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A CONTRIBUTION TO THE DEVELOPMENT OF FUNCTIONAL THINKING RELATED TO CONVEXITY

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Abstract. When a liquid (water) flows into a vessel at the constant inflow rate, then the height filling function is convex or concave depending on the way how the level of the liquid changes. When the level changes accelerating or slowing down, the function is convex or concave, respectively. This vivid interpretation holds in general, namely we prove that given a strictly increasing convex (concave) continuous function, then there exists a vessel such that its height filling function is equal to the given function. (A fact that seems to be new.)

We also hope that our paper could exemplify the case of a research project to be assigned to excellent students.

ZDM Subject Classification: I24; *AMS Subject Classification*: 97I20.

Key words and phrases: Height filling function; convex and concave functions.

1. Introduction

Suppose that a liquid (water) flow has a constant inflow rate v_0 into a vessel, which has the form of a surface of revolution, and suppose that this procedure begins at moment $t = 0$ and ends at moment $t = T$.

We study the dependence of the height $h = \boldsymbol{h}(t)$, which will be called the height filling function (following Eisenmann [5]), of the liquid level at time t. As we have indicated in the abstract, we give a geometric visual characterization of monotone convex functions that can be considered as the main result of this paper. Additionally, we also provide material that can serve as motivation and illustration for the deeper understanding of basic concepts and ideas of the differential and integral calculus, as well as the further development of functional thinking in teaching mathematics.

The content of the paper is as follows. In Section 2, we consider examples which include a circular cylinder, a circular cone, a surface composed of cone and a cylinder linked to it, and in Section 3 a surface composed of a finite number of cylinders. These examples give motivation to consider a vessel, which has the form of a surface of revolution, and study the properties of the height filling function.

After writing a recent version of our text, we received the text by Eisenmann [5] who considered similar questions and we realized that our approach is related to the concept of functional thinking. Among other things we are motivated by PISA test and a text, cf. [7].

In our considerations, we identify a surface of revolution with a vessel. If the generatrix of surface of revolution is continuous function we call it elementary surface of revolution.

In Section 3 we consider the following question which has been suggested by Professor M. Marjanović:

QUESTION 1. *Suppose that* $\boldsymbol{h}$ *is a strictly increasing convex continuous function on* $[0, T]$ *with* $\boldsymbol{h}(0) = 0$*. Is there a vessel such that* $\boldsymbol{h}$ *is the corresponding height filling function?*

It turns out that the question is very interesting and related to geometric characterization of convexity. We first study the case when the height filling function is smooth. Namely, we show that

A) the height filling function $\boldsymbol{h}$ is continuous on $[0, T]$ and continuously differentiable on $(0, T]$, where $\boldsymbol{h}(0) = 0$, $0 < \boldsymbol{h}'(t) < \infty$ on $(0, T]$, and $0 < \lim_{t\to 0+} \boldsymbol{h}'(t) \leq +\infty$

   if and only if

B) there is a corresponding elementary surface of revolution.

Since convex functions have derivative everywhere except at most on a countable set, there is no solution of the question in the class of elementary surfaces of revolution. It is surprising that we can extend the class of elementary surfaces to get a solution. The suggestion was to approximate $\boldsymbol{h}$ by a sequence of polygonal lines $\boldsymbol{l}_n$. We show that $\boldsymbol{l}_n'$ and the corresponding generatrix $\boldsymbol{r}_n$ converge except on a countable set and give a positive answer to the question.

Our approach is based on the notion of extended graph for monotone functions. Namely, graph of monotone (in general discontinuous) function $\boldsymbol{f}$ can be extended (in a natural way) by adding at most countable family of segments to get Jordan path which we denote by $\Gamma^*_{\boldsymbol{f}}$ and call it extended graph of $\boldsymbol{f}$. Using the formula (3) below one can prove, roughly, that if a height filling function $\boldsymbol{h}$ is strictly increasing convex continuous function on $[0, T]$ with $\boldsymbol{h}(0) = 0$, then there is an elementary surface of revolution (a vessel) in extended sense (whose generatrix is extended graph of a decreasing function) such that $\boldsymbol{h}$ is the corresponding height filling function.

For the sake of convenience of readers we will consider only the case when $\boldsymbol{h}$ has derivative on the whole interval $[0, T]$ except, maybe, at a finite set.

In Section 4 we collect some properties of monotone and convex functions.

2. The analytical properties of height filling function

Let us first consider a few examples. Suppose that a liquid (water) flows with the constant inflow rate v_0 into the vessels that have a form as in Fig. 1. Denote by T the time for which vessels are filled, and denote by $\boldsymbol{h}(t)$ the height of liquid in vessel at the moment t. Note that the vessels in Fig. 1. have a form of a surface of revolution, generated by rotation of graph of a function $\boldsymbol{r}\colon [0, H] \to \mathbb{R}$ about

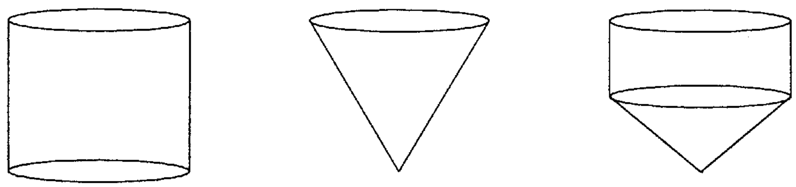

Fig. 1

x-axis. Suppose that then the vessels are placed so that their rotation axes are in vertical position.

EXAMPLE 1. For the first vessel in Fig. 1 the generatrix is

$$\boldsymbol{r}(x) = 1, \quad x \in [0, H]$$

and

$$\boldsymbol{h}(t) = \frac{v_0}{\pi} t, \quad t \in [0, T].$$

EXAMPLE 2. For the second vessel in Fig. 1 the generatrix is

$$\boldsymbol{r}(x) = x, \quad x \in [0, H]$$

and

$$\boldsymbol{h}(t) = \sqrt[3]{\frac{3v_0 t}{\pi}}, \quad t \in [0, T].$$

EXAMPLE 3. For the third vessel in Fig. 1 the generatrix is

$$\boldsymbol{r}(x) = \begin{cases} x & \text{if } x \in [0, \frac{H}{2}] \\ \frac{H}{2} & \text{if } x \in [\frac{H}{2}, H], \end{cases}$$

and

$$\boldsymbol{h}(t) = \begin{cases} \sqrt[3]{\frac{3v_0 t}{\pi}} & \text{if } t \in [0, \frac{\pi}{v_0}\frac{H^3}{24}] \\ \frac{v_0}{\pi}\frac{4}{H^2}\left(t + \frac{\pi}{v_0}\frac{H^3}{12}\right) & \text{if } t \in [\frac{\pi}{v_0}\frac{H^3}{24}, \frac{\pi}{v_0}\frac{H^3}{6}]. \end{cases}$$

Now, consider a general case. Let $H > 0$ and let a function $\boldsymbol{r}\colon [0, H] \to \mathbb{R}$ with following be given properties:

(r1) $\boldsymbol{r}$ is continuous on $[0, H]$ and

(r2) $\boldsymbol{r}(x) > 0$ for $x \neq 0$.

Rotating the curve

$$c_{\boldsymbol{r}} = \{(0, y) : y \in [0, \boldsymbol{r}(0)]\} \cup \{(x, \boldsymbol{r}(x)) : x \in [0, H]\}$$

about x-axis we get the surface $\sigma = \sigma_{\boldsymbol{r}}$. The surfaces we get in this way are called the *elementary surfaces of revolution*, see Fig. 2.

It is convenient to identify the inner "wall" of the vessel (and the whole vessel) by $\sigma_{\boldsymbol{r}}$. In that sense we will consider *the vessel* to be the part of the space bounded

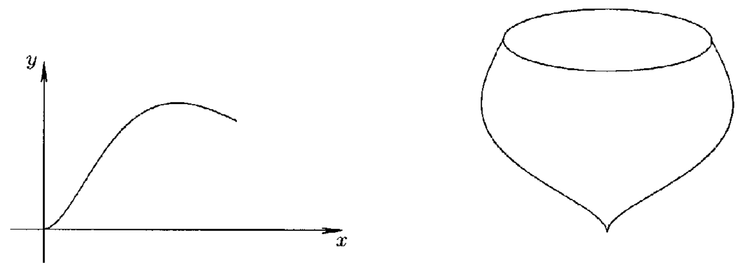

Fig. 2

by $\sigma_{\boldsymbol{r}}$ and the plane $x = H$, while by the *axis of rotation of the vessel* we will consider the rotation axis of the surface $\sigma_{\boldsymbol{r}}$. Function $\boldsymbol{r}$ (respectively the graph of $\boldsymbol{r}$) will be called *generatrix* of surface $\sigma_{\boldsymbol{r}}$.

Place the vessel so that its rotation axis is in vertical position and pour liquid evenly through the upper part so that the speed of the change of the liquid volume is constant and equal to v_0. Suppose that this procedure begins at moment $t = 0$. Denote by $\boldsymbol{h}(t)$ the height of the liquid level at the time t. Then, if we express the liquid volume at the moment t in two ways, we get the following equality:

$$v_0 t = \pi \int_0^{\boldsymbol{h}(t)} \boldsymbol{r}^2(x)\, dx. \tag{1}$$

We will examine the properties of function $\boldsymbol{h}$. Since the height of the vessel is H, the total volume of the liquid that can fit the vessel $\sigma_{\boldsymbol{r}}$, is

$$\pi \int_0^H \boldsymbol{r}^2(x)\, dx.$$

If by T we denote the total time needed for the vessel $\sigma_{\boldsymbol{r}}$, to be filled in, then

$$v_0 T = \pi \int_0^H \boldsymbol{r}^2(x)\, dx.$$

The time for the liquid level to reach the height $h \in [0, H]$ is

$$t = \frac{1}{v_0}\pi \int_0^h \boldsymbol{r}^2(x)\, dx.$$

It is convenient to introduce the function $\boldsymbol{g}\colon [0, H] \to \mathbb{R}$ which is defined by the formula

$$\boldsymbol{g}(h) = \frac{1}{v_0}\pi \int_0^h \boldsymbol{r}^2(x)\, dx.$$

The range of $\boldsymbol{g}$ is $[0, T]$ and by (1), $t = \boldsymbol{g}(\boldsymbol{h}(t))$, for every $t \in [0, T]$. Thus $\boldsymbol{g}$ is the inverse function of $\boldsymbol{h}$, i.e. $h = \boldsymbol{g}^{-1}(t)$.

Using the basic properties of definite integrals we get that the function $\boldsymbol{g}$ is strictly increasing and continuous on $[0, H]$, and that

$$\boldsymbol{g}'(h) = \frac{1}{v_0}\pi \boldsymbol{r}^2(h).$$

Hence we conclude that $\boldsymbol{g}$ is continuously differentiable on $[0, H]$.

Since the function $\boldsymbol{g}$ is continuous and strictly increasing on $[0, H]$, we conclude that the inverse function $\boldsymbol{h}$ is also continuous and strictly increasing on $[0, T]$. Particularly, $\boldsymbol{h}(0) = 0$. Also, using the theorem of the first derivative of inverse function we get

$$\boldsymbol{h}'(t) = \frac{v_0}{\pi \boldsymbol{r}^2(\boldsymbol{h}(t))} \tag{2}$$

for $t \in (0, T]$ ($t \in [0, T]$, if $\boldsymbol{r}(0) \neq 0$). From the last formula we conclude that $\boldsymbol{h}$ is continuously differentiable on $[0, T]$ (if $\boldsymbol{r}(0) \neq 0$), respectively $\boldsymbol{h}$ is continuously differentiable on $(0, T]$ and $\lim_{t \to 0+} \boldsymbol{h}'(t) = +\infty$ (if $\boldsymbol{r}(0) = 0$). Of course, at the ends of intervals we consider one side derivatives.

Notice that $\boldsymbol{h}'$ has no zeros.

Therefore we have proved:

Theorem 1. *If the vessel (formed in the previously described way) is being evenly filled with some liquid by the constant rate v_0, then the function that describes the dependence of liquid level of the time, the function $\boldsymbol{h}$ has the following properties:*

(h1) $\boldsymbol{h}$ *is strictly increasing and continuous on* $[0, T]$,

(h2) $\boldsymbol{h}(0) = 0$,

(h3a) $\boldsymbol{h}$ *is continuously differentiable on* $[0, T]$ *and for every* $t \in [0, T]$, $\boldsymbol{h}'(t) > 0$ (*if* $\boldsymbol{r}(0) \neq 0$),

or

(h3b) $\boldsymbol{h}$ *is continuously differentiable on* $(0, T]$, $\lim\limits_{t \to 0+} \boldsymbol{h}'(t) = +\infty$ *and for every* $t \in (0, T]$, $\boldsymbol{h}'(t) > 0$ (*if* $\boldsymbol{r}(0) = 0$).

We will call the class of all functions that satisfy conditions (h1),(h2) and (h3a) or (h3b) *the class of height filling functions* and denote by $\mathcal{H} = \mathcal{H}[0, T]$.

Moreover if we assume that the function $\boldsymbol{r}$ is differentiable on $(0, H)$, then we get

$$\boldsymbol{h}''(t) = -\frac{2v_0}{\pi} \frac{\boldsymbol{r}'(\boldsymbol{h}(t))\boldsymbol{h}'(t)}{\boldsymbol{r}^3(\boldsymbol{h}(t))},$$

that is

$$\boldsymbol{h}''(t) = -\frac{2v_0^2}{\pi^2} \frac{\boldsymbol{r}'(\boldsymbol{h}(t))}{\boldsymbol{r}^5(\boldsymbol{h}(t))}.$$

Using the last formula we prove

Theorem 2. *If $\boldsymbol{r}$ is increasing then $\boldsymbol{r}' \geq 0$ and $\boldsymbol{h}'' \leq 0$, that is $\boldsymbol{h}$ is concave on $(0, T)$. If $\boldsymbol{r}$ is decreasing then $\boldsymbol{r}' \leq 0$ and $\boldsymbol{h}'' \geq 0$, that is $\boldsymbol{h}$ is convex on $(0, T)$.*

3. Finding of a vessel if a height filling function is given

3.1. The solution if height filling function is continuously differentiable

For a given function $\boldsymbol{r}$, we denote by $\mathcal{S}\boldsymbol{r}$ the corresponding height filling function for the vessel σ_r.

In the previous section we showed that if function $\boldsymbol{r}$ has properties (r1) and (r2), then the corresponding height filling function $\boldsymbol{h} = \mathcal{S}\boldsymbol{r}$ has the properties (h1),(h2) and (h3a) or (h3b). Now, let us see if the opposite is true, i.e. if $\boldsymbol{h}$ has the properties (h1),(h2) and (h3a) or (h3b), is there the function $\boldsymbol{r}$ with the properties (r1) and (r2) such that $\boldsymbol{h} = \mathcal{S}\boldsymbol{r}$?

Note that the formula (3) has important role in our investigation.

Before we proceed, we leave to the reader to consider the following example:

EXAMPLE 4. For the function $\boldsymbol{h} : [0, T] \to \mathbb{R}$ defined by $\boldsymbol{h}(t) = t^2$ there is no the corresponding vessel. However there is “the corresponding unbounded vessel”. Note this function does not satisfy $(h3a)$ or $(h3b)$ and $(ch3)$ below; namely for this function $\boldsymbol{h}'_+(0) = 0$.

Let a function $\boldsymbol{h} : [0, T] \to \mathbb{R}$ with the properties (h1), (h2) and (h3a) or (h3b) be given. We readily see that if we put $H = \boldsymbol{h}(T)$, then $\boldsymbol{h}$ maps $[0, T]$ to $[0, H]$ bijectively. Let us set up the problem to find a function $\boldsymbol{r} : [0, H] \to \mathbb{R}$, with the properties (r1) and (r2) such that if we form the elementary surface of revolution (the vessel) $\sigma_{\boldsymbol{r}}$ and fill it with liquid at the constant rate v_0 then the dependence of the liquid level on time is described exactly by the function $\boldsymbol{h}$.

For the function $\boldsymbol{r}$ it must be true that

$$v_0 t = \pi \int_0^{\boldsymbol{h}(t)} \boldsymbol{r}^2(x)\, dx$$

for every $t \in [0, T]$. Differentiating the last equation we get that for such $\boldsymbol{r}$ (we yet need to define) it must be true that

$$v_0 = \pi \boldsymbol{r}^2(\boldsymbol{h}(t))\boldsymbol{h}'(t)$$

for every $t \in [0, T]$, that is

$$\boldsymbol{r}(\boldsymbol{h}(t)) = \sqrt{\frac{v_0}{\pi \boldsymbol{h}'(t)}} \tag{3}$$

for every $t \in [0, T]$. Since $\boldsymbol{h} : [0, T] \to [0, H]$ is a bijection, it follows that there is $\boldsymbol{h}^{-1} : [0, H] \to [0, T]$. Let us define the sought function $\boldsymbol{r} : [0, H] \to \mathbb{R}$ in the following way

$$\boldsymbol{r}(x) = \sqrt{\frac{v_0}{\pi \boldsymbol{h}'(\boldsymbol{h}^{-1}(x))}}.$$

In addition if $\boldsymbol{h}$ satisfies the condition (h3b) define $\boldsymbol{r}(0) = 0$. It can be checked directly that the function $\boldsymbol{r}$ defined like this satisfies the required conditions (details are left to the reader to check). Therefore we have shown

THEOREM 3. *Let a function* $\boldsymbol{h} : [0, T] \to \mathbb{R}$*, with properties (h1),(h2) and (h3a) or (h3b) be given. Then there is a corresponding elementary surface of revolution, i.e. there is a function* $\boldsymbol{r}$ *with the properties (r1) and (r2) such that the dependence of the liquid level in the vessel* $\sigma_{\boldsymbol{r}}$ *is described exactly by the function* $\boldsymbol{h}$*.*

EXAMPLE 5. If $\boldsymbol{h}(t) = \sqrt{t}$, then $\boldsymbol{r}(x) = \sqrt{\frac{2v_0 x}{\pi}}$.

3.2. The solution if height filling function is convex

3.2.1. EXAMPLES. Now, we consider examples which give solutions of Question 1 in special cases.

Let $\boldsymbol{h} : [0, T] \to \mathbb{R}$ be a function with the properties:

(ch1) $\boldsymbol{h}$ is continuous, convex and strictly increasing on $[0, T]$.

(ch2) $\boldsymbol{h}(0) = 0$.

(ch3) $\boldsymbol{h}'_+(0) > 0$ and $\boldsymbol{h}'_-(T) < +\infty$.

The class of all functions that satisfy the conditions (ch1),(ch2) and (ch3) we denote by $\mathcal{C}on = \mathcal{C}on[0, T]$.

Denote $H = \boldsymbol{h}(T)$. Because $\boldsymbol{h}$ is strictly increasing and continuous on $[0, T]$ we get that $\boldsymbol{h}$ maps $[0, T]$ onto $[0, H]$, bijectively. Examine if for a given $\boldsymbol{h} \in \mathcal{C}on$ there is a function $\boldsymbol{r} \colon [0, T] \to \mathbb{R}$ such that $\boldsymbol{h}$ is the height filling function for vessel $\sigma_{\boldsymbol{r}}$.

In previous sections we proved that there is injective correspondence between the class of generatrices of elementary surfaces of revolution and the class $\mathcal{H}$ of height filling functions. Since there is a function that satisfies the conditions (ch1), (ch2) and (ch3) which does not belong to the class of height filling functions, we conclude that for such function there is no corresponding generatrix $\boldsymbol{r}$ such that $\sigma_{\boldsymbol{r}}$ is an elementary surface of revolution. However it makes sense to consider the following examples, in which we consider a function with properties (ch1), (ch2) and (ch3):

EXAMPLE 6. Let $a > 0$ and $\boldsymbol{h}(t) = at$; then if we assume that there is a function $\boldsymbol{r}$ with required properties and if we express the volume of the liquid in the vessel $\sigma_{\boldsymbol{r}}$ at the moment $t \in [0, T]$ in two ways we get:

$$v_0 t = \pi \int_0^{at} \boldsymbol{r}^2(x)\, dx.$$

Differentiating the last equation we get

$$v_0 = \pi \boldsymbol{r}^2(at) a,$$

that is

$$\boldsymbol{r}(at) = \sqrt{\frac{v_0}{\pi a}}$$

for every $t \in [0, T]$. That is the function $\boldsymbol{r}$ must be given by

$$\boldsymbol{r}(x) = \sqrt{\frac{v_0}{\pi a}} \qquad x \in [0, aT].$$

It can be checked directly that if $\boldsymbol{r}$ is defined in this way we really get that the dependence of the liquid level in the vessel $\sigma_{\boldsymbol{r}}$ is given by the function $\boldsymbol{h}$.

EXAMPLE 7. Let $a_2 > a_1 > 0$ and

$$\boldsymbol{h}(t) = \begin{cases} a_1 t & \text{if } t \in \left[0, \frac{T}{2}\right] \\ a_2 t + (a_1 - a_2)\frac{T}{2} & \text{if } t \in \left[\frac{T}{2}, T\right]. \end{cases}$$

If we assume that there is a function $\boldsymbol{r}$ with the required conditions and if the volume of the liquid poured in the vessel, generated by the function $\boldsymbol{r}$, until the moment $t \in [0, T]$, is expressed in two ways we get the following equalities

$$v_0 t = \pi \int_0^{a_1 t} \boldsymbol{r}^2(x)\, dx, \qquad t \in \left[0, \frac{T}{2}\right],$$

that is,

$$v_0 t = \pi \int_0^{\frac{a_1 T}{2}} \boldsymbol{r}^2(x) dx + \pi \int_{\frac{a_1 T}{2}}^{a_2 t + (a_1 - a_2)\frac{T}{2}} \boldsymbol{r}^2(x)\, dx, \qquad t \in \left[\frac{T}{2}, T\right].$$

By differentiating this with respect to t, we get

$$v_0 = \pi \boldsymbol{r}^2(a_1 t) a_1, \qquad t \in \left[0, \frac{T}{2}\right],$$

that is,

$$v_0 = \pi \boldsymbol{r}^2\Big(a_2 t + (a_1 - a_2)\frac{T}{2}\Big) a_2, \qquad t \in \left[\frac{T}{2}, T\right].$$

Therefore, the function $\boldsymbol{r}$ must be given by

$$\boldsymbol{r}(x) = \begin{cases} \sqrt{\frac{v_0}{\pi a_1}} & \text{if } x \in \left[0, \boldsymbol{h}(\frac{T}{2})\right] \\ \sqrt{\frac{v_0}{\pi a_2}} & \text{if } x \in \left[\boldsymbol{h}(\frac{T}{2}), \boldsymbol{h}(T)\right]. \end{cases}$$

Notice that the function $\boldsymbol{r}$ is not well-defined for $x = \boldsymbol{h}(\frac{T}{2})$, i.e. the function $\boldsymbol{r}$ has two values at the point $x = \boldsymbol{h}(\frac{T}{2})$.

EXAMPLE 8. Let $a_n > \cdots > a_2 > a_1 > 0$, $0 = t_0 < t_1 < \cdots < t_n = T$ and

$$\boldsymbol{h}(t) = \begin{cases} a_1 t & \text{if } t \in [0, t_1] \\ a_2 t - a_2) t_1 + a_1 t_1 & \text{if } t \in [t_1, t_2] \\ a_3 t - a_3 t_2 + a_2(t_2 - t_1) + a_1 t_1 & \text{if } t \in [t_2, t_3] \\ \cdots & \cdots \\ a_n t - a_n t_{n-1} + a_{n-1}(t_{n-1} - t_{n-2}) + \cdots + a_1 t_1 & \text{if } t \in [t_{n-1}, t_n]\,. \end{cases}$$

If we assume that there is a function $\boldsymbol{r}$ with the required conditions we get (analogously to the previous examples) that

$$\boldsymbol{r}(x) = \begin{cases} \sqrt{\frac{v_0}{\pi a_1}} & \text{if } x \in [0, \boldsymbol{h}(t_1)] \\ \sqrt{\frac{v_0}{\pi a_2}} & \text{if } x \in [\boldsymbol{h}(t_1), \boldsymbol{h}(t_2)] \\ \cdots & \cdots \\ \sqrt{\frac{v_0}{\pi a_n}} & \text{if } x \in [\boldsymbol{h}(t_{n-1}), \boldsymbol{h}(t_n)]\,. \end{cases}$$

Notice that $\boldsymbol{r}$ is not well-defined at the points

$$x = \boldsymbol{h}(t_1), \ldots, \boldsymbol{h}(t_{n-1}),$$

i.e. $\boldsymbol{r}$ is multi-valued because it maps each of the points $\boldsymbol{h}(t_1), \ldots, \boldsymbol{h}(t_{n-1})$ to two different values (see Fig. 3).

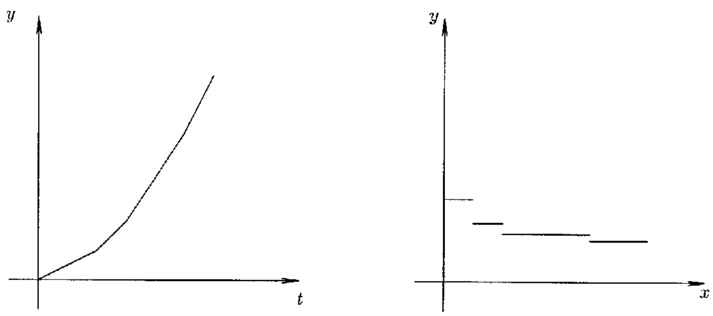

Fig. 3

However, regardless of this "problem" with the function $\boldsymbol{r}$, it is natural to conclude that the required surface of revolution (vessel) is the one in Fig. 4. Therefore we can consider $\boldsymbol{r}$ as a multi-valued function that maps each point $x_k = \boldsymbol{h}(t_k)$ to the segment $[\boldsymbol{r}(x_k+), \boldsymbol{r}(x_k-)]$, $k = 1, \ldots, n-1$. As the generatrix of the sought vessel we can regard the curve

$$\{(0, y) : y \in [0, \boldsymbol{r}(0)]\} \cup \Gamma^*_{\boldsymbol{r}}.$$

Here $\Gamma^*_{\boldsymbol{r}}$ is the "graph" of the multi-valued function $\boldsymbol{r}$. This example gives a motivation for defining a notion of an extended graph of a monotone function in general case.

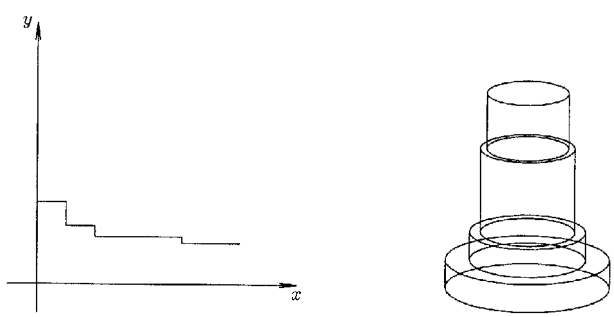

Fig. 4

3.2.2. The general case. Now, we outline an answer to the Question 1. We will consider two approaches:

The first approach. For a given function $\boldsymbol{h} \in \mathcal{C}on[0,T]$, we will use function $\boldsymbol{r}$ given by the formula (3):

$$\boldsymbol{r}(\boldsymbol{h}(t)) = \sqrt{\frac{v_0}{\pi \boldsymbol{h}'(t)}}.$$

Since convex function $\boldsymbol{h}$ has derivative everywhere except on at most countable set D, $\boldsymbol{r}$ is defined except on the set $\boldsymbol{h}(D)$.

By Theorem 1 (statement (h3a) or (h3b)), in general there is no solution of the question in the class of elementary surfaces of revolution. It is surprising that we can extend the class of elementary surfaces of revolution to get a solution.

The second approach. We will approximate $\boldsymbol{h}$ with a sequence of polygonal lines $\boldsymbol{l}_n$ and show that $\boldsymbol{l}'_n$ and corresponding sequence of generatrices $\boldsymbol{r}_n$ converge pointwise except on countable set.

Note that we do not suppose that $\boldsymbol{h}$ is continuously differentiable in general and that it is not a priori clear that the extended graph $\Gamma^*_{\boldsymbol{r}}$ of monotone function $\boldsymbol{r}$ is a Jordan path, i.e. that it has continuous one-to-one parametrization. The interested reader can try to prove it using the following result (which we denote by C)):

C) A convex function is differentiable if and only if it is continuously differentiable, cf. [11].

We will try to keep the consideration on elementary level. Now we will elaborate the first approach.

Step 1. We first introduce the notion of extended graph for monotone function. Example 8 gives a motivation for that. Let us assume we have an interval I and an increasing function $\boldsymbol{f}\colon I \to \mathbb{R}$. Then for all x, y, $x < y$ implies $\boldsymbol{f}(x) < \boldsymbol{f}(y)$. Therefore, $\boldsymbol{f}$ can have only one type of simple discontinuity, where the right and left limit are not equal to each other. More precisely, if $\boldsymbol{f}$ has a discontinuity at point $p \in I$ then $\boldsymbol{f}(p-) < \boldsymbol{f}(p+)$, where $\boldsymbol{f}(p-)$ indicates the limit from the left and $\boldsymbol{f}(p+)$ indicates the limit from the right. Therefore, for discontinuity at point p we can assign the interval $I_p = [\boldsymbol{f}(p-), \boldsymbol{f}(p+)]$.

Let D be the set in I of points at which $\boldsymbol{f}$ is discontinuous. Notice that the set D is at most countable. Define $J_p = \{(p, y) : y \in I_p\}$, $D_{\boldsymbol{f}} = \bigcup_{p \in D} J_p$ and $\Gamma^*_{\boldsymbol{f}} = \Gamma_{\boldsymbol{f}} \cup D_{\boldsymbol{f}}$, where $\Gamma_{\boldsymbol{f}}$ is the graph of $\boldsymbol{f}$; we call $\Gamma^*_{\boldsymbol{f}}$ the *extended graph of* $\boldsymbol{f}$. We can consider $\Gamma^*_{\boldsymbol{f}}$ as a Jordan path.

Similarly we can define the extended graph for a decreasing function. Note that an extended graph can contain horizonal and vertical segments.

Step 2. The function $\boldsymbol{r}$ is determined by the formula

$$\boldsymbol{r}(\boldsymbol{h}(t)) = \sqrt{\frac{v_0}{\pi \boldsymbol{h}'(t)}}.$$

More precisely, we define function $\boldsymbol{r}\colon [0, H] \setminus \boldsymbol{h}(D) \to \mathbb{R}$, where D is a set of points $t \in [0, T]$ such that $\boldsymbol{h}'(t)$ does not exist (for simplicity the reader can assume that the set D is finite). Since the function $\boldsymbol{h}$ is convex we get that function $\boldsymbol{r}$ is decreasing and the extended graph $\Gamma^*_{\boldsymbol{r}}$ of $\boldsymbol{r}$ is defined.

Rotating the curve

$$\{(0,y) : y \in [0, \boldsymbol{r}(0)]\} \cup \Gamma^*_{\boldsymbol{r}}$$

about x-axis we get a surface $\Sigma = \Sigma_{\Gamma^*_{\boldsymbol{r}}}$. The surfaces which we get in this way are called *extended elementary surfaces of revolution*. Similarly as we have defined a vessel (using the corresponding elementary surface of revolution) we define an *extended vessel* (using the corresponding extended elementary surface of revolution) and identify an extended elementary surface of revolution and the extended vessel. As an illustration of this procedure the reader can consider Example 8.

The interested reader can check that the given function $\boldsymbol{h}$ is the corresponding height filling function for the extended vessel $\Sigma_{\Gamma^*_{\boldsymbol{r}}}$ obtained in this way.

EXAMPLE 9. If inflow rate v_0 is equal to π and if function $\boldsymbol{h}\colon [0, 4\ln 2] \to \mathbb{R}$ is defined by formula

$$\boldsymbol{h}(t) = \begin{cases} e^{t/3} - 1 & \text{if } t \in [0, 3\ln 2] \\ 2t - 6\ln 2 + 1 & \text{if } t \in [3\ln 2, 4\ln 2], \end{cases}$$

then $\boldsymbol{h}$ satisfies the conditions (ch1),(ch2) and (ch3), and we get

$$\boldsymbol{r}(x) = \begin{cases} \frac{1}{\sqrt{\frac{1}{3}(x+1)}} & \text{if } x \in [0,1] \\ \frac{1}{\sqrt{2}} & \text{if } x \in [1, 2\ln 2 + 1]. \end{cases}$$

See Fig. 5 and Fig. 6.

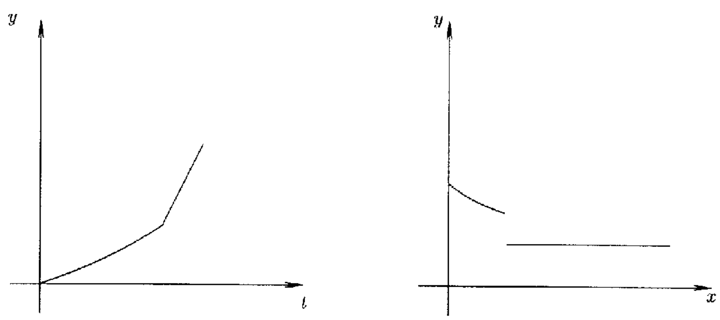

Fig. 5

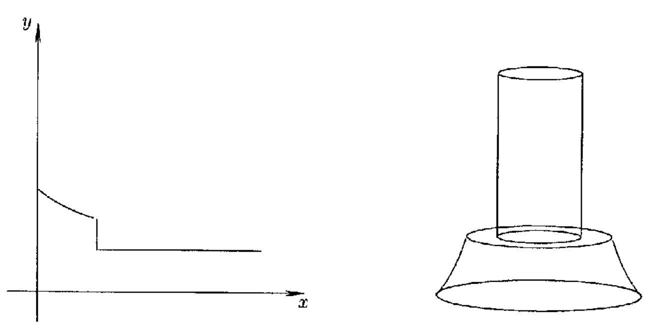

Fig. 6

3.3. Convergence of sequence of derivatives.

We approximate $\boldsymbol{h}$ by a sequence of polygonal lines $\boldsymbol{l}_n$. We show that $\boldsymbol{l}'_n$ and the corresponding generatrix $\boldsymbol{r}_n$ converge except on a countable set. This approach gives also a positive answer to the question and leads to further results. We plan to study it in a forthcoming paper with more details. We will present a solution of the problem in a few steps.

Step 1. In this step we approximate a given function $\boldsymbol{f}$ by a sequence of polygonal lines $\boldsymbol{l}_n$ and consider the convergence of sequence of derivatives $\boldsymbol{l}'_n$. We need the definition of partition. For $n \in \mathbb{N}$ a *partition* P of $[a,b]$ is a finite sequence $a = x_0 < x_1 < \cdots < x_n = b$. Let $d_i = x_i - x_{i-1}$ be the width of subinterval $I_i = [x_{i-1}, x_i]$, then the *parameter* of such a partition is the width of the largest subinterval formed by the partition i.e. $\lambda(P) = \max_{i=1,\dots,n} d_i$. A sequence of partitions $(P_n)_{n\in\mathbb{N}}$ is *successive* if P_{k+1} contains all points of P_k for $k \geq 1$.

Suppose that $\boldsymbol{f}\colon [a,b] \to \mathbb{R}$. We consider sequences of successive partitions $(P_n)_{n\in\mathbb{N}}$ of the interval $[a,b]$ for which $\lambda(P_n) \to 0$, when $n \to \infty$. Let Λ_n be a polygonal line given by points $M_0, \dots, M_n$, where $M_k = (x_k, \boldsymbol{f}(x_k))$ $(k = 0,1,2,\dots,n)$. With this polygonal line Λ_n we will associate the function $\boldsymbol{l}_n\colon [0,T] \to \mathbb{R}$ so that the polygonal line Λ_n is the graph of $\boldsymbol{l}_n$. Of course, ${\boldsymbol{l}'}^{+}_n(x)$ (the right derivative of $\boldsymbol{l}_n$ at x) exists for all $x \in [a,b)$ and ${\boldsymbol{l}'_n}^{-}(x)$ (the left derivative of $\boldsymbol{l}_n$ at x) exists for all $x \in (a,b]$. We will prove the next proposition.

Proposition 1. *If* $\boldsymbol{f}$ *has a derivative at* $x \in [a,b]$*, then* $\lim_{n\to+\infty} {\boldsymbol{l}'_n}^{+}(x) = \boldsymbol{f}'(x)$ *and* $\lim_{n\to+\infty} {\boldsymbol{l}'_n}^{-}(x) = \boldsymbol{f}'(x)$.

For $x \in [a,b)$ there is a sequence of segments $[y_n, z_n]$, such that $y_n \leq x < z_n$ for all $n \in \mathbb{N}$ and $y_n, z_n \in P_n$ and there are no points of partition P_n in (y_n, z_n). We consider two cases:

1. If $y_n < x < z_n$ for all n, then

$$\boldsymbol{f}(z_n) - \boldsymbol{f}(x) = \boldsymbol{f}'(x)(z_n - x) + \varepsilon_n(x)(z_n - x)$$

and

$$\boldsymbol{f}(y_n) - \boldsymbol{f}(x) = \boldsymbol{f}'(x)(y_n - x) + \epsilon_n(x)(y_n - x),$$

where $\varepsilon_n(x), \epsilon_n(x) \to 0$ when $n \to \infty$. Therefore, we find

$$\lim_{n\to\infty} \frac{\boldsymbol{f}(z_n) - \boldsymbol{f}(y_n)}{z_n - y_n} = \boldsymbol{f}'(x).$$

Hence, since ${\boldsymbol{l}'_n}^{+}(x) = \frac{\boldsymbol{f}(z_n) - \boldsymbol{f}(y_n)}{z_n - y_n}$, we get the desired result.

2. There is m such that $y_n = x$, for $n \geq m$. We use $\boldsymbol{f}(z_n) - \boldsymbol{f}(x) = \boldsymbol{f}'(x)(z_n - x) + \varepsilon_n(x)(z_n - x)$, where $\varepsilon_n(x) \to 0$, when $n \to \infty$. Therefore, we find

$$\lim_{n\to\infty} \frac{\boldsymbol{f}(z_n) - \boldsymbol{f}(y_n)}{z_n - y_n} = \boldsymbol{f}'(x).$$

Hence, since ${\boldsymbol{l}'_n}^{+}(x) = \frac{\boldsymbol{f}(z_n) - \boldsymbol{f}(y_n)}{z_n - y_n}$, we get the desired result.

Similarly, for $x \in (a,b]$ we obtain $\lim_{n\to\infty} \boldsymbol{l}_n'^{-}(x) = \boldsymbol{f}'(x)$.

Step 2. If the graph of function $\boldsymbol{h}$, which satisfies the condition (ch1),(ch2) and (ch3), is a polygonal line, then the solution is presented in Example 8.

Step 3. Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of successive partition of $[0,T]$ for which $\lambda(P_n) \to 0$ and $\boldsymbol{l}_n$ is a function whose graph is the corresponding polygonal line for P_n and $\boldsymbol{h}$. As in Example 8, for every $\boldsymbol{l}_n$ we define $\boldsymbol{r}_n$. Because $\boldsymbol{r}_n$ is decreasing there is an extended graph $\Gamma^*_{\boldsymbol{r}_n}$ of $\boldsymbol{r}_n$. The surface $\Sigma_n = \Sigma_{\Gamma^*_{\boldsymbol{r}_n}}$ which we get in this way is called the *extended cylindric surface*. Similarly as we defined a vessel (using the corresponding elementary surface of revolution) we define an *extended cylindric vessel* (using the corresponding extended cylindric surface) and identify an extended cylindric surface and the extended cylindric vessel.

We study the limit of Σ_n via the limit of $\boldsymbol{r}_n$ when $n \to \infty$.

In example 8, we have shown that

$$\boldsymbol{r}_n(\boldsymbol{h}(t)) = \sqrt{\frac{v_0}{\pi \boldsymbol{l}_n'^{+}(t)}}, \qquad t \in [0,T), \tag{4}$$

respectively

$$\boldsymbol{r}_n(\boldsymbol{h}(T)) = \sqrt{\frac{v_0}{\pi \boldsymbol{l}_n'^{-}(T)}}.$$

Hence, if $\boldsymbol{h}'(t)$ exists, we get

$$\lim_{n\to\infty} \boldsymbol{r}_n(\boldsymbol{h}(t)) = \lim_{n\to\infty} \sqrt{\frac{v_0}{\pi \boldsymbol{l}_n'^{+}(t)}} = \sqrt{\frac{v_0}{\pi \boldsymbol{h}'(t)}}\,.$$

Therefore, we can define function $\boldsymbol{r}\colon [0,H]\setminus \boldsymbol{h}(D) \to \mathbb{R}$, where D is the set of points $t \in [0,T]$ such that $\boldsymbol{h}'(t)$ does not exist (for simplicity the reader can assume that set D is finite). The function $\boldsymbol{r}$ is determined by the formula

$$\boldsymbol{r}(\boldsymbol{h}(t)) = \sqrt{\frac{v_0}{\pi \boldsymbol{h}'(t)}}.$$

Since the function $\boldsymbol{h}$ is convex we conclude that $\boldsymbol{r}$ is decreasing, and we define an extended graph $\Gamma^*_{\boldsymbol{r}}$ of $\boldsymbol{r}$ and get the extended elementary surface $\Sigma = \Sigma_{\Gamma^*_{\boldsymbol{r}}}$.

We can first assume that convex function $\boldsymbol{h}$ has derivatives except at most finite set and the interested reader can check that

a) function $\boldsymbol{h}$ is the corresponding height filling function for the extended vessel $\Sigma_{\Gamma^*_{\boldsymbol{r}}}$ obtained in this way and that

b) $\Sigma_{\Gamma^*_{\boldsymbol{r}}}$ is also the limit of the corresponding sequence of extended cylindric vessels.

Step 4. Using the above approach and the fact that a convex function has a derivative everywhere except on at most countable set, we can find the corresponding surface of revolution if $\boldsymbol{h}$ is a general convex function. More precisely we can show that $\Sigma_{\Gamma^*_{\boldsymbol{r}}}$ is the corresponding surface of revolution.

We leave details to the interested reader. We collect the obtained results in the following statement:

THEOREM 4. *Suppose that a height filling function* $\boldsymbol{h}$ *satisfy the conditions* $(ch1)$, $(ch2)$ *and* $(ch3)$. *There is a vessel such that* $\boldsymbol{h}$ *is the corresponding height filling function. Let* $(P_n)_{n\in\mathbb{N}}$ *be a sequence of successive partition of* $[0,T]$ *for which* $\lambda(P_n) \to 0$, *when* $n \to \infty$ *and* $\boldsymbol{l}_n$ *the function whose graph is the corresponding polygonal line for* P_n *and* $\boldsymbol{h}$. *Then* $\lim_{n\to\infty} {\boldsymbol{l}'_n}^+(t) = \boldsymbol{h}'(t)$ *for* $t \in [0,T) \setminus D$, *where* D *is a countable set. If* $\boldsymbol{r}_n$ *is defined by formula* (4), *then* $\Gamma^*_{\boldsymbol{r}_n}$ *is the generatrix of extended cylindric surface. Then* $\boldsymbol{r}_n(x)$ *converges to a* $\boldsymbol{r}(x)$ *for* $x \in [0,H] \setminus \boldsymbol{h}(D)$; $\boldsymbol{r}$ *is a decreasing function. If* Σ *is a surface (a vessel) with generatrix* $\Gamma^*_{\boldsymbol{r}}$, *then* $\boldsymbol{h}$ *is the corresponding height filling function.*

Since the volume and time are proportional magnitudes ($V = v_0 t$), we can consider the height-volume function which describes dependence $h = \boldsymbol{h}(V)$ and the its inverse function which we call the volume-height function and which describes the dependence $V = \boldsymbol{V}(h)$. In this way we eliminate a liquid (water) flow (with the constant inflow rate v_0) into a vessel from considerations and consider a pure mathematical problem. We leave details to the interested reader.

At the end, let us note that the case of decreasing functions corresponds to the situation when the liquid is left to flow out of a vessel. Similarly, the condition $\boldsymbol{h}(0) = 0$ can be changed into $\boldsymbol{h}(0) = a$ by supposing that some liquid already exists at the level a and the change of the interval $[0,T]$ is a simple matter of shifting the time.

4. Monotone and convex functions

In this section we collect some properties of monotone and convex functions.

4.1.Monotone functions

In calculus, a function $\boldsymbol{f}$ defined on a subset of the real numbers with real values is called *monotonically increasing* (also *increasing*, or *non-decreasing*), if for all x and y such that $x \le y$ one has $\boldsymbol{f}(x) \le \boldsymbol{f}(y)$, so $\boldsymbol{f}$ preserves the order. Likewise, a function is called *monotonically decreasing* (also *decreasing*, or *non-increasing*) if $\boldsymbol{f}(x) \ge \boldsymbol{f}(y)$, whenever $x \le y$, so $\boldsymbol{f}$ reverses the order.

If the order $\le$ in the definition of monotonicity is replaced by the strict order $<$, then one obtains a stronger requirement. A function with this property is called strictly increasing. Again, by inverting the order symbol, one finds a corresponding concept called strictly decreasing. Functions that are strictly increasing or decreasing are one-to-one (because for x not equal to y, either $x < y$ or $x > y$ and so, by monotonicity, either $\boldsymbol{f}(x) < \boldsymbol{f}(y)$ or $\boldsymbol{f}(x) > \boldsymbol{f}(y)$, thus $\boldsymbol{f}(x)$ is not equal to $\boldsymbol{f}(y)$). The terms non-decreasing and non-increasing avoid any possible confusion with strictly increasing and strictly decreasing, respectively.

Some basic applications and results:

The following properties are true for a monotonic function $\boldsymbol{f}\colon \mathbb{R} \to \mathbb{R}$.

(m1) $\boldsymbol{f}$ has limits from the right and from the left at every point of its domain;

(m2) $\boldsymbol{f}$ has a limit at infinity (either ∞ or $-\infty$) of either a real number, ∞, or $-\infty$;

(m3) $\boldsymbol{f}$ can only have jump discontinuities;

(m4) $\boldsymbol{f}$ can only have countably many discontinuities in its domain.

These properties are the reason why monotonic functions are useful in technical work in analysis. Two facts about these functions are:

(m5) if $\boldsymbol{f}$ is a monotonic function defined on an interval I, then $\boldsymbol{f}$ is differentiable almost everywhere on I, i.e. the set of numbers x in I such that $\boldsymbol{f}$ is not differentiable in x has Lebesgue measure zero.

(m6) if $\boldsymbol{f}$ is a monotonic function defined on an interval $[a,b]$, then $\boldsymbol{f}$ is Riemann integrable on interval $[a,b]$.

4.2. Convex functions

A function $\boldsymbol{f}\colon (a,b)\to\mathbb{R}$ is called *convex* if for all $x,y\in(a,b)$ and $\lambda\in[0,1]$ the inequality

$$\boldsymbol{f}(\lambda x+(1-\lambda)y)\le\lambda\boldsymbol{f}(x)+(1-\lambda)\boldsymbol{f}(y)$$

holds. A function $\boldsymbol{f}$ is called *concave* if the function $-\boldsymbol{f}$ is convex.

Some basic applications and results:

The following properties are true for a convex function $\boldsymbol{f}\colon\mathbb{R}\to\mathbb{R}$.

(c1) for all $x,y,z\in\mathbb{R}$ such that $y<x<z$ holds

$$\frac{\boldsymbol{f}(x)-\boldsymbol{f}(y)}{x-y}\le\frac{\boldsymbol{f}(z)-\boldsymbol{f}(x)}{z-x};$$

(c2) for all $x\in\mathbb{R}$ there exist left derivative at x, which we denote by $\boldsymbol{f}'_-(x)$ and right derivative at x, which we denote by $\boldsymbol{f}'_+(x)$;

(c3) for all $x,y\in\mathbb{R}$ such that $x<y$

$$\boldsymbol{f}'_-(x)\le\boldsymbol{f}'_+(x)\le\boldsymbol{f}'_-(y)\le\boldsymbol{f}'_+(y)$$

holds. Therefore the functions $\boldsymbol{f}'_-$ and $\boldsymbol{f}'_+$ are increasing;

(c4) the set of points x for which $\boldsymbol{f}'_-(x)\ne\boldsymbol{f}'_+(x)$ is at most countable;

(c5) for all $x\in\mathbb{R}$ there exists the finite limit (the Schwarz's derivative) [3, pp. 66–67],

$$\lim_{h\to0}\frac{\boldsymbol{f}(x+h)-\boldsymbol{f}(x-h)}{2h}=\boldsymbol{f}'_*(x)\,;$$

(c6) $\boldsymbol{f}'_*$ is an increasing function;

(c7) for all $a,b\in\mathbb{R}$ such that $a<b$, $(\boldsymbol{f}'_*)'\in L^1(a,b)$ holds.

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