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LIMITS OF COMPOSITE FUNCTIONS

Milosav M. Marjanović and Zoran Kadelburg

Abstract. Finding $\lim_{x_0} g(f(x))$, first the following two limits

$$\text{(i)} \quad \lim_{x_0} f(x) = y_0 \qquad \text{(ii)} \quad \lim_{y_0} g(y) = \alpha$$

are found and then, it is taken that $\lim_{x_0} g(f(x)) = \alpha$. The existence of the limits under (i) and (ii) is the basis for this method, which is not legitimate in general. In this notice we give necessary and sufficient conditions for the legitimacy of this method.

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1. The limit of a composite function $g\circ f$ at a point x_0 is often found by finding

$$\text{(i)} \quad \lim_{x_0} f(x) = y_0 \qquad \text{(ii)} \quad \lim_{y_0} g(y) = \alpha$$

and then, by taking that $\lim_{x_0} g(f(x)) = \alpha$. This procedure certainly establishes a method of finding the limit of a composite function. In some books on analysis this method is applied without any explicit considerations of its legitimacy, in some others sufficient conditions of this or that form are given (see, e.g., [1]) and, as far as we can follow it, the book [2] was the first place where necessary and sufficient conditions for the applicability of this method were given. In the cited book, the limit of a function at a point was taken as the possibility of its continuous extension to that point. Also the cases of isolated points were included. That all made the corresponding proofs somewhat complicated.

In this notice we omit the (trivial) cases of isolated points and provide straighter proofs based on the Heine's definition of limit. Let us note that setting of our considerations in the frame of metric spaces makes no difference when compared with the very specific case of real functions (whose domains and ranges are subsets of $\mathbf{R}$). Let us also add that for a set A, A' will be the set of its accumulation points.

2. As a definition of the limit we take the following Cauchy's version.

Definition. Let (M_1, d_1) and (M_2, d_2) be metric spaces, $A \subset M_1$, $f\colon A \to M_2$ and $x_0 \in A'$. An element $y_0 \in M_2$ is said to be the limit of f at the point x_0 if

(C) for each real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that $x \in A \setminus \{x_0\}$ and $d_1(x, x_0) < \delta$ imply $d_2(f(x), y_0) < \varepsilon$.

The notations $\lim_{x_0} f(x) = y_0$ or $\lim_{x_0} f = y_0$ are used to denote that y_0 is the limit of f at x_0.

It is easy to show that the Cauchy's condition (C) is equivalent to the following Heine's condition:

(H) for each sequence (x_n) in $A \setminus \{x_0\}$, when $x_n \to x_0$ (in M_1), then $f(x_n) \to y_0$ (in M_2).

Indeed, let the condition (C) hold true and let (x_n) be a sequence in $A \setminus \{x_0\}$ such that $x_n \to x_0$. Then, for the given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in A \setminus \{x_0\}$ and $d_1(x, x_0) < \delta$ imply $d_2(f(x), y_0) < \varepsilon$. Since $x_n \to x_0$, there exists a natural number n_0 such that $d_1(x_n, x_0) < \delta$ for each $n > n_0$. It follows that, for $n > n_0$, $d_2(f(x_n), y_0) < \varepsilon$, what proves that $f(x_n) \to y_0$. Therefore, the condition (H) holds true.

To prove the converse, suppose that the condition (C) is not satisfied. Then, there exists a number $\varepsilon > 0$ such that for each $\delta > 0$, there exists $x \in A \setminus \{x_0\}$ such that $d_1(x, x_0) < \delta$ and $d_2(f(x), y_0) \geq \varepsilon$. In particular, we can take $\delta = 1/n$ for each natural n and then find $x_n \in A \setminus \{x_0\}$ such that $d_1(x_n, x_0) < 1/n$ and $d_2(f(x_n), y_0) \geq \varepsilon$. Hence, $x_n \to x_0$ and $f(x_n) \nrightarrow y_0$, what means that the Heine's condition is not satisfied, either.

Thus, the equivalence of conditions (C) and (H) has been proved.

The following example shows that the described method of finding the limit of a composite function does not work without additional requirements.

EXAMPLE 1. Let $f\colon \mathbf{R} \to \mathbf{R}$ and $g\colon \mathbf{R} \to \mathbf{R}$ be the functions defined by

$$f(x) = 0, \quad \text{for each } x \in \mathbf{R} \qquad \text{and} \qquad g(x) = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then, obviously, $\lim\limits_{0} f(x) = 0$ and $\lim\limits_{0} g(y) = 1$, but $g\circ f(x) = 0$ for each $x \in \mathbf{R}$ and thereby $\lim\limits_{0} g\circ f(x) = 0$. Thus, we have

$$\lim_{x_0} f(x) = y_0 \quad \text{and} \quad \lim_{y_0} g(y) = \alpha, \quad \text{but} \quad \lim_{x_0} g\circ f(x) \neq \alpha.$$

The theorem that follows gives the necessary and sufficient conditions under which the method is legitimate.

THEOREM 1. *Let M_1, M_2, M_3 be metric spaces, $A \subset M_1$, $B \subset M_2$ and let $x_0 \in A'$, $y_0 \in B'$. Further on, let $f\colon A \to B$ and $g\colon B \to M_3$ be functions such that*

$$\text{(I)} \quad \lim_{x_0} f(x) = y_0 \quad \textit{and} \quad \text{(II)} \quad \lim_{y_0} g(y) = \alpha \in M_3.$$

Then

$$\lim_{x_0} g\circ f(x) = \alpha$$

if and only if one of the following conditions holds:

(1) *$g(y_0) = \alpha$,*

(2) *there exists $\delta > 0$ such that for each $x \in A \setminus \{x_0\}$, $d_1(x, x_0) < \delta$ implies $f(x) \neq y_0$.*

Proof. Suppose first that condition (1) holds. Let (x_n) be a sequence in $A \setminus \{x_0\}$ such that $x_n \to x_0$. Assumption (I) implies that $f(x_n) \to y_0$. Consider the following two subsequences of the sequence $(f(x_n))$—the first is formed by those terms (y_n) which are different from y_0 and the second by those terms (z_n) which are equal to y_0. (One of these subsequences might be finite; and then it can be neglected.) Now, $y_n \in B \setminus \{y_0\}$ and $y_n \to y_0$, so, applying assumption (II), we obtain that $g(y_n) \to \alpha$. For the subsequence (z_n), condition (1) implies that $g(z_n) \to \alpha$. Hence, both of the possible subsequences of $(g\circ f(x_n))$ tend to α, and so $g\circ f(x_n) \to \alpha$. Hence, we conclude that $\lim\limits_{x_0} g\circ f(x) = \alpha$.

Suppose now that condition (2) holds and let us fix the corresponding real number δ. Let again (x_n) be a sequence in $A \setminus \{x_0\}$ such that $x_n \to x_0$. Then, there exists $n_0 \in \mathbf{N}$ such that for $n > n_0$, $d_1(x_n, x_0) < \delta$, and so, by (2), $f(x_n) \neq y_0$. It means that $f(x_n) \in B \setminus \{y_0\}$ (possibly with a finite number of exceptions), and we can apply assumption (II) to conclude that $g\circ f(x_n) \to \alpha$. It follows that $\lim\limits_{x_0} g\circ f(x) = \alpha$ is valid in this case, as well.

Finally, we have to show that validity of one of the conditions (1) and (2) is necessary for the conclusion $\lim\limits_{x_0} g\circ f(x) = \alpha$. Suppose the contrary, that neither of them is valid. Let us notice first that $y_0 \in B$, for otherwise the condition (2) would be satisfied. Then $g(y_0) \neq \alpha$ and for each $\delta > 0$ there exists $x \in A \setminus \{x_0\}$ such that $d_1(x, x_0) < \delta$ and $f(x) = y_0$. Take, for each $n \in \mathbf{N}$, $\delta_n = 1/n$, and find the corresponding $x_n \in A \setminus \{x_0\}$ such that $d_1(x_n, x_0) < 1/n$ and $f(x_n) = y_0$. Then (x_n) is a sequence in $A \setminus \{x_0\}$ such that $x_n \to x_0$. But, $(g\circ f(x_n))$ is a constant sequence with terms equal to $g(y_0)$, hence different from α. It means that $g\circ f(x_n) \not\to \alpha$ and thereby $\lim\limits_{x_0} g\circ f(x) = \alpha$ does not hold. This completes the proof of the theorem. ∎

Note that neither of conditions (1) and (2) was fulfilled in Example 1.

A standard procedure of applying Theorem 1 for finding limits can be formulated in the following way.

Corollary 1. *Let $f\colon A \to B$ be an invertible function and $\lim\limits_{y_0} f^{-1}(y) = x_0$; further on, let $\lim\limits_{x_0} g\circ f(x) = \alpha$. Then $\lim\limits_{y_0} g(y) = \alpha$.*

$$\begin{array}{l} M_2 \supset B \xrightarrow{g} M_3 \\ \quad f^{-1} \downarrow\uparrow f \\ M_1 \supset A \end{array}$$

Proof. Indeed, (I) $\lim\limits_{y_0} f^{-1}(y) = x_0$, (II) $\lim\limits_{x_0} g\circ f(x) = \alpha$ and the condition

(2) there exists $\varepsilon > 0$ such that $y \in B \setminus \{y_0\}$ and $d_2(y, y_0) < \varepsilon$ imply $f^{-1}(y) \neq x_0$

is fulfilled (because of the invertibility of f). Hence, by Theorem 1, we have

$$\alpha = \lim_{y_0} (g\circ f)(f^{-1}(y)) = \lim_{y_0} g(y). \quad ∎$$

In this form, the method is usually called "introduction of a new variable".

Before considering further examples, note that when we deal with real functions $f\colon A \to \mathbf{R}$ ($A \subset \mathbf{R}$), points $\pm\infty$ can be included in consideration as elements of a metric space $(\overline{\mathbf{R}}, \bar{d})$. Here $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ and a possible choice of metric $\bar{d}$ on $\overline{\mathbf{R}}$ is given in Exercise 3 at the end of the paper. Neighbourhoods of the point $+\infty$ have the form of open intervals $(M, +\infty)$, and similarly for the point $-\infty$.

In particular, a sequence (x_n) in $\mathbf{R}$ converges to $\alpha \in \overline{\mathbf{R}}$ if and only if the function $x\colon \mathbf{N} \to \mathbf{R}$ defined by $x(n) = x_n$ has the limit α in the metric space $(\overline{\mathbf{R}}, \bar{d})$.

Example 2. (a) $\lim\limits_{+\infty}\Big(1+\dfrac{1}{x}\Big)^x = e$; (b) $\lim\limits_{-\infty}\Big(1+\dfrac{1}{x}\Big)^x = e$; (c) $\lim\limits_{0}(1+y)^{1/y} = e$.

Proof. (a) Let $B = \mathbf{N}$ and $g_1, g_2\colon B \to \mathbf{R}$ be given by

$$g_1(n) = \left(1 + \frac{1}{n+1}\right)^n, \quad g_2(n) = \left(1 + \frac{1}{n}\right)^{n+1}.$$

It follows easily from the definition of the number e that $\lim\limits_{\infty} g_1(n) = \lim\limits_{\infty} g_2(n) = e$.

Let $A = \{x \in \mathbf{R} \mid x \geqslant 1\}$, and $f\colon A \to \mathbf{N}$ be given as $f(x) = \lfloor x \rfloor$. Obviously, $\lim\limits_{+\infty} f(x) = \infty$. Condition (2) of Theorem 1 is fulfilled—for each $x \in A$ ($x \neq +\infty$), $f(x) \neq \infty$. Therefore the functions

$$(g_1 \circ f)(x) = \left(1 + \frac{1}{\lfloor x \rfloor + 1}\right)^{\lfloor x \rfloor} \quad \text{and} \quad (g_2 \circ f)(x) = \left(1 + \frac{1}{\lfloor x \rfloor}\right)^{\lfloor x \rfloor + 1}$$

have the limit equal to e at the point $+\infty$. Since for each $x > 1$ we have

$$\left(1 + \frac{1}{\lfloor x \rfloor + 1}\right)^{\lfloor x \rfloor} < \left(1 + \frac{1}{x}\right)^x < \left(1 + \frac{1}{\lfloor x \rfloor}\right)^{\lfloor x \rfloor + 1},$$

the conclusion (a) follows from the Sandwich Theorem.

(b) Using the substitution $t = -x$ we obtain that

$$\lim_{-\infty}\left(1 + \frac{1}{x}\right)^x = \lim_{+\infty}\left(1 - \frac{1}{t}\right)^{-t} = \lim_{+\infty}\left(1 + \frac{1}{t-1}\right)^t.$$

Now the substitution $u = t - 1$ gives

$$\lim_{+\infty}\left(1 + \frac{1}{t-1}\right)^t = \lim_{+\infty}\left(1 + \frac{1}{u}\right)^{u+1} = \lim_{+\infty}\left(1 + \frac{1}{u}\right)^u \cdot \lim_{+\infty}\left(1 + \frac{1}{u}\right) = e.$$

Both of these introductions of new variables can be easily justified. Hence, the conclusion (b) follows.

(c) Let $A = B = \{x \in \mathbf{R} \mid x > 0\}$ and $f\colon A \to B$ be given by $f(x) = 1/x$. Then f is invertible, $f^{-1}(y) = 1/y$ and $\lim\limits_{0+} f^{-1}(y) = +\infty$. Let $g\colon B \to \mathbf{R}$ be given by $g(y) = (1+y)^{1/y}$. Then $g \circ f\colon B \to \mathbf{R}$ acts like $g \circ f(x) = \Big(1 + \dfrac{1}{x}\Big)^x$ and, by (a), $\lim\limits_{+\infty} g \circ f(x) = e$. Applying Corollary 1 we obtain that $\lim\limits_{0+}(1+y)^{1/y} = e$.

In a similar way, using (b), it can be proved that $\lim\limits_{0-}(1+y)^{1/y} = e$, hence the conclusion (c) follows. ■

Example 3. Let $f\colon (-1,+\infty) \to (0,+\infty)$ be given by $f(x) = (1+x)^{1/x}$ and $g\colon (0,+\infty) \to \mathbf{R}$ by $g(y) = \log_a y$. Then, according to the previous example, $\lim\limits_{0} f(x) = e\ (= y_0)$, $\lim\limits_{e} g(y) = \log_a e\ (= \alpha)$. Moreover, condition (1) of Theorem 1 is fulfilled—$g(y_0) = g(e) = \log_a e = \alpha$. Therefore

$$\lim_{0} \frac{\log_a(1+x)}{x} = \lim_{0} \log_a(1+x)^{1/x} = \log_a e.$$

In particular, for $\ln = \log_e$, we have $\lim\limits_{0} \dfrac{\ln(1+x)}{x} = 1$.

Example 4. In order to find $\lim\limits_{0} \dfrac{a^y - 1}{y}$, put $y = \log_a(1+x) = f(x)$. Then f is an invertible function with $f^{-1}(y) = a^y - 1$, and $\lim\limits_{0} f^{-1}(y) = 0$. Furthermore, according to Example 3,

$$g{\circ}f(x) = \frac{a^{\log_a(1+x)} - 1}{\log_a(1+x)} = \frac{x}{\log_a(1+x)} \quad \text{and} \quad \lim_{0} g{\circ}f(x) = \frac{1}{\log_a e} = \ln a.$$

Applying Corollary 1, we obtain that $\lim\limits_{0} g(y) = \lim\limits_{0} \dfrac{a^y - 1}{y} = \ln a$.

We conclude with an example involving functions in two variables.

Example 5. Find $\lim\limits_{(+\infty,a)} \left(1 + \dfrac{1}{x}\right)^{\frac{x^2 y}{x+y}}$, for $a \in \mathbf{R}$.

We can write the given expression as

$$\left(1 + \frac{1}{x}\right)^{\frac{x^2 y}{x+y}} = e^{\frac{xy}{x+y} \ln \left(1 + \frac{1}{x}\right)^x}.$$

Since $\lim\limits_{(+\infty,a)} \dfrac{xy}{x+y} = a$ and using the continuity of logarithmic and exponential functions (i.e., that $\lim\limits_{y_0} \ln y = \ln y_0$ for $y_0 > 0$ and $\lim\limits_{y_0} e^y = e^{y_0}$ for $y_0 \in \mathbf{R}$), we obtain that the given limit is equal to e^a.

Exercises

1. Let M_1 and M_2 be metric spaces, $A \subset M_1$, $B \subset M_2$ and $f\colon A \to \mathbf{R}$, $g\colon B \to \mathbf{R}$. Let $\lim\limits_{x_0} f = \alpha$, $\lim\limits_{y_0} g = \beta$ and let $h\colon \mathbf{R}^2 \to \mathbf{R}$ be a continuous function. Then

$$\lim_{(x_0,y_0)} h(f(x), g(y)) = h(\alpha, \beta).$$

In particular,

$$\lim_{(x_0,y_0)}[f(x)+g(y)] = \lim_{x_0} f(x) + \lim_{y_0} g(y),$$
$$\lim_{(x_0,y_0)}[f(x)\cdot g(y)] = \lim_{x_0} f(x) \cdot \lim_{y_0} g(y).$$

2. (a) $\displaystyle\lim_{(0,0,\dots,0)} \frac{e^{x_1\cdot x_2\cdot\ldots\cdot x_n}-1}{x_1\cdot x_2\cdot\ldots\cdot x_n} = 1.$

(b) $\displaystyle\lim_{(\infty,\infty)} \left(n\cdot m\cdot \sin\frac{1}{nm}\right) = 1.$

3. The function $\varphi\colon \mathbf{R}\to(-1,1)$ given by $\varphi(x)=\dfrac{2}{\pi}\arctan x$ is bijective and both functions φ and φ^{-1} are continuous.

(a) The function $d\colon \mathbf{R}\times\mathbf{R}\to[0,+\infty)$ given by $d(x,y)=|\varphi(x)-\varphi(y)|$ is a metric on $\mathbf{R}$.

(b) A sequence (x_n) in $\mathbf{R}$ converges to $\alpha\in\mathbf{R}$ if and only if the function $x(n)=x_n$ has the limit α in the metric space $(\mathbf{R},d)$.

(c) Let $\varphi(-\infty)=-1$ and $\varphi(+\infty)=1$. Then

$$\bar d(x,y)=|\varphi(x)-\varphi(y)|$$

defines a metric on $\overline{\mathbf{R}}$ and $(\mathbf{R},d)$ is a subspace of $(\overline{\mathbf{R}},\bar d)$.

(d) Let $f\colon A\to M$. Then $\lim\limits_{+\infty} f$, as it is usually defined, exists, where $A\subset\mathbf{R}$, if and only if $\lim\limits_{+\infty} f$ exists, where A is taken to be a subset of $(\overline{\mathbf{R}},\bar d)$, and these two limits are equal.

(e) Let $A\subset M$. Then $\lim\limits_{x_0} f=+\infty$, as it is usually defined, where $f\colon A\to\mathbf{R}$, if and only if $\lim\limits_{x_0} f=+\infty$, where $f\colon A\to\overline{\mathbf{R}}$, with the metric $\bar d$.

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Milosav M. Marjanović, SANU, Knez Mihailova 35, 11000 Belgrade, Serbia
E-mail: milomar@beotel.rs
Zoran Kadelburg, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia
E-mail: kadelbur@matf.bg.ac.rs