THE TEACHING OF MATHEMATICS
2009, Vol. XII, 1, pp. 7–14

NEW CLOSE FORM APPROXIMATIONS OF $\ln(1 + x)$

Sanjay Kumar Khattri

Abstract. Based on Newton-Cotes and Gaussian quadrature rules, we develop several closed form approximations to $\ln(1 + x)$. We also compare our formulae to the Taylor series expansion. Another objective of our work is to inspire students to formulate other better approximations by using the presented approach. Because of the level of mathematics, the presented work is easily embraceable in an undergraduate class.

ZDM Subject Classification: N45; *AMS Subject Classification*: 00A35.

Key words and phrases: Quadrature rules; closed form approximation; logarithm.

1. Introduction

The expression $\ln(1 + x)$ is an important expression in mathematics. It shows up surprisingly at many places [2, 4, 5, 6]. Generally, we approximate $\ln(1 + x)$ through a finite sum of an infinite series [3]. But, the number of terms in a finite sum can make the algebra very complicated. Thus, we develop some simple but robust closed form approximations to $\ln(1 + x)$ through quadrature rules.

The Taylor series expansion of $\ln(1 + x)$ is

$$\ln(1+x) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{x^i}{i} \qquad \text{for} \quad -1 < x \leq 1. \tag{1}$$

First year undergraduate students are exposed to concepts of limits and quadrature. Through these concepts, we develop closed form approximations. The work presented in this paper is crucial for enhancing concepts such as convergence and approximation of undergraduate students. The work will encourage students to formulate other relations for representing $\ln(1 + x)$ by using other quadrature rules, for example, Lobatto [9, 11]. Based on our work, teacher can ask students to formulate even better approximations to the mathematical expression $\ln(1 + x)$.

Figure 1 presents a graph of the function $1/x$. The area under the graph and between the vertical lines $x = n$ and $x = n + 1$ is given as

$$\int_n^{n+1} \frac{1}{x}\, dx.$$

The exact value of this integral is $\ln(1 + \frac{1}{n})$. For forming various closed form approximations, we will approximate this integral through different quadrature rules.

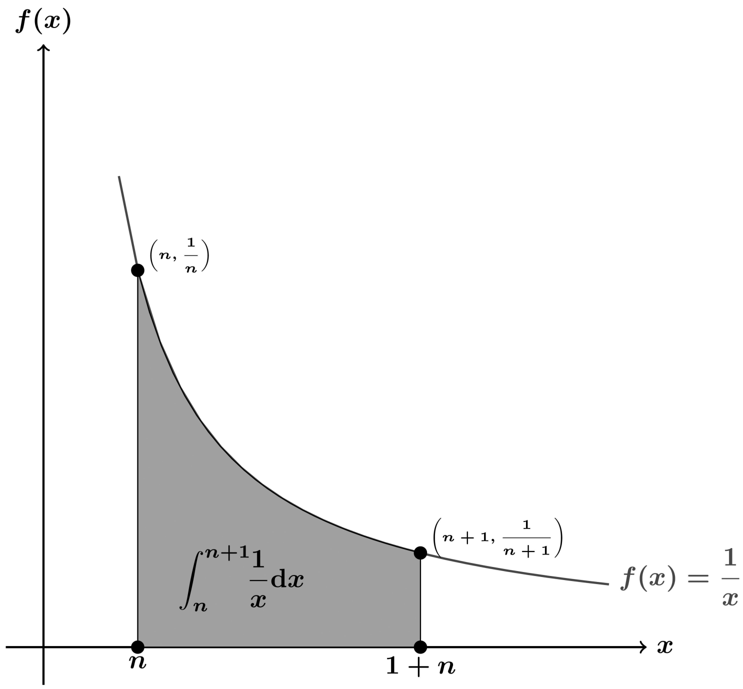

Fig. 1

2. Approximation through trapezoidal quadrature rule

Trapezoidal quadrature rule [14] is given as

$$\int_n^{n+1} \frac{1}{x}\, dx \approx \frac{h}{2}\left[f(x_1) + f(x_2)\right].$$

Here, $h = 1$, $x_1 = n$ and $x_2 = n + 1$. Thus,

$$[\ln x]_n^{n+1} \approx \frac{1}{2}\left[\frac{1}{n} + \frac{1}{n+1}\right],$$

$$\ln\left(1 + \frac{1}{n}\right) \approx \frac{1}{n}\left[\frac{n + \frac{1}{2}}{n+1}\right] = \frac{1}{n}\left[\frac{1 + \frac{0.5}{n}}{1 + \frac{1}{n}}\right].$$

Now replacing $\frac{1}{n}$ by x in the above equation we get the expression of $\ln(1 + x)$ through the trapezoidal rule

$$\ln(1 + x) \approx x \left[\frac{1 + 0.5\, x}{1 + x}\right]. \tag{2}$$

Let us call this expression, the Trapezoidal Euler's Log (TELOG).

3. Approximation through Simpson's quadrature rule

The Simpson's $\frac{1}{3}$-quadrature rule [14] for approximating integrals is given as

$$\int_n^{n+1} \frac{1}{x}\, dx \approx \frac{h}{3}\, \left[f(x_0) + 4f(x_1) + f(x_2)\right].$$

Here, $h = \frac{1}{2}$, $x_0 = n$, $x_1 = \frac{2n+1}{2}$ and $x_2 = n+1$,

$$\int_n^{n+1} \frac{1}{x}\, dx \approx \frac{1}{6}\left[\frac{1}{n} + \frac{8}{2n+1} + \frac{1}{n+1}\right],$$

$$\ln\left(1+\frac{1}{n}\right) \approx \frac{1}{6}\left[\frac{1}{n} + \frac{\frac{8}{n}}{2+\frac{1}{n}} + \frac{\frac{1}{n}}{1+\frac{1}{n}}\right]. \tag{3}$$

Now replacing $\frac{1}{n}$ by x in the above equation we get the expression for $\ln(1+x)$ through the Simpson's quadrature rule as

$$\ln(1+x) \approx \frac{1}{6}\left[x + \frac{8x}{2+x} + \frac{x}{1+x}\right]. \tag{4}$$

Let us call this expression the $\frac{1}{3}$-Simpson Euler's Log ($\frac{1}{3}$-SELOG).

4. Aproximation through Simpson's $\frac{3}{8}$-quadrature rule

Approximation of the integral through Simpson's $\frac{3}{8}$-quadrature rule [14] is

$$\int_n^{n+1} \frac{1}{x}\, dx \approx \frac{3h}{8}\, \left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right].$$

Here, $h = \frac{1}{3}$, $x_0 = n$, $x_1 = \frac{3n+1}{3}$, $x_2 = \frac{3n+2}{3}$ and $x_3 = n+1$

$$\int_n^{n+1} \frac{1}{x}\, dx \approx \frac{1}{n}\left[\frac{36n^3+54n^2+20n+1}{(36n^3+54n^2+20n+1)+18n^2+24n+7}\right],$$

$$\ln\left(1+\frac{1}{n}\right) \approx \frac{1}{8}\left[\frac{1}{n} + \frac{\frac{9}{n}}{3+\frac{1}{n}} + \frac{\frac{9}{n}}{3+\frac{2}{n}} + \frac{\frac{1}{n}}{1+\frac{1}{n}}\right]. \tag{5}$$

Now replacing $\frac{1}{n}$ by x in the above equation we get the expression of $\ln(1+x)$ through the Simpson's quadrature rule

$$\ln(1+x) \approx \frac{1}{8}\left[x + \frac{9x}{3+x} + \frac{9x}{3+2x} + \frac{x}{1+x}\right]. \tag{6}$$

Let us call this expression the $\frac{3}{8}$-Simpson Euler's Log ($\frac{3}{8}$-SELOG).

5. Approximation through Boole's quadrature rule

The Boole's quadrature rule [13] is

$$\int_n^{n+1} \frac{1}{x}\, dx \approx \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right].$$

Here, $h = \frac{1}{4}$, $x_0 = n$, $x_1 = \frac{4n+1}{4}$, $x_2 = \frac{4n+2}{4}$, $x_3 = \frac{4n+3}{4}$ and $x_4 = n+1$,

$$\ln\left(1+\frac{1}{n}\right) \approx \frac{2}{4\times 45}\left[\frac{7}{n} + \frac{32\times 4}{4n+1} + \frac{12\times 4}{4n+2} + \frac{32\times 4}{4n+3} + \frac{7}{n+1}\right].$$

Now replacing $\frac{1}{n}$ by x in the above equation we get the expression of $\ln(1+x)$ through the Boole's rule

$$\ln(1+x) \approx \frac{1}{90}\left[7x + \frac{128x}{4+x} + \frac{48x}{4+2x} + \frac{128x}{4+3x} + \frac{7x}{1+x}\right]. \tag{7}$$

Let us call this expression the Boole Euler's Log (BELOG).

6. Approximation through Gauss-Legendre 2 point quadrature

The two point Gauss-Legendre Quadrature [12] is

$$\int_n^{n+1} \frac{1}{x}\,dx \approx k\,[w_1\,f(x_1) + w_2\,f(x_2)].$$

Here, $k = \frac{n+1-n}{2} = \frac{1}{2}$, $x_1 = \frac{2n+1}{2} + \frac{1}{2\sqrt{3}}$ and $x_3 = \frac{2n+1}{2} - \frac{1}{2\sqrt{3}}$. Weights are $w_1 = 1$ and $w_2 = 1$.

$$\int_n^{n+1} \frac{1}{x}\,dx \approx \frac{1}{2}\left[\frac{2\sqrt{3}}{(2n+1)\sqrt{3}+1} + \frac{2\sqrt{3}}{(2n+1)\sqrt{3}-1}\right],$$

$$\ln\left(1+\frac{1}{n}\right) \approx \frac{6n+3}{6n^2+6n+1} = \frac{\frac{6}{n}+\frac{3}{n^2}}{6+\frac{6}{n}+\frac{1}{n^2}}$$

Now replacing $\frac{1}{n}$ by x in the above equation we get the expression for $\ln(1+x)$ through the two point Gauss-Legendre quadrature rule as

$$\ln(1+x) \approx \frac{6x+3x^2}{6+6x+x^2} \tag{8}$$

Let us call this expression the two point Gauss-Legendre Log (2P-GLLOG).

7. Approximation through Gauss-Legendre 3 point quadrature

Three point Gauss-Legendre quadrature rule [12] is given as

$$\int_n^{n+1} \frac{1}{x}\,dx \approx k\,[w_1\,f(x_1) + w_2\,f(x_2) + w_3\,f(x_3)].$$

The weights w_i and points x_i are given as

$$w_1 = \frac{8}{9}, \qquad x_1 = \frac{2n+1}{2},$$

$$w_2 = \frac{5}{9}, \qquad x_2 = \frac{(2n+1)\sqrt{5}+\sqrt{3}}{2\sqrt{5}},$$

$$w_3 = \frac{5}{9}, \qquad x_3 = \frac{(2n+1)\sqrt{5}-\sqrt{3}}{2\sqrt{5}}.$$

Thus,

$$\ln\left(1+\frac{1}{n}\right) \approx \frac{1}{9}\left[6\,\frac{60n^2+60n+11}{(2n+1)\left(2\sqrt{5}n+\sqrt{5}+\sqrt{3}\right)\left(2\sqrt{5}n+\sqrt{5}-\sqrt{3}\right)}\right]$$
$$= \left[\frac{60\,n^2+60\,n+11}{60\,n^3+90\,n^2+36\,n+3}\right] = \left[\frac{\frac{60}{n}+\frac{60}{n^2}+\frac{11}{n^3}}{60+\frac{90}{n}+\frac{36}{n^2}+\frac{3}{n^3}}\right].$$

Now replacing $\frac{1}{n}$ by x in the above equation we get the expression $\ln(1+x)$ through the three point Gauss-Legendre quadrature rule as

$$\ln(1+x) \approx \left[\frac{60x+60x^2+11x^3}{60+90x+36x^2+3x^3}\right] \tag{9}$$

Let us call this definition, the three point Gauss-Legendre Log (3P-GLLOG).

8. Approximation through Gauss-Legendre 4 point quadrature

The four point Gauss-Legendre quadrature rule [12] is given as

$$\int_n^{n+1} \frac{1}{x}\,dx \approx k\,\left[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + w_4 f(x_4)\right].$$

Here, $k=\frac{1}{2}$. Weights w_i and points x_i are given as

$$w_1 = \frac{18+\sqrt{30}}{36}, \qquad x_1 = \frac{(2n+1)\sqrt{7}+\sqrt{3-2\sqrt{\frac{6}{5}}}}{2\sqrt{7}},$$

$$w_2 = \frac{18+\sqrt{30}}{36}, \qquad x_2 = \frac{(2n+1)\sqrt{7}-\sqrt{3-2\sqrt{\frac{6}{5}}}}{2\sqrt{7}},$$

$$w_3 = \frac{18-\sqrt{30}}{36}, \qquad x_3 = \frac{(2n+1)\sqrt{7}+\sqrt{3+2\sqrt{\frac{6}{5}}}}{2\sqrt{7}},$$

$$w_4 = \frac{18-\sqrt{30}}{36}, \qquad x_4 = \frac{(2n+1)\sqrt{7}-\sqrt{3-2\sqrt{\frac{6}{5}}}}{2\sqrt{7}}.$$

Thus,

$$\ln\left(1+\frac{1}{n}\right) \approx \frac{420n^3+630n^2+260n+25}{420n^4+840n^3+540n^2+120n+6}$$
$$= \frac{\frac{420}{n}+\frac{630}{n^2}+\frac{260}{n^3}+\frac{25}{n^4}}{\frac{420}{1}+\frac{840}{n}+\frac{540}{n^2}+\frac{120}{n^3}+\frac{6}{n^4}}.$$

Now replace $\frac{1}{n}$ by x in the above equation. The expression for $\ln(1+x)$ through the four point Gauss-Legendre quadrature rule is

$$\ln(1+x) \approx \frac{420x+630x^2+260x^3+25x^4}{420+840x+540x^2+120x^3+6x^4} \tag{10}$$

Let us call this expression the four point Gauss-Legendre Log (4P-GLLOG).

9. Approximation through Gauss-Legendre 5 point quadrature

The five point Gauss-Legendre quadrature rule [12] is given as

$$\int_n^{n+1} \frac{1}{x}\, dx \approx k\, \left[w_1\, f(x_1) + w_2\, f(x_2) + w_3\, f(x_3) + w_4\, f(x_4) + w_5\, f(x_5)\right].$$

Here, $k = \frac{1}{2}$. Weights w_i and points x_i are given as

$$\begin{aligned}
w_1 &= \frac{128}{225}, & x_1 &= n + \frac{1}{2}, \\
w_2 &= \frac{161}{450} + \frac{13}{900}\sqrt{70}, & x_2 &= n + \frac{1}{2} + \frac{1}{42}\sqrt{245 - 14\sqrt{70}}, \\
w_3 &= \frac{161}{450} + \frac{13}{900}\sqrt{70}, & x_3 &= n + \frac{1}{2} - \frac{1}{42}\sqrt{245 - 14\sqrt{70}}, \\
w_4 &= \frac{161}{450} + \frac{13}{900}\sqrt{70}, & x_4 &= n + \frac{1}{2} + \frac{1}{42}\sqrt{245 + 14\sqrt{70}}, \\
w_5 &= \frac{161}{450} - \frac{13}{900}\sqrt{70}, & x_5 &= n + \frac{1}{2} - \frac{1}{42}\sqrt{245 + 14\sqrt{70}}.
\end{aligned}$$

$$\begin{aligned}
\ln\left(1 + \frac{1}{n}\right) &\approx \frac{7560n^4 + 15120n^3 + 9870n^2 + 2310n + 137}{7560n^5 + 18900n^4 + 16800n^3 + 6300n^2 + 900n + 30} \\
&= \frac{\frac{7560}{n} + \frac{15120}{n^2} + \frac{9870}{n^3} + \frac{2310}{n^4} + \frac{137}{n^5}}{\frac{7560}{1} + \frac{18900}{n} + \frac{16800}{n^2} + \frac{6300}{n^3} + \frac{900}{n^4} + \frac{30}{n^5}}.
\end{aligned}$$

Now replacing $\frac{1}{n}$ by x in the above equation the expression for $\ln(1+x)$ through the five point Gauss-Legendre quadrature rule is obtained

$$\ln(1+x) \approx \frac{7560x + 15120x^2 + 9870x^3 + 2310x^4 + 137x^5}{7560 + 18900x + 16800x^2 + 6300x^3 + 900x^4 + 30x^5} \tag{11}$$

Let us call this expression the five point Gauss-Legendre Log (5P-GLLOG).

10. Numerical work

For performing computations to great accuracy, we use the high precision C^{++} library **ARPREC** [4]. In almost every calculus book, the mathematical expression $\ln(1+x)$ is given by the following infinite series:

$$\begin{aligned}
\ln(1+x) &= \sum_{i=1}^{\infty} (-1)^{i-1}\, \frac{x^i}{i} \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots
\end{aligned} \tag{12}$$

Let us now find out the error in computing $\ln 2$ by various of our formulae and infinite series. For exact value of $\ln 2$, we are using the library **ARPREC**. For computing $\ln 2$ by the infinite series, we are considering first 10 terms. Table 1 presents error in approximating $\ln 2$ through different closed form approximations. Here, error is equal to the exact value of the mathematical constant $\ln 2$ minus the value given by different approximations. From Table 1, it can be inferred that our approximations are more accurate.

Formulae	Error
Infinite series	$-5.248\,774 \times 10^{-2}$
$\frac{1}{3}$-SELOG	$-1.297\,264 \times 10^{-3}$
2P-GLLOG	$8.394\,883 \times 10^{-4}$
3P-GLLOG	$2.548\,744 \times 10^{-5}$
4P-GLLOG	$7.631\,145 \times 10^{-7}$
5P-GLLOG	$2.270\,691 \times 10^{-8}$

Table 1. Error (exact-formulae) by different closed form approximations. We are taking first ten terms of the infinite series.

11. Conclusions

In this work, we have developed some new closed form approximations for the expression $\ln(1+x)$. Numerical work authenticates the robustness of these closed form approximations. One big advantage of the formulae over series is that the formulae can be easily programmed even on a calculator.

Basic mathematics is being used to derive these relations. Thus, the presented strategy is easily adopted in an undergraduate class. It will encourage students in formulating even more improved formulae for $\ln(1+x)$.

Acknowledgments. We thank reviewers for their time, and for their valuable comments.

REFERENCES

1. J. A. Knox and H. J. Brothers, *Novel series based approximations to e*, The College Mathematics Journal, **30** (4), 1999.
2. E. Maor, *e: The Story of a Number*, Princeton University Press, 1994.
3. H. J. Brothers and J. A. Knox, *New closed-form approximations to the logarithmic constant e*, Mathematical Intelligencer, **20** (4), 1998.
4. D. H. Bailey, Y. Hida, X. S. Li and B. Thompson, *ARPREC: An Arbitrary Precision Computation Package*, LBNL-53651, Available `http://www.osti.gov/bridge/servlets/purl/817634-sZjOm2/native/817634.PDF`, Accessed 20 August 2008.
5. T. N. T. Goodman, *Maximum products and* $\lim\left(1+\frac{1}{n}\right)^n = e$, American Mathematical Monthly, **93** (9), 1986.
6. C. L. Wang, *Simple inequalities and old limits*, American Mathematical Monthly, **96** (4), 1989.
7. H. Yang and H. Yang, *The Arithmetic-Geometric Mean Inequality and the constant e*, Mathematics Magazine, **74** (4), 2001.

8. C. W. Barnes, *Euler's constant and e*, American Mathematical Monthly, **96** (7), 1984.
9. S. Wasowicz, *On error bounds for Gauss-Legendre and Lobatto Quadrature Rules*, J. Inequalities Pure Appl. Math., **7** (3): Article 84, http://www.emis.de/journals/JIPAM/images/004_06_JIPAM/004_06_www.pdf, Accessed 30 August 2008.
10. R. Johnsonbaugh, *The Trapezoid Rule, Stirling's Formula and Euler's constant*, American Mathematical Monthly, **88** (9), 1981.
11. E. W. Weisstein, *Lobatto Quadrature*, MathWorld, A WolframWeb Resource, http://mathworld.wolfram.com/LobattoQuadrature.html, Accessed 30 August 2008.
12. E. W. Weisstein, *Legendre-Gauss Quadrature*, MathWorld, A Wolfram Web Resource, http://mathworld.wolfram.com/Legendre-GaussQuadrature.html, Accessed 01 February 2009.
13. E. W. Weisstein, *Boole's Rule*, MathWorld, A Wolfram Web Resource, http://mathworld.wolfram.com/BoolesRule.html, Accessed 01 February 2009.
14. E. W. Weisstein, *Newton-Cotes Formulas*, MathWorld, A Wolfram Web Resource, http://mathworld.wolfram.com/Newton-CotesFormulas.html, Accessed 01 February 2009.

Stord/Haugesund University College, Bjørnsonsgt. 45 Haugesund 5528, Norway

E-mail: sanjay.khattri@hsh.no