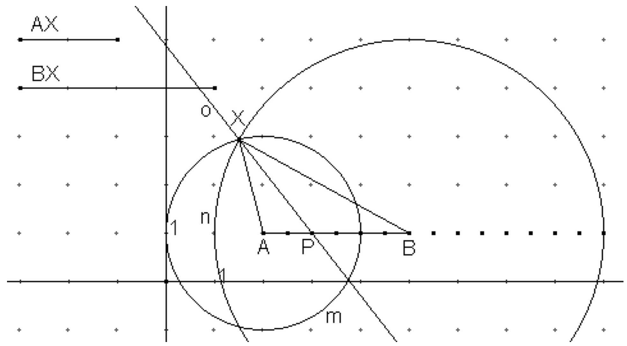

Fig. 6

There is a coordinate system with grid points of integer coordinates used. The lengths of horizontal segments AX, BX determine the lengths of the sides of triangle ABX. Therefore the radii of circles m, n are interconnected with them. The right end-points of horizontal segments AX, BX and the point B can be dragged only on the grid points in horizontal direction. The midpoints between two adjacent grid points are made on hidden ray AB. Students can easily construct different triangles with integer sides and focus on the cases when the point P lies in some of the marked points on side AB. Now they can return to triangles with the length of side AB equal to 5 cm, which were constructed in paper and pencil environment. Exploration of triangles with different side lengths should lead students to formulate statements corresponding to theorem (1).

This dynamic construction can be also used for solving other tasks which enable students' deeper understanding of relationships between the ratios of the lengths of the segments in the triangles. The first task involves a construction of a triangle ABX in which a length of side BX is not given: *Construct triangle ABX with sides* $AB = 6$ cm *and* $AX = 2$ cm *so that angle bisector of interior angle $\angle X$ intersects side AB in the point P which is at distance of* 1.5 cm *from the point A.* Now, side is divided into 12 equal parts and the following equation holds: $AP/PB = 3/9$. According to theorem (1) we have $2/BX = 3/9$ and the length of side is equal to 6 cm. After this calculation students can change the lengths of the horizontal segments AX, BX and the length of side AB in the dynamic construction (see Fig. 7).

The lengths of two sides of a triangle are not given in the second task: *Construct a triangle ABX with $AX = 4$ cm so that angle bisector of interior angle $\angle X$ intersects side in the point P which is at distance of* 3 cm *from the point A.* Students can use the lengths of the segments expressed in cm in solving this task. An isosceles triangle with AX and BX equal to 4 cm and AB equals to 6 cm is a trivial solution of the task. We use the ratios of the lengths of segments in a

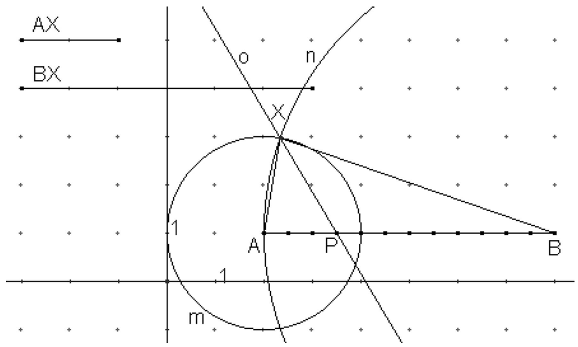

Fig. 7

triangle to find other solutions: $3/PB = 4/BX$. Expressing the ratio PB/BX we have: $PB/BX = 3/4$. The case $PB = 3\,\mathrm{cm}$, $BX = 4\,\mathrm{cm}$ leads to the above-mentioned trivial solution. From the equation $PB/BX = 6/8$ we have $PB = 6\,\mathrm{cm}$ and $BX = 8\,\mathrm{cm}$. If we substitute the ratio 3/4 by the ratios 9/12, 12/16, ... then we get other solutions.

The third dynamic construction can be used for the initial verification of theorem (1) with the help of relationships between similar triangles.

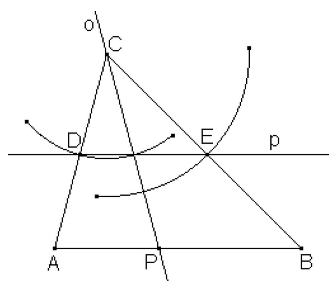

Fig. 8

We present to students a dynamic construction (see Fig. 8) with points D, E on sides of triangle ABC satisfying $CD = AP$ and $CE = PB$. If equality $CD/AC = CE/BC$ holds, then triangles CDE and CAB are similar according to test SAS. How could we verify by adding other figures or by determining properties of constructed figures, whether theorem (1) holds for different triangles ABC? For example, if we draw a line $p \parallel AB$ through the point D, then p has to pass through the point E. When we move the vertices of triangle ABC, line p always passes through the point E. Although the described method is not a proof of theorem (1), such tasks can develop students' abilities to use properties of figures and relationships between them for making good mathematical arguments. The purpose of this dynamic construction is also to direct students' attention to use triangle similarity in proof of theorem (1).

We have used this dynamic construction for exploration of different triangles in search of a counterexample, which would contest the general statement. If the general statement were false, we would probably quickly succeed in finding of a counterexample as a result of exploration of different particular cases with the help of IGS. Dynamic properties of IGS allow reasoning at a new level based on providing immediate feedback on errors. As Chazan [4] noted: the development of Cabri-like environments, on the contrary, contributed to the support of changes in teaching by stimulating conjecturing and the renewal of dialectical relationships between proofs and refutations. Of course the fact, that we find no counterexample, does not mean that the statement holds, and therefore rigorous proof has to follow.

The students of the experimental class used the second dynamic construction (see Fig. 6) to also try the examples when the length of side AB was 5 cm and $AX + BX = 20\,\text{cm}$ (for example $(8, 12)$) and $AX + BX = 15\,\text{cm}$ (for example $(9, 6)$). The majority of students were able to formulate a statement corresponding to theorem (1) after these explorations. The proof of theorem (1) was performed under the teacher's guidance using relationships (6) for areas of triangles APX and PBX in the last phase of the lesson.

4. Proofs based on similarity of triangles

Division of a segment into two parts in the given ratio displayed in figure 1 can be used for a proof of theorem (1). Without loss of generality we can assume that $AC \leq BC$ holds in triangle ABC. It can be completed by the point D, which lies in an intersection of ray BC and a line drawn through the point A parallel to the interior angle bisector o (see Fig. 9). These relations hold: $\angle CAD = \angle ACP$ and $\angle PCB = \angle ADC$.

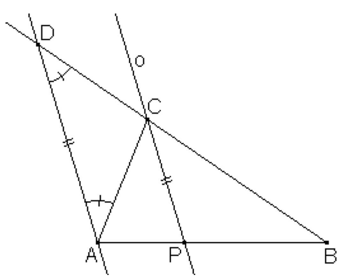

Fig. 9

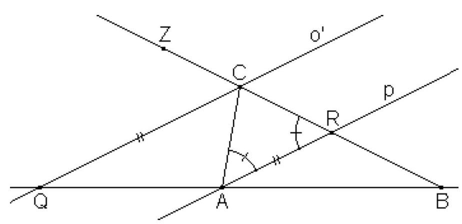

Fig. 10

Since $\angle ACP = \angle PCB$, so interior angles in triangle DAC on side are equal and triangle DAC is isosceles ($AC = DC$). Thus the segment is divided by the point C in the ratio BC/AC, which is equal PB/AP.

Triangle ABD can be also used for the proof of the converse theorem of theorem (1). Let us assume that a line o divides interior angle $\angle C$ into two parts and it intersects side AB in the point P, for which $PB/AP = BC/AC$. Then we can show that a line o is the angle bisector of interior angle $\angle C$. Since $AD \parallel BC$, triangles BPC and BAD are similar according to test AA. Thus segment BD

is divided by the point C in the ratio PB/AP. According to the assumption, we have $PB/AP = BC/AC$, hence $CD = AC$. Triangle DAC is isosceles and $\angle CAD = \angle ADC$, but then also $\angle ACP = \angle BCP$.

For a better understanding of a proof of theorem (1), we can show that analogous theorem holds for exterior angle bisector at the vertex C in triangle ABC:

(2) *The angle bisector o' of exterior angle at the vertex C in triangle ABC intersects line AB in the point Q, for which $QA/QB = AC/BC$ holds.*

To prove this theorem we again draw line p through the point A parallel to angle bisector o' (see Fig. 10). Let us label points $Q \in o' \cap AB$, $R \in p \cap BC$. Analogously as in the proof of theorem (1), there are the following relations: $\angle QCZ = \angle ARC$ and $\angle QCA = \angle CAR$.

Since angle bisector o' divides angle ACZ into two equal angles, triangle ARC is isosceles, and we have $AC = RC$. Triangle similarity $ABR \sim QBC$ implies: $AB/QB = BR/BC$. We obtain:

$$\frac{QB - QA}{QB} = \frac{BC - CR}{BC} \iff 1 - \frac{QA}{QB} = 1 - \frac{AC}{BC} \iff \frac{QA}{QB} = \frac{AC}{BC}.$$

Analogously as for theorem (1) we can show that converse theorem of theorem (2) also holds: *If a line q passes through the vertex C in triangle ABC and it intersects line AB in the point Q, which does not lie on side AB, and $QA/QB = AC/BC$ holds, then the line q is the angle bisector of exterior angle at the vertex C.*

5. Proofs with the use of trigonometric law of sines and areas of triangles

The interior angles in triangle ABC are labelled as in figure 11.

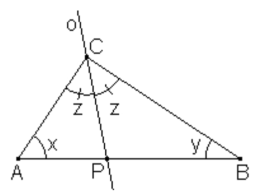

Fig. 11

The trigonometric law of sines can be applied to express the ratio of the lengths AC, BC:

$$\frac{BC}{\sin x} = \frac{AC}{\sin y}; \qquad \frac{\sin x}{\sin y} = \frac{BC}{AC}. \tag{3}$$

Similarly, the following relations hold for triangles APC and PBC:

$$\frac{PC}{\sin x} = \frac{AP}{\sin z}; \qquad \frac{PC}{\sin y} = \frac{PB}{\sin z}. \tag{4}$$

After expressing PC from both equations (4) and comparing the obtained expressions we have:

$$\frac{\sin x}{\sin y} = \frac{PB}{AP}. \tag{5}$$

Using relations (3) and (5) we finally get the equation $BC/AC = PB/AP$ which completes the proof of theorem (1).

We focus on the areas of triangles APC and PBC (see Fig. 11) to find another way to prove theorem (1). Since triangles APC and PBC have identical altitudes constructed from vertex C, so we have for the ratio of their areas $S_{\triangle APC}/S_{\triangle PBC} = AP/PB$. These areas can be also expressed with the help of sines of the interior angles at vertex C. We use the following formulas for the calculation of areas of triangles APC and PBC:

$$S_{\triangle APC} = \frac{1}{2} AC \cdot PC \sin(\angle ACP) \quad \text{and} \quad S_{\triangle PBC} = \frac{1}{2} PC \cdot BC \sin(\angle PCB). \tag{6}$$

Since line o is the angle bisector of interior angle $\angle C$, so we obtain with the help of the relations (6) $S_{\triangle APC}/S_{\triangle PBC} = AC/BC$ and thus $AP/PB = AC/BC$.

6. Proofs by analytic method

For other way of proving theorem (1), we can use analytic method based on the basic elements of vector calculus. A starting point for the proof is the choice of a suitable coordinate system and expression of relationships between objects. We choose oblique coordinate system in a plane. The coordinate axes are determined by line AB and the angle bisector of interior angle $\angle C$ in triangle ABC. Figure 12 shows the coordinates of the vertices of triangle ABC in the coordinate system.

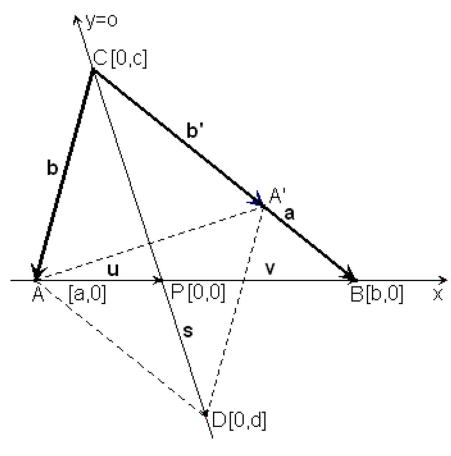

Fig. 12

The vertices of triangle ABC and the point $P \in o \cap AB$ are used for the definition of vectors: $\boldsymbol{b} = \overrightarrow{CA}$, $\boldsymbol{a} = \overrightarrow{CB}$, $\boldsymbol{u} = \overrightarrow{AP}$ and $\boldsymbol{v} = \overrightarrow{PB}$. A vector $\boldsymbol{b}'$ is the reflected image of the vector $\boldsymbol{b}$ reflected across the angle bisector o, and vector $\boldsymbol{s} = \overrightarrow{CD} = \boldsymbol{b} + \boldsymbol{b}'$. The vectors $\boldsymbol{u}$, $\boldsymbol{v}$ defined in this way are linearly dependent, and also $\boldsymbol{b}'$, $\boldsymbol{a}$ are linearly dependent vectors:

$$\boldsymbol{v} = k \cdot \boldsymbol{u}, \quad \boldsymbol{a} = t \cdot \boldsymbol{b}', \quad \text{and} \quad k, t \in \mathbf{R}^+. \tag{7}$$

To prove theorem (1) we have to show, that vectors $\boldsymbol{v}, \boldsymbol{a}$ are equal real multiples of vectors $\boldsymbol{u}, \boldsymbol{b}'$, thus:

$$k = t. \tag{8}$$

We calculate the coordinates of these vectors:

$$\begin{aligned} \boldsymbol{u} &= PA = (-a, 0), & \boldsymbol{v} &= BP = (b, 0), \\ \boldsymbol{a} &= BC = (b, -c), & \boldsymbol{b} &= AC = (a, -c), \\ \boldsymbol{s} &= DC = (0, d - c), & \boldsymbol{b}' &= \boldsymbol{s}\boldsymbol{b} = (-a, d). \end{aligned}$$

We have:

$$\left.\begin{array}{l} \boldsymbol{v} = k \cdot \boldsymbol{u} \iff b = k \cdot (-a), \text{ and } 0 = k \cdot 0 \iff k = b/(-a) \\ \boldsymbol{a} = t \cdot \boldsymbol{b}' \iff b = t \cdot (-a), \text{ and } c = t \cdot d \iff t = b/(-a) = -c/d \end{array}\right\} \implies k = t.$$

Since $a < 0$, and $b > 0$, k and t are equal positive real numbers and validity of theorem (1) is proved.

7. Transition from the ratios of the lengths of segments to Apollonius circle

At the end of the lesson, we return to the question which was raised at the end of the Hands-on exploration. A simple dynamic construction can be made according to the construction displayed in Fig. 3. We construct angle bisector o of interior angle $\angle X$, the point $P \in o \cap AB$, the midpoint S of side AB and a midpoint of AS. Students will be required to move the point X, and to find such triangles ABX, that the point P lies at the midpoint of AS. Figure 13 shows several such positions of the point X in one half-plane determined by line AB. Since $AP/PB = 1/3$, so the ratios AX/BX in these triangles also equal to $1/3$ according to theorem (1). *In what figure do the vertices X lie in triangles ABX?*

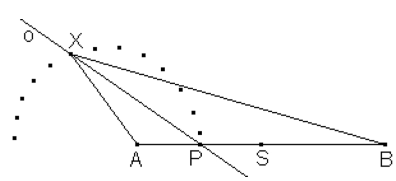

Fig. 13

To find an answer to this question, students can use advantages of IGS, and they may subsequently visualize unknown figure by means of manipulation of the dynamic construction. After location of a suitable position of the point X, it is possible to create a new free point and to place it on the point X. When dragging the point X, free point stays in its position. By means of experimentation with a dynamic construction, students should be able to find out that the points X lie on a circle. At the initial stage of solving such problems, IGS enables students to discover unknown figures and to make conjectures. Experimentally discovered relationships and features of examined objects are the basis for additional stages of problem solving focused on logical justification of the statement and generalization of the solution.

To make dynamic construction for detailed drawing of unknown figure, the technique for the construction of segments with the given ratio of their lengths can be used. The basis of the dynamic construction is not experimental finding of suitable positions for the point X, but the construction of such points X for which $AX/BX = 1/3$. Students might independently make a construction for drawing unknown figure. Activation of trace for intersection points of two circles, which determine required locations of the point X, can be used for visualization of unknown figure. Figure 14 shows one of the possibilities of making such dynamic construction.

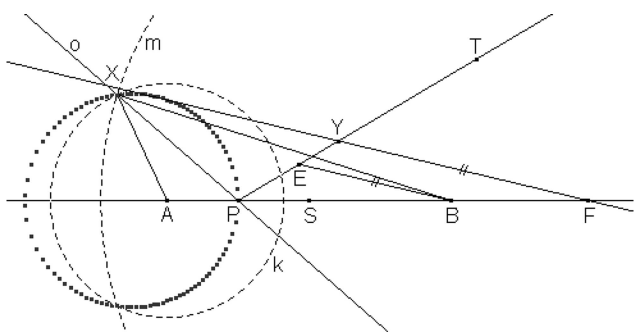

Fig. 14

Similarity of triangles is used to determine the lengths of segments AX, BX for which $AX/BX = 1/3$. An arbitrary additional ray PT is drawn after construction of the points A, B, S, P and line AB. The point P is the midpoint of segment AS. The point E is constructed on ray PT so that $PE = AP$. The segments with the ratio of their lengths equal to 1/3 can be easily constructed with the help of triangle PBE. The point Y is an arbitrary specific point on ray PT. The length of segment PY represents the length of segment AX. We can draw a line parallel to segment BE through the point Y for the determination of corresponding length of segment BX. The point F satisfying $PY/PF = 1/3$ lies in the intersection of this line with line AB. Then we construct the circles $k(A, PY)$ and $m(B, PF)$. The intersection points of these circles determine suitable positions of the point X. Movement of the point Y along ray PT evokes re-drawing the circles k, m and we obtain different positions of the point X. We turn on trace for both intersection points of the circles k, m to draw unknown figure.

With the help of the dynamic construction, we have constructed the points of a circle, which is called Apollonius circle. This circle intersects line AB in the points P and Q. We have $AP/PB = QA/QB$. Segment QP is the diameter of Apollonius circle. We have come to the conclusion:

(9) *The set of all points in plane having constant ratio of distances to two fixed points A and B is a circle.*

To prove theorem (9) we use triangle ABC with the angle bisector o of interior angle $\angle C$. According to theorem (1) we have for the point $P \in o \cap AB$, $AP/PB = AC/BC$. Since theorem (2) and its converse theorem hold, so the point Q differing from the point P exists on line AB and $QA/QB = AC/BC$. The point Q lies on the angle bisector o' of exterior angle at the vertex C in triangle $ABCE$ (see Fig. 15).

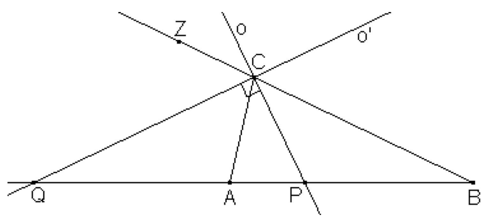

Fig. 15

We have $\angle ACZ + \angle ACB = 180°$. Since angle bisectors o and o' bisect the corresponding angles, then $\angle QCP = \angle ACZ/2 + \angle ACB/2 = 90°$. We have found out that pairs of the lines passed through arbitrary points having constant ratio of distances to two fixed points A, B and through the points P, Q make a right angle. According to Thales's theorem, the set of all such points forms a circle with diameter QP, without the points P, Q. But the points P, Q have the same ratio of distances to two fixed points A, B as other points of the circle, therefore they belong to Apollonius circle.

8. Conclusion

Our paper is aimed at presenting the fact that geometrical problems may offer interesting possibilities of using IGS for the purpose of conducting such activities that will enable students to discover and justify relationships between geometrical objects. By exploring the ratio of the distances of the point from two given points we get to the circle. The described approach can be used and developed in investigation of sums and differences of the distances of the points from two given points. The lesson plan based on this approach could enable students to go from the circles to other types of conics.

Important objectives of mathematics teaching are to support changes in teaching by means of stimulating conjecturing, and to develop students abilities to logical justification of discovered relationships. The exploration of particular cases, making of conjectures, generalization of discovered relationships, and rigorous proofs of the formulated theorems should be used together as complementary processes in order to enable deeper understanding of mathematical knowledge.

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