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GEOMETRICAL DEFINITION OF π AND ITS APPROXIMATIONS BY NESTED RADICALS

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Abstract. In this paper the length of the arc of a circle and the area of a circular sector are defined in an elementary way. From this we derive the geometrical definition of the number π, in two equivalent ways. Formulas for the area and the perimeter of a circle are proved. Also, the number π is represented as a limit of several sequences involving nested radicals. From this some approximations of π are obtained. One of them is in terms of Golden ratio.

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1. Introduction

The number π is one of the five most important numbers in mathematics, the other four being 0, 1, i and e. These four numbers are relatively easy to define. This is not the case with π. Within the frame of Mathematical analysis it is usually defined as the least positive zero of the function $y = \sin x$, defined as

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Then, using integrals, one easily obtains the formulas for the perimeter of a circle and the area of a disc.

This is probably the reason why, even in the books which are entirely devoted to π, for instance [1], there are no geometrical definitions of π.

But, there are strong methodological reasons for such a definition. Above all, the number π arises from geometry. We first meet it when we learn the formula for the perimeter or the area of a circle, and these formulas are interesting for rather broad range of people, even for many which are not mathematicians.

Thus, the question arises: Must we wait to learn series, derivatives and integrals to explain why the area of the unit disc is π?

We shall try in this paper to explain these questions in an elementary way, within the frame of elementary geometry. Beside some elementary properties of triangles, we shall use the notion of limit of a sequence, which is, more or less, equivalent to the method of exhaustion broadly used by Archimedes, which was the first mathematician that offered accurate approximations of π. This consideration also introduces the notion of the radian measure of angles, which is the fundamental notion in the development of trigonometry.

We also think that this consideration might be useful as the first meeting with following problem: How something that is curved may be measured by something that is straight? From this question integrals arise.

2. Definition of the area of a circular sector and the length of a circular arc

Suppose that A and B are points on the circle of radius r (Fig. 1). If $A = M_0$, $A_1, \dots\ M_n = B$ are points on the arc $\overset{\frown}{AB}$ taken in cyclic order then we say that the polygon $OM_0M_1 \dots M_nO$ is inscribed into the circular sector OAB. We also say that the polygonal line $M_0, \dots, M_n$ is inscribed in the arc $\overset{\frown}{AB}$. The polygonal line $A = N_0, N_1, \dots, N_k = B$, such that line segments N_iN_{i+1} $(i = 0, 1, \dots, k-1)$ belong to the tangent lines on the circle in the points M_i is said to be circumscribed about the arc $\overset{\frown}{AB}$. Similarly the polygon $ON_0N_1 \dots N_kO$ is said to be circumscribed about the circular sector $\overset{\frown}{AOB}$.

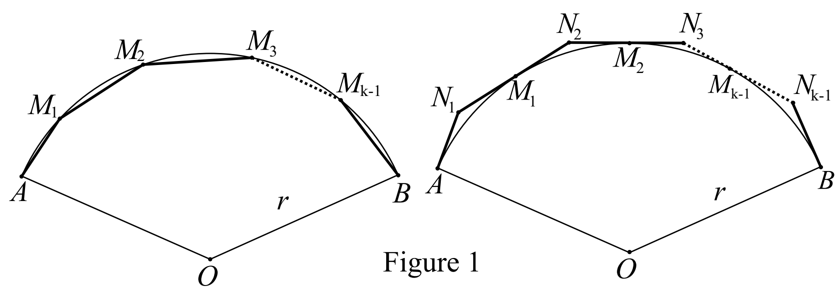

Figure 1

We shall prove that lengths of inscribed lines and areas of inscribed polygons are bounded from above, and that lengths of circumscribed lines and areas of circumscribed polygons are bounded from below.

For this it is enough to prove that the length of each inscribed line is less than the length of arbitrary circumscribed line, and the same for areas of circular sectors.

We shall first prove the following: If C is a point of the line OA such that $OA \le OC$, and D a point of OB with $OB \le OD$ (Fig. 2) then

$$AB \le CD. \tag{1}$$

Take the point M at the middle of the chord AB, and let P be the intersection point of the lines CD and OM. Finally, N and Q are points of the line OM such that NC and QD are perpendicular to OM.

It follows from similarity of triangles OAM and OCN that $AM \le CN$ and, similarly, $BM \le DQ$, which yields

$$AB = AM + MB \le CN + QD \le CP + PD = CD,$$

since CP and PD are hypotenuses of right triangles whose legs are CN and DQ.

Note that the inequality is sharp under the condition that either $A \neq C$ or $B \neq D$.

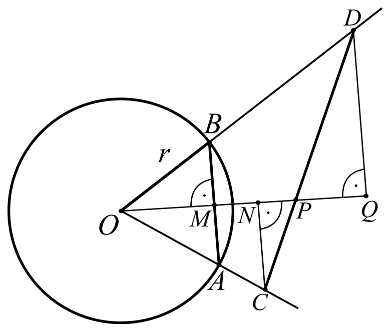

Figure 2

Suppose now that AB is one of the line segments forming an inscribed polygonal line Σ_i, and that C and D are points of intersections of OA and OB with a circumscribed polygonal line Σ_s. In this way we may bijectively map Σ_i onto a polygonal line Σ_i' whose vertices lie on Σ_s. It follows that the length of Σ_i' is less or equal to the length of Σ_s. On the other hand, (1) implies that the length of Σ_i is less than the length of Σ_i'.

Thus the length of a polygonal line inscribed in $\overset{\frown}{AB}$ is less than the length of a polygonal line circumscribed over $\overset{\frown}{AB}$. In the same way we prove that the area of a polygon inscribed in a circular sector is less than the area of a polygon circumscribed about this segment.

It follows that these two sets are bounded from above and thus they have the least upper bounds, which will be denoted by l and p respectively. In the same way the set of lengths of polygonal lines circumscribed about an arc and the set of areas of polygons circumscribed about a circle sector are bounded from below and thus have the greatest lower bound, which we shall denote by L and P respectively.

We thus may define the length of an arc and the area of a circular sector in two ways.

Definition 1. The length of an arc is the least upper bound of the set of lengths of polygonal lines inscribed into the arc. The area of a circular sector is the least upper bound of the set of areas of polygons which are inscribed into the segment.

Definition 2. The length of an arc is the greatest lower bound of the set of lengths of circumscribed polygonal lines. The area of a circular sector is the greatest lower bound of the set of areas of polygons which are circumscribed about the sector.

3. Archimedes Doubling Method

In this section we shall prove that Definitions 1 and 2 are equivalent. Namely, it will be proved that the length of an arc and the area of a sector my be obtained by the so called Archimedes doubling method.

It is a well known fact that the first accurate approximations for π were obtained by Archimedes using regular 6, 12, 24, 48, 96-gons. We shall apply this idea to arbitrary arcs and circular sectors. Let A and B be two points on the circle of radius r. Denote $x = d(A, B)$, the length of the chord AB, and take the point C of the circle that halves the arc $\overset{\frown}{AB}$ (Fig. 3).

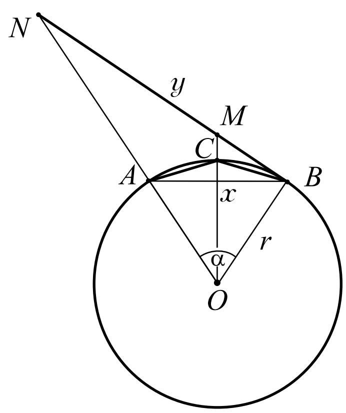

Figure 3

If we denote $a_0(r, x) = x$, $a_1(r, x) = d(B, C)$ then by Pythagoras rule we have

$$d(B,C)^2 = \frac{x^2}{4} + \left(r - \sqrt{r^2 - \frac{x^2}{4}}\right)^2,$$

that is

$$a_1(r,x) = \sqrt{2r^2 - r\sqrt{4r^2 - a_0(r,x)^2}}.$$

Define the sequence of nested radicals $\{a_n(r, x) : n = 1, 2, \dots\}$ in the following way:

$$a_{n+1}(r,x) = \sqrt{2r^2 - r\sqrt{4r^2 - a_n^2(r,x)}}, \qquad n = 1, 2, \dots$$

It follows that $a_n(r, x)$ is the side of the regular 2^n-gonal line $A = P_0, P_1, \dots, P_{2^n} = B$ inscribed in the arc $\overset{\frown}{AB}$. Then obviously

$$l_n(r,x) = 2^n a_n(r,x), \qquad n = 0, 1, 2, \dots$$

is its length.

It holds

$$a_2(r,x) = \sqrt{2r^2 - r\sqrt{4r^2 - a_1^2(r,x)}} = \sqrt{2r^2 - r\sqrt{2r^2 + r\sqrt{4r^2 - x^2}}}.$$

Continuing in the same way one obtains

$$\text{(2)} \qquad a_n(r,x) = \sqrt{2r^2 - r\sqrt{2r^2 + r\sqrt{2r^2 + \cdots + r\sqrt{2r^2 + \sqrt{4r^2 - x^2}}}}}.$$

On the right-hand side of the preceding equation square roots appear exactly $n+1$ times.

By denoting

$$f_n(r,x) = \sqrt{2r^2 + \cdots + r\sqrt{2r^2 + r\sqrt{4r^2 - x^2}}}, \qquad n = 1, 2, \ldots$$

where in the expression on the right-hand side radicals appear exactly n times, we obtain

$$\text{(3)} \qquad a_n(r,x) = \sqrt{2r^2 - rf_n(r,x)}, \qquad n = 1, 2, 3, \ldots .$$

For the lengths $l_n(r,x)$ we have now

$$\text{(4)} \qquad l_n(r,x) = 2^n\sqrt{2r^2 - rf_n(r,x)}.$$

Define also the sequence $\{p_n(r,x) : n = 0, 1, \ldots\}$ such that

$$p_0(r,x) = \frac{x}{2}\sqrt{4r^2 - x^2}$$

is the area of triangle OAB and $p_n(r,x)$, $(n = 1, 2, \ldots)$ are the areas of 2^n-gons $O, P_0, P_1, \ldots, P_{2^n}, O$.

It follows that

$$\begin{aligned} p_1(r,x) &= 2 \cdot \frac{a_1(r,x)}{2}\sqrt{4r^2 - a_1^2(r,x)} \\ &= \sqrt{2r^2 - r\sqrt{4r^2 - x^2}}\sqrt{2r^2 + r\sqrt{4r^2 - x^2}} = rx. \end{aligned}$$

In the same way, for $n = 2, 3, \ldots$ we obtain

$$p_n(r,x) = 2^n\frac{a_n(r,x)}{2}\sqrt{4r^2 - a_n^2(r,x)} = r2^{n-1}a_{n-1}(r,x).$$

Using (3) it follows that

$$\text{(5)} \qquad p_n(r,x) = r2^{n-1}\sqrt{2r^2 - f_{n-1}(r,x)} \qquad n = 2, 3, \ldots .$$

According to (4) i (5) we conclude that

$$\text{(6)} \qquad rl_n(r,x) = 2p_{n+1}(r,x), \qquad n = 0, 1, 2, \ldots .$$

It is clear that $\{l_n(r,x)\}$ and $\{p_n(r,x)\}$ are increasing, and being bounded from above they are convergent. But, until now, we only know that $\lim_{n\to\infty} l_n(r,x) \le l$, and $\lim_{n\to\infty} p_n(r,x) \le p$, where l, p are from Section 2.

We shall now construct a sequence of circumscribed polygonal lines about the arc $\widehat{AB}$.

Take the tangent line BN of the circle at the point B (Fig. 3), where N is the intersection point of the tangent line and the line OA. Let M be the point on the tangent line such that OM halves the angle AOB. Denote $BN = y$. From elementary geometry we know that $BM : MN = r : ON$. This yields

$$b_1(r,y) = \frac{rb_0(r,y)}{r+\sqrt{r^2+y^2}},$$

where $b_0(r,y) = y$, $b_1(r,y) = d(M,B)$.

Define the sequence $\{b_n(r,y) : n = 1,2,\dots\}$ such that

$$b_n(r,y) = \frac{rb_{n-1}(r,y)}{r+\sqrt{r^2+b_{n-1}(r,y)^2}}.$$

If we also define the sequence $\{g_n(r,y) : n = 1,2,3\dots\}$ such that

$$g_1(r,y) = r+\sqrt{r^2+y^2},$$
$$g_n(r,y) = g_{n-1}(r,y)+\sqrt{g_{n-1}(r,y)^2+y^2}, \qquad n = 2,4,\dots$$

then we have

$$b_1(r,y) = \frac{ry}{g_1(r,y)},\ b_2(r,y) = \frac{ry}{g_2(r,y)}.$$

Using induction we obtain

$$b_n(r,y) = \frac{ry}{g_n(r,y)}, \qquad n = 2,\dots.$$

Triangles OAM and OBM are congruent, which implies that $AM = BM$, and that the line segment AM lies on the line tangent to the circle at the point A. It follows that

$$P_1(r,y) = 2\cdot\frac{r}{2}b_1(r,y) = rb_1(r,y)$$

is the area of the polygon $OAMBO$ circumscribed about the sector OAB. In the same way the area $P_n(r,y)$ of the polygon consisting of n right triangles with one leg equal 1 and the other equal $b_n(r,y)$ is

$$P_n(r,y) = r2^{n-1}b_n(r,y).$$

For the length $L_n(r,y)$ of the circumscribed polygonal line consisting of n line segments of the lengths equal $b_n(r,y)$ we have

$$L_n(r,y) = 2^n b_n(r,y). \tag{7}$$

Two preceding equations imply

$$rL_n(r,y) = 2P_n(r,y). \tag{8}$$

Sequences $\{P_n(r,x) : n = 0,1,2,\dots\}$ and $\{L_n(r,x) : n = 0,1,2,\dots\}$ are decreasing and bounded from below, that is, they are convergent. It holds

$$L(r,x) \le \lim_{n\to\infty} L_n(x,r), \quad P(r,x) \le \lim_{n\to\infty} P_n(x,r),$$

where $L(r,x)$ and $P(r,x)$ are as in Section 2.

Until now we have proved the following inequalities:

$$l_n(r,x) < l(r,x) \le L(r,x) < L_n(r,x), \qquad n = 1,2,\dots \tag{9}$$

$$p_n(r,y) < p(r,y) \le P(r,y) < P_n(r,y), \qquad n = 1,2,\dots . \tag{10}$$

We shall now find the connection between $a_n(r,x) = d(A,B)$ and $b_n(r,y) = d(B,N)$ (Fig. 3).

From the right triangle OBM we obtain $\frac{x}{2} = \frac{rMB}{\sqrt{MB^2+r^2}}$, which implies

$$MB = \frac{rx}{\sqrt{4r^2-x^2}}.$$

Next, from $NM : MB = ON : r$ it follows $MB = \frac{ry}{r+\sqrt{r^2+y^2}}$. From two preceding equations we conclude that

$$\frac{y}{r+\sqrt{r^2+y^2}} = \frac{x}{\sqrt{4r^2-x^2}}.$$

Accordingly, for $n = 1,2,\dots,$

$$\frac{b_n(r,y)}{r+\sqrt{r^2+b_n(r,y)^2}} = \frac{a_n(r,x)}{\sqrt{4r^2-a_n(r,x)^2}}$$

holds true. Multiplying by 2^n we obtain

$$\frac{L_n(r,y)}{r+\sqrt{r^2+b_n(r,y)^2}} = \frac{l_n(r,x)}{\sqrt{4r^2-a_n(r,x)^2}}. \tag{11}$$

According to the fact that $\lim_{n\to\infty} a_n(r,x) = \lim_{n\to\infty} l_n(r,y) = 0$ we finally have

$$\lim_{n\to\infty} l_n(r,x) = \lim_{n\to\infty} L_n(r,y).$$

From (8) and (6) it also follows

$$\lim_{n\to\infty} p_n(r,y) = \lim_{n\to\infty} P_n(r,y).$$

We have thus proved that

$$\lim_{n\to\infty} l_n(r,x) = \lim_{n\to\infty} L_n(r,y) = l(r,x) = L(r,y),$$

and

$$\lim_{n\to\infty} p_n(r,x) = \lim_{n\to\infty} P_n(r,y) = p(r,x) = P(r,y),$$

which means that Definitions 1 and 2 are equivalent.

From (6) or (8) we obtain the following

THEOREM 1. *The area $\mathcal{P}$ of the circular sector of a circle with radius r determined by an arc of the length l is*

$$\mathcal{P} = \frac{r \cdot l}{2}.$$

Now we are able to define π in two equivalent way.

DEFINITION 3. The number π is the area of the unit circle.

DEFINITION 4. The number π is the half of the perimeter of the unit circle.

It remains to express the length of arc and the area of circular sector as functions of π.

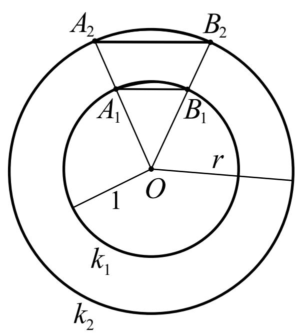

Figure 4

Consider two concentric circles k_1 and k_2 whose radii are 1 and r respectively (Fig. 4). Correspondence $A_1 \mapsto A_2$ is a bijection of k_1 onto k_2. Accordingly, there is also a bijection between polygonal lines inscribed in corresponding arcs $\overset{\frown}{A_1B_1}$ and $\overset{\frown}{A_2B_2}$. Since

$$d(A_2B_2) : d(A_1B_1) = r : 1$$

the lengths of the corresponding polygonal lines have also the same ratio. This implies that the lengths of the arcs $\overset{\frown}{A_2B_2}$ and $\overset{\frown}{A_1B_1}$ are in the same ratio.

We have thus proved the following:

THEOREM 2. *The perimeter $\mathcal{O}$ and the area $\mathcal{P}$ of the circle with radius r are*

$$\mathcal{O} = 2 \cdot r \cdot \pi, \quad \mathcal{P} = r^2\pi.$$

We finally define the radian measure of an angle.

DEFINITION 5. Radian measure of an angle α is the ratio $\frac{l}{r}$, where l is the length of the arc determined by this angle and r is radius of the circle.

4. Some formulas for π in terms of nested radicals

All results that are proved until now are of existential type. But we do not know yet a basic fact about π, namely, that $\pi = 3.141592\ldots$.

From our considerations it is easy to obtain various approximations for π. For this we shall use formulas, obtained in Section 3.

Taking $r = 1$, $x = 2$ in (4) we obtain

$$\pi = \lim_{n\to\infty} 2^{n+1} \sqrt{2 - \underbrace{\sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}_{n-\text{times}}},$$

which is the formula (66) in [3] (Pi formulas, 2000.g.)

Taking $x = 1$ in the same formula yields

$$\pi = 3 \lim_{n\to\infty} 2^{n+1} \sqrt{2 - \underbrace{\sqrt{2 + \cdots + \sqrt{2 + \sqrt{3}}}}_{n-1-\text{times}}}.$$

A few of the first terms of the sequence on the right-hand side give well-known Archimedes lower approximations of π.

Take finally

$$x = \sqrt{\frac{5 - \sqrt{5}}{2}},$$

which is the side of the regular pentagon inscribed in the unit circle. We obtain

$$\sqrt{4 - x^2} = \frac{\sqrt{5} + 1}{2},$$

and thus we obtain approximations of π in terms of Golden ratio,

$$\pi = 5 \cdot \lim_{n\to\infty} 2^n \sqrt{2 - \underbrace{\sqrt{2 + \cdots + \sqrt{2 + \sqrt{2 + \frac{\sqrt{5}+1}{2}}}}}_{n-1-\text{times}}}.$$

A few of the first approximations are given in [2].

Putting $r = 1$, $y = 1$ in (7) we obtain

$$\pi = \lim_{n\to\infty} \frac{2^{n+2}}{(1 + \sqrt{2}) + \sqrt{(1 + \sqrt{2})^2 + 1} + \sqrt{\left(1 + \sqrt{2} + \sqrt{(1 + \sqrt{2})^2 + 1}\right)^2 + 1} + \cdots}$$

where the expression in the denominator has n summands. For $k = 2, 3, \ldots, n$ the k-th term a_k in the sum is obtained by taking square root of $\left(\sum_{i=1}^{k-1} a_i\right)^2 + 1$.

For $r = 1$, $y = \sqrt{3}$ we obtain

$$\pi = \lim_{n\to\infty} \frac{3\sqrt{3}\cdot 2^n}{3 + 2\sqrt{3} + \sqrt{(3+2\sqrt{3})^2 + 3} + \sqrt{\left(3 + 2\sqrt{3} + \sqrt{(3+2\sqrt{3})^2+3}\right)^2 + 3} + \cdots},$$

where, again, the expression in the denominator has n summands, obtained similarly as in the preceding formula.

First four terms in the sequence on the right-hand side give Archimedes upper approximations of π.

We shall finish the paper by an immediate consequence of (7). Namely, in this equation y is an arbitrary positive real number. It implies that we may take $y \to +\infty$ to obtain

$$\frac{\pi}{2} = \lim_{y\to\infty} \lim_{n\to\infty} \frac{2^n y}{g_n(y)}.$$

Taking particularly $y = n$ one obtains

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2^n n}{g_n(n)},$$

that is,

$$\pi = \lim_{n\to\infty} \frac{2^{n+1} n}{1 + \sqrt{3} + \sqrt{(1+\sqrt{3})^2 + 4} + \sqrt{\left(1 + \sqrt{3} + \sqrt{(1+\sqrt{3})^2+4}\right)^2 + 9} + \cdots}.$$

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