

## THE DEFINITE INTEGRAL AND COMPUTER

Dragoslav Herceg and Đorđe Herceg

**Abstract.** In this paper the results of the learning concept of definite integral and numeric integration with the computer is presented. The tested students attend Šabac Chemical Technological college. The aim of this test was to check the student's theoretical, visual and practical knowledge of definite integral.

*ZDM Subject Classification:* I54; *AMS Subject Classification:* 97D40, 97D80, 97U70.

*Key words and phrases:* CAS; DGS; dynamic software; teaching-learning process; integration; numeric integration.

### 1. Introduction

In almost all secondary schools definite integral and its applications are studied. The concept of definite integral is almost always introduced as the Riemann integral, which is in turn defined in terms of Riemann sums, and its geometric interpretation. For secondary school pupils, as well as for high school and university students, this definition is hard to understand. The courses of Numerical mathematics at all levels contain topics on numerical integration, which is partly based on Riemann sums. With the aid of mathematical software for visualization and computation of approximate integrals, the notion of definite integral and its calculation is more easily adopted by pupils and students.

We present one possible approach to lecturing on approximate integration, based on our experience with the students of Šabac Chemical technological college and the students of Computer Science at the Faculty of Sciences, Novi Sad.

We start by introducing the definite integral. The concept is explained by calculation of the area of a figure, and it is then formalized by using the Riemann sums and its geometrical interpretation. Left, middle and right rectangular rule are introduced from the start. Trapezoidal and Simpson's rule are introduced soon afterwards. Simpson's rule is first explained in its simple form, and later in the complex form. The computer was used to produce graphs and pictures, and substantial attention was devoted to developing interactive illustrations which are animated by changing parameters.

In the class we used **GeoGebra** and **Mathematica** for visualization. While **Mathematica** satisfies most of our computing needs, it lacks interactivity, which **GeoGebra** supplements in more than one way. Besides, **GeoGebra** is much simpler to use, and it is available in many languages, including pupils' native Serbian. Of no lesser importance is the fact that **GeoGebra** is free, while **Mathematica** is not.

We used applets made in *Mathematica*, which were published on the accompanying CD with the book [2], and on the web, [www.herceg.rs/](http://www.herceg.rs/). *GeoGebra* applets were also used, [www.herceg.rs/](http://www.herceg.rs/). The students used the same applets in their work. The CD contained additional examples, which were not in the book. This allowed the students to practice individually, study our examples and make their own ones.

Two generations of high school students were tested after the topic on numerical integration was completed. Every generation took two tests. Test results show that the use of software has had significant positive impact on student's knowledge gained in this course.

## 2. The area problem

In integral calculus the area problem leads us to formulate idea of definite integral in terms of limits. If we want to use a computer in teaching we may follow suggestions given by Tall [5]. Tall suggested how concepts of the Calculus could be approached globally using computer. The idea of area under a graph presents a fundamentally great problem. Each calculation of the approximation of area under a graph requires many intermediate calculations. Using algebraic methods the summation in all but the simplest examples becomes exceedingly difficult. The calculator enables area approximations to be computed numerically. A computer with flexible computer language can allow much more powerful methods to be used. This leads to far more accurate calculations and can be most helpful for inducing algebraic formulae from numerical results, [5].

The area problem can be formulated as follows: Find area  $A$  of the region  $S$  bounded by the graph of function  $f$ , the vertical lines  $x = a$  and  $x = b$ , and the  $x$ -axis, where  $f(x) \geq 0$  for  $x \in [a, b]$ . The region

$$S = \{ (x, y) \mid a \leq x \leq b, 0 \leq y \leq f(x) \}$$

is illustrated in Figure 1.

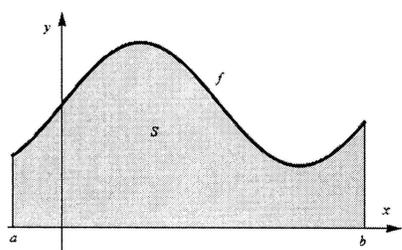


Fig. 1

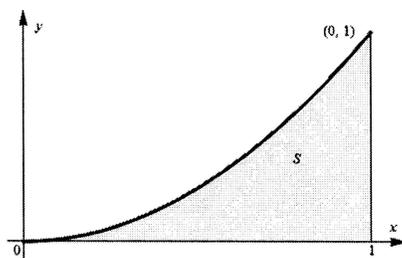


Fig. 2

Let us try to find area under parabola  $y = x^2$  from  $a = 0$  to  $x = b$ , the region  $S = \{ (x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2 \}$ , Figure 2. One method of approximating the desired area is to divide the interval  $[0, 1]$  into subintervals of equal length and consider the rectangles whose bases are these subintervals and whose heights are

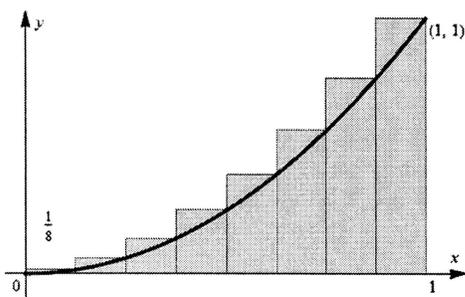


Fig. 3

the values of the function at the right-hand endpoints of these subintervals. Figure 3 shows the approximation of the parabolic region by eight rectangles.

Let  $A_n$  be the sum of areas of  $n$  rectangles. Each rectangle has width  $1/n$  and the heights are  $(i/n)^2$ ,  $i = 1, 2, \dots, n$ . Thus

$$\begin{aligned} A_n &= \sum_{k=1}^n \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} (1^2 + 2^2 + \dots + n^2) \\ &= \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}. \end{aligned}$$

Using this we obtain the following tables:

$n$	2	4	8	20	40	200	1000
$A_n$	$\frac{5}{8}$	$\frac{15}{32}$	$\frac{51}{128}$	$\frac{287}{800}$	$\frac{1107}{3200}$	$\frac{26867}{80000}$	$\frac{6767}{20000}$

Table 1

$n$	2	4	8	20	40	200	1000
$A_n$	0.625	0.46875	0.39844	0.35865	0.34594	0.33584	0.33835

Table 2

It looks as if  $A_n$  becomes closer to  $\frac{1}{3}$  as  $n$  increases. In fact,

$$\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$$

So, we accept

$$A = \lim_{n \rightarrow \infty} A_n = \frac{1}{3}.$$

Using these algebraic methods one can consider the summation in all but the simplest examples. If we use dynamic mathematics software, we can offer more.

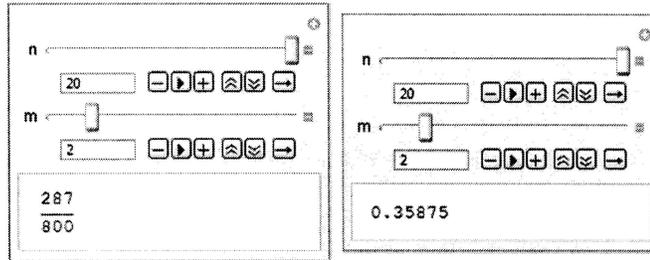


Fig. 4

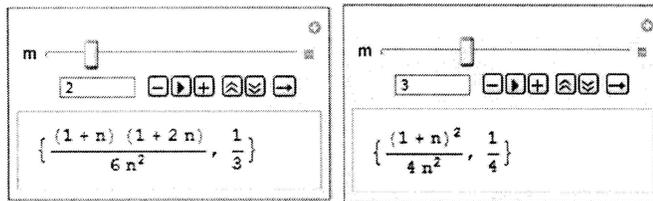


Fig. 5

Let us try to find approximations of the area under parabola  $y = x^m$  from  $x = 0$  to  $x = 1$  for different values of  $m$ . On Figure 4 we present use of right endpoint approximation. Results are presented as fractions and as decimal numbers. On Figure 5 we give the approximation of area under parabola  $y = x^m$  from  $x = 0$  to  $x = 1$  for different values of  $m$  obtained using right endpoint approximation with  $n$  subintervals of equal length and corresponding limit values as  $n \rightarrow \infty$ .

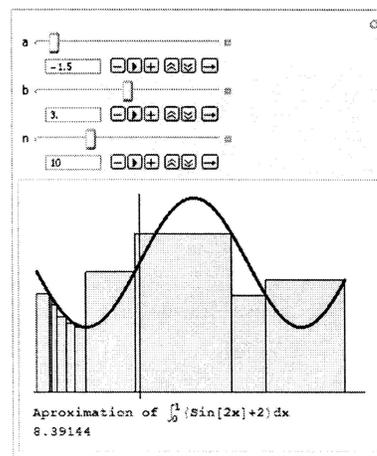


Fig. 6

Firstly, we can consider  $S$  as region which depends on  $a$  and  $b$ . Figure 6

dynamically visualizes the concept of the Riemann integral sum. In addition to graph of functions  $f$  two points  $a$  and  $b$  are shown that can be moved along the  $x$ -axis in order to modify the considered interval. Using a slider, say  $n$ , one can change the number of rectangles used to calculate the right-hand sum in respect to function  $f$  within this interval. The values of this sum are displayed as dynamic text which automatically adapts to modifications.

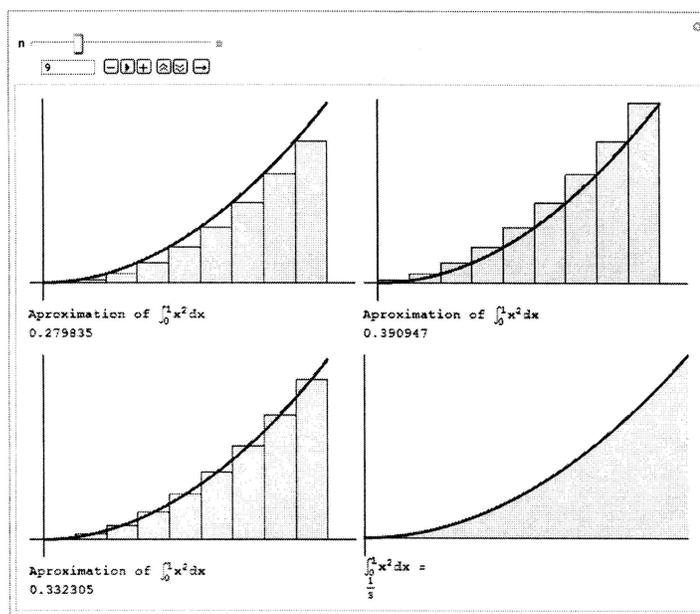


Fig. 7

In a similar way we can consider left-hand sum and midpoint sum. All sums, right-hand, left-hand and midpoint, one can see at once, Figure 7. Using *Mathematica* one can investigate how these sums change when  $n$  tends to infinity. In our CD (in the book [2]) the routines for graphical presentations of left-hand, right-hand and midpoint sum are given. These routines may be changed so that we can use slider to change the number of subintervals, see [www.herceg.rs/](http://www.herceg.rs/).

In applying idea of this example, where we considered the area under parabola  $y = x^2$ , to more general region  $S$  of Figure 1, we have no need to use rectangles of equal width. We start by subdividing the interval  $[a, b]$  into  $n$  subintervals  $[x_{i-1}, x_i]$  by choosing partition points  $x_0, x_1, \dots, x_n$  so that

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$

This subdivision is called a partition of  $[a, b]$  and we denote it by  $P$ . Let us define  $\Delta x_i = x_i - x_{i-1}$  and  $\lambda(P) = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$ . By drawing the lines

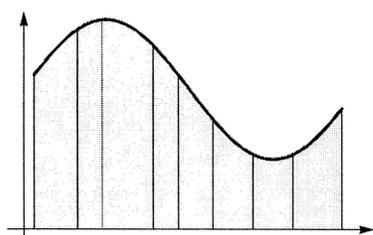


Fig. 8

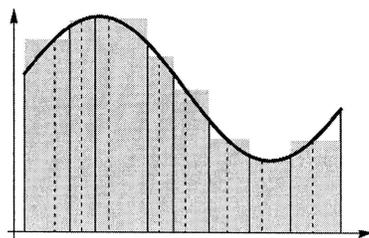


Fig. 9

$x = x_i$ ,  $i = 0, 1, \dots, n$  we use the partition  $P$  to divide region  $S$  into strips  $S_1, S_2, \dots, S_n$ , as in Figure 8.

If we choose a number  $c_i \in [x_{i-1}, x_i]$ ,  $i = 1, 2, \dots, n$  and construct a rectangle  $R_i$  with the base  $\Delta x_i$  and height  $f(c_i)$ , we can approximate strip  $S_i$  by rectangle  $R_i$ , Figure 9.

Each  $c_i$  can be anywhere in its subinterval. The area of rectangle  $R_i$  is

$$A_i = f(c_i) \Delta x_i.$$

The  $n$  rectangles  $R_1, R_2, \dots, R_n$  form a polygonal approximation to the region  $S$ . The area  $A$  of region  $S$  is approximated by sum of the areas of these rectangles, i.e.

$$A \approx A_n = \sum_{i=1}^n f(c_i) (x_i - x_{i-1}),$$

Using our `Mathematica` routine one can see how this approximation changes when  $n$  tends to infinity. This approximation appears to become better and better as the strips become thinner and thinner, i.e. as  $\lambda(P) \rightarrow 0$ . Therefore we define the area  $A$  of the region  $S$  as

$$(1) \quad \lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

if this limit exists. The preceding discussion and our dynamic figure show that the definition of area corresponds to our intuitive feeling of what area ought to be.

The limit of the form (1) arises when we compute an area. It turns out that the same type of limit occurs in a wide variety of situations. We therefore give this type of limit a special name and notation. So we have the following definition.

**DEFINITION 1.** If  $f$  is a function defined in  $[a, b]$ , let  $P$  be a partition of  $[a, b]$  with partition points  $x_0, x_1, \dots, x_n$  where  $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ . Let us denote  $\Delta x_i = x_i - x_{i-1}$ ,  $i = 1, 2, \dots, n$ ,  $\lambda(P) = \max \{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$  and let  $c_i$  be an arbitrary point in  $[x_{i-1}, x_i]$ . Then the definite integral of  $f$  from  $a$  to  $b$  is

$$\int_a^b f(x) dx = \lim_{\lambda(P) \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

if this limit exists.

### 3. Numerical integration

If we need to calculate an approximation of a definite integral, we can use some Riemann sums. Recall that definite integral is defined as the limit of Riemann suma, so any Riemann sum could be used as an approximation to the integral. In particular, if we take the Riemann sum

$$A_n = \sum_{i=1}^n f(c_i)(x_i - x_{i-1})$$

with  $c_i = x_{i-1}$ , then

$$\int_a^b f(x) dx \approx L_n = \sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1}).$$

The approximation  $L_n$  is called left endpoint approximation.

The second choice of  $c_i$  is  $c_i = x_i$ . In this case we have

$$\int_a^b f(x) dx \approx R_n = \sum_{i=1}^n f(x_i)(x_i - x_{i-1})$$

and this approximation is called right endpoint approximation.

If we choose  $c_i = \frac{1}{2}(x_{i-1} + x_i)$ , we obtain midpoint approximation

$$\int_a^b f(x) dx \approx M_n = \sum_{i=1}^n f\left(\frac{1}{2}(x_{i-1} + x_i)\right)(x_i - x_{i-1}).$$

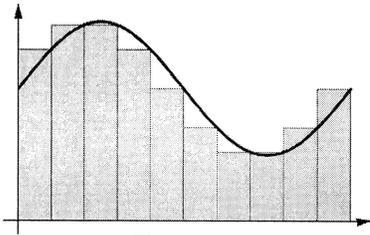


Fig. 10

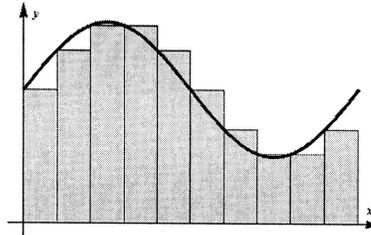


Fig. 11

In a special case, when we take a partition of  $[a, b]$  into  $n$  subintervals of equal length  $h = (b - a)/n$ , we have

$$x_i = a + ih = a + i \frac{b-a}{n}, \quad i = 0, 1, \dots, n$$

and the following three approximations

$$\int_a^b f(x) dx \approx L_n = \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + i \frac{b-a}{n}\right),$$

$$\int_a^b f(x) dx \approx R_n = \frac{b-a}{n} \sum_{i=1}^n f\left(a + i \frac{b-a}{n}\right),$$

$$\int_a^b f(x) dx \approx M_n = \frac{b-a}{n} \sum_{i=1}^n f\left(a + i \frac{b-a}{n} - \frac{b-a}{2n}\right).$$

Figures 10, 11 and 12 show  $L_n$ ,  $R_n$  and  $M_n$  for  $n = 10$ ,  $a = 0$ ,  $b = 1$  and  $f(x) = \sin x + 2$  respectively.

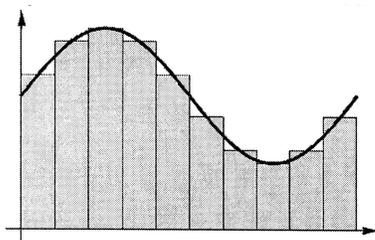


Fig. 12

Another approximation results from averaging the approximations  $L_n$  and  $R_n$ :

$$\int_a^b f(x) dx \approx T_n = \frac{1}{2} \left( \sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1}) + \sum_{i=1}^n f(x_i)(x_i - x_{i-1}) \right)$$

$$= \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2} (x_i - x_{i-1}).$$

The formula  $T_n$  is called Trapezoidal Rule. The reason for this name can be seen from Figures 13, where we give Trapezoidal Rule for  $n = 1$ . Figure 14 shows Composite Trapezoidal Rule, i.e. the case  $n > 1$ ,

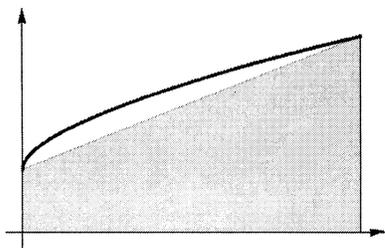


Fig. 13

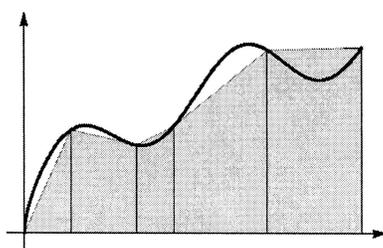


Fig. 14

which illustrates the case  $f(x) \geq 0$ . The area of trapezoid that lies above the  $i$ -th subinterval is

$$\frac{f(x_{i-1}) + f(x_i)}{2} (x_i - x_{i-1})$$

and if we add the areas of all these trapezoids we get  $T_n$ . In equidistant case, i.e. if  $x_i = a + i\frac{b-a}{n}$ ,  $i = 0, 1, \dots, n$ , we obtain Trapezoidal Rule in the form

$$T_n = \frac{b-a}{2n} \left( f(a) + 2 \sum_{i=1}^{n-1} f\left(a + i\frac{b-a}{n}\right) + f(b) \right).$$

Let us consider a partition of interval  $[a, b]$  into  $n$  subintervals, but this time we assume that  $n$  is an even number, where  $a = x_0 < x_2 < x_4 < \dots < x_{n-2} < x_n = b$  and  $x_{2i-1} = \frac{1}{2}(x_{2i-2} + x_{2i})$ . In this case we can consider approximation of a definite integral as weighted average of  $T_{n/2}$  and  $M_{n/2}$ :

$$S_n = \frac{1}{3}T_{n/2} + \frac{2}{3}M_{n/2},$$

where  $T_{n/2}$  and  $M_{n/2}$  denote trapezoidal formula and midpoint formula for partition  $a = x_0 < x_2 < x_4 < \dots < x_{n-2} < x_n = b$ , respectively. So, we obtain

$$\begin{aligned} S_n &= \frac{1}{3} \sum_{i=1}^{n/2} \frac{f(x_{2i-2}) + f(x_{2i})}{2} (x_{2i} - x_{2i-2}) \\ &\quad + \frac{2}{3} \sum_{i=1}^{n/2} f\left(\frac{1}{2}(x_{2i-2} + x_{2i})\right) (x_{2i} - x_{2i-2}) \\ &= \frac{1}{6} \sum_{i=1}^{n/2} (f(x_{2i-2}) + f(x_{2i}) + 4f(x_{2i-1})) (x_{2i} - x_{2i-2}). \end{aligned}$$

The approximation  $S_n$  is called Simpson's rule. In equidistant case, i.e. if  $x_i = a + i\frac{b-a}{n}$ , approximation  $S_n$  ( $n$  an even positive integer) is given by

$$\begin{aligned} S_n &= \frac{b-a}{3n} \left( f(a) + 4 \sum_{i=1}^{n/2} f\left(a + (2i-1)\frac{b-a}{n}\right) \right. \\ &\quad \left. + 2 \sum_{i=1}^{n/2-1} f\left(a + 2i\frac{b-a}{n}\right) + f(b) \right). \end{aligned}$$

The Simpson's rule can be derived in various ways. One derivation replaces the integrand  $f$  by the quadratic polynomial, i.e. the parabola, which takes the same values as  $f$  at the end points  $a$  and  $b$  and the midpoint  $m = (a+b)/2$ , Figure 15.

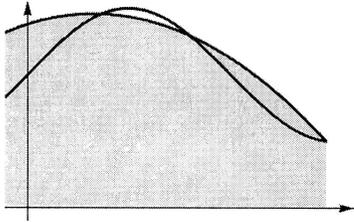


Fig. 15

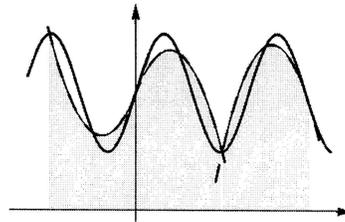


Fig. 16

If the interval of integration  $[a, b]$  is “small” in some sense, then Simpson’s rule will provide an adequate approximation to the exact integral. By small, what we really mean is that the function being integrated is relatively smooth over the interval  $[a, b]$ . For such a function, a smooth quadratic interpolate like the one used in Simpson’s rule will give good results.

In many cases, Simpson’s rule may give very poor results. One common way of handling this problem is by breaking up the interval  $[a, b]$  into a number of small subintervals. Simpson’s rule is then applied to each subinterval, with the results being summed to produce an approximation for the integral over the entire interval. This sort of approach is termed the composite Simpson’s rule, Figure 16.

#### 4. Experience

We taught Numerical analysis to the students of the Chemical technological college in Šabac during school years 2006/07 and 2007/08. The students were divided into two groups each year. During the previous school year, the students have taken an exam in Mathematics, which not all had passed. The number of students which have passed the exam, as well as their grade average, and the number of student which have failed, is shown in Table 3. The grades range from 5 (fail), 6 (worst) to 10 (best).

Group, School year	Passed / Grade average	Failed
First, 2006/07	13 (59%) / 7.77	8 (41%)
Second, 2006/07	21 (52%) / 7.76	19 (48%)
First, 2007/08	18 (54%) / 7.89	15 (46%)
Second, 2007/08	38 (56%) / 7.92	29 (44%)

Table 3.

Based on this data one can conclude that the two groups were well balanced, with regard to students’ proficiency.

The topic on numerical integration was based on the definition of the definite integral. Special attention was paid to the Riemann sums and the limit of Riemann sums. Our intention was to show that the Riemann sums should be thought of as the approximate value of the definite integral, and to demonstrate the behavior of the integration error.

Classes for the first group of students were based on the materials from the book [2]. Calculation of the Riemann sums with random partitioning of the integration interval was repeated multiple times, and supported with the graphical representation. Left, right and middle Riemann sums were considered, as they serve as the basis for quadrature formulas. Various integrands were shown, with different domains of integration and different numbers of points in the partitions. The Trapezoid rule and Simpson’s rule were also shown together with their graphical representation, first for simple integrands and then for more complex ones. It is important to point out that the graphs generated for Simpson’s rule were very useful in class, and that they are not easily drawn without software.

Classes for the second group of students covered the same topics, but they were taught in a conventional way, using chalk and blackboard. The graphs were the same as for the first group, but they were drawn by hand. The classical teaching method was employed, meaning that the lectures followed the textbooks and the numerical examples were previously prepared. Numerical results were presented to the students, with no calculations actually performed in class. The parameters could not be varied. The drawings used were just sketches, and did not always accurately represent the chosen data. The sketches were not always aesthetically satisfactory, regardless of our effort. For example, we could only show one Riemann sum with previously chosen partition and a pre-calculated result.

After the material has been explained, a test was conducted in both groups, which contained six questions:

1. Write the definition of definite integral. (20 points)
2. Write the definition of Riemann sum for  $\int_a^b f(x) dx$ . (20 points)
3. Write formula for left endpoint approximation of  $\int_a^b f(x) dx$ . (15 points)
4. Illustrate Composite Trapezoidal Rule for  $\int_a^b f(x) dx$ . (15 points)
5. Illustrate simple Simpson's Rule for  $\int_a^b f(x) dx$ . (15 points)
6. Write the remainder term for the composite Simpson's rule for  $\int_a^b f(x) dx$ . (15 points)

Table 4 shows point average by groups for generations of 2006/07 and 2007/08. Table 5 shows the results from the test. It can be easily seen that the group which experimented with a computer with *Mathematica* and *GeoGebra* and additional programs, achieved better results.

	Group 1		Group 2	
	Count	Point average	Count	Point average
2006/07	21	59.52	40	51.38
2007/08	33	58.64	67	47.16
Sum	54	58.98	107	48.74

Table 4.

	Group 1	Group 2
Count	54	107
Average	58.9815	48.7383
Standard deviation	22.7865	21.8603
Minimum	15	15
Maximum	100	100
Confidence interval	[52.7619, 65.201]	[44.5485, 52.9282]

Table 5. Summary statistics

We use a *t*-test to compare the means of the two groups of students. It also constructs confidence intervals or bounds for each mean. Since the computed

$t = 2.76744$  and  $p$ -value ( $p = 0.00632$ ) is less than 0.05, we can reject the null hypothesis (the difference between the two means equals 0.0) in favor of the alternative (the difference does not equal 0.0), i.e., there is a statistically significant difference between the means of the two samples at the 95,0% confidence level.

## 5. Conclusion

The advantages introduced by a computer are not only in quicker calculation and drawing. The computer does all the tedious work, which leaves the teacher and the pupils with enough time to discuss the problem, try out multiple ideas and approaches to solving, and, finally, compare and analyze them. The method of solving a problem is as important as its solution. Use of a computer is particularly important when working with pupils who have difficulties understanding all the aspects of solving a mathematical problem. They are freed from uninspiring and time-consuming solving by hand, so they have more time to learn the important points.

Starting from the problem of determining the area under a curve and using the Riemann sums we defined the definite integral. **GeoGebra** and **Mathematica** applets were used for visualisation. Algebraic approach was also used, as we calculated intermediate values and the final values.

Our goal was to demonstrate how numerical integration formulae can be derived from the Riemann sums. It is essential for students to understand and master the concept of definite integral defined by limit of Riemann sums. Numerical integration formulae then follow as a special case of Riemann sums and their linear combinations.

Testing results show that the described method of derivation of numerical integration formulae has a positive impact on the students' knowledge and skills.

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Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad  
*E-mail*: hercegd@im.ns.ac.rs