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A BROADER WAY THROUGH THEMAS OF ELEMENTARY SCHOOL MATHEMATICS, I

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Abstract. As a basis for treatment of elementary school mathematical themas, a phenomenological scheme representing concepts as three component entities is given. Three related components: class of examples, mental image and name, serve very well as a piece of general language used in subject analysis. With regard to iconic representation of concepts, "leaf" used in that representation is considered as being a premodel of geometric plane. To underline that idea and hence, to avoid some often encountered discrepancies, the following rule which governs such representation is formulated: two icons occupying different places are different.

At the end, the first topic "Sets and Counting" is sketched, setting forth the way how the meaning of words "set" and "element" can be assimilated at the initial stage.

AMS Subject Classification: 00 A 35

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1. Introduction

Collecting a number of primary school textbooks in mathematics from several countries and turning over their leaves, one feels like being exposed to the cross winds of various trends and approaches. Variegated spirit of these books does not project clearly the modes of treatment of traditional themas of arithmetic and geometry which sometimes seem to be melted by the heat of the newly proclaimed objectives.

Not much resemblance is seen when one inspects the secondary school textbooks. At this level, the teaching themas are clearly outlined, and no pretensions to develop intellectual abilities of students out of the very paths of mathematics exist. From the time of the I Congress of Mathematicians (Zürich, 1897) onwards the secondary mathematics instruction has been guided and influenced by groups of experts consisting of teachers of mathematics from universities and secondary schools. On the contrary, the mathematical curriculum in elementary schools has ever been under direct or indirect influences of general educationalists. And it is quite natural to suppose that those of them who have never mastered the subject matters could not be capable to emphasize the spirit in which they should be taught.

On the other side, many a professional mathematician is inclined to underrate this fundamental level of cognition at which, the mathematical concepts start to be synthesized from "nothing". Neglecting important questions of where and how the mathematical concepts begin to exist, such a professional is of little help to the elementary school teachers (though sometimes seen as their supervisor).

Performing instruction at a fixed level, the instructor has to be acquainted with the adjacent levels. An elementary school teacher should know how the concepts initiated at the first level are developed at the next higher and the curriculum planners must know even much more, they have to be subject experts.

Clarification of the spirit in which the topics of elementary school mathematics are to be taught and the scope of their treatment, we often see left for consideration to the various forms of the "moot courts". "Democratically" open, such forums could eventually bring "good" conclusions and "nice" resolutions.

As a matter of fact, each clarification has to start with the subject analysis of the selected topics, presented "in ink" and exposed to the objective criticism. Inasmuch as the traditional didactics of mathematics does not provide such an analysis, we plan to write a series of papers going through several elementary school teaching themas, treating their constituent topics in detail. Therefore, expressing openly the inspiration drawn from H. Freudenthal's book [4], we shall be weeding and sowing in a somewhat uncared – for garden.

Hoping that our papers will be of interest not only for professional mathematicians specialized in education but also for didactics of mathematics experts, psychologists and gifted school teachers, we shall be using only the rudiments of basic school mathematics. Whenever we skip that frame, the exposition will be set aside in the form of clearly marked addenda. Omitting them, there will be no break in the text.

Note also that we use here the term "secondary school" following the prototypes of the German Gymnasium and the French Lycée. The elementary school, as the lowest, extends variously from four to eight years and here we shall usually concentrate our attention to the first four years.

2. Influences on educational practice

Since the time of Socrates until the beginnings of modern psychology, art of teaching had exclusively been based upon philosophy. Marked formalities in methods of teaching in the Middle Ages were the product of scholasticism.

No doubt, the great innovations of Jan Amos Komensky (1592–1670) were under the direct influence of philosophical works of Francis Bacon (1561–1626). Considering sences as the main source of the basic knowledge, Komensky's lessons in his famous book "`Orbis Sensualium Pictus`" (The Visible World in Pictures) start with woodcuts representing the real world phenomena and then, they are followed by sentences describing them. Though pictured numbers and geometric drawings were used in the periods of ancient civilizations, Komensky's „`orbis pictus`" is the first systematic example of iconic representation.

The doctrine of Komensky's sensualism as a limited approach was a starting point for Johann Heinrich Pestalozzi (1761–1827) to elaborate it further. According to Pestalozzi, a child should learn how to think, proceeding gradually from observation to comprehension, to the formation of clear concepts. Perhaps Pestalozzi also was under the influence of Kant's view that perception without conception is blind and conception without perception is empty.

In the course of the 19th century, Pestalozzi's followers proclaimed the following two didactical points of view:

1. The concept of number must be formed on the ground providing meaning and evidence.

2. That ground must not be turned into a mere playing.

Expressing the second point, these educators stand against a possibly existing pedagogical tendency of that time to dodge all exertion.

Since the beginning of this century the mathematical education has been guided and controlled in an internationally organized way. This is especially true as far as the secondary school level is concerned. From that level some second-hand ideas were reflected towards the elementary school curricula which, accordingly, have also been changed. The Merano programs (from 1905) served as a basis of the reform during the first half of the century. The most important innovation of that period was the introduction of the concept of function into the secondary school subjects. Emphasizing the dominant place of that concept in contemporary mathematics, F. Klein pleaded for penetrating the whole subject matter by this idea which would help developing the functional thinking of pupils.

Germs and some corresponding technical preparations of this concept found in early stages of teaching are still the points of dispersal. An example is the wrong understanding of the role of letters in arithmetic teaching where they are seen as means of composing equations by which the word problems are solved. On the contrary, their right role is to link arithmetic and algebra, as S. L. Sobol'ev pointed out at the II International Congress on Mathematical Education (Exeter, 1972). And as Vieta's `logistica speciosa` has ultimately been based on axioms derived from arithmetical rules so school algebra linked with arithmetic is no longer taught using rules without reason.

With the middle fifties, a very intensive modernization of school programs in mathematics started. Seeing the development of mathematical science in the 20th century marked by the tendency toward unification, the creators of this movement emphasized an adequate unity in school programs and, on the other side, their supporters showed an unusual zeal for realization. As a result, we could see the school books full of set theory which, often not being properly embedded in the traditional material, was overloading the syllabi. Combined with some ideas of conception of the natural number which were coming from the soft terrains of experimentation, varieties of diagrams with commanding arrows were sticking out of the elementary mathematics books. A strange period followed, when everyone was involved in discussing teaching problems: scientists, instructors, psychologists, pedagogs, journalists etc. – those who knew something and those who knew nothing about it.

The first constructive reaction to this reform was the Freudenthal's demand

(Bologna, 1961) for new methodology which would fix the right place and lead to an appropriate didactical shaping of the new material (and, anyway, the traditional methodology of teaching mathematics was generally considered as too much preaching and inferior). The already mentioned Exeter Congress is taken to be an important turning point. Of what happened then and there, we select to mention R. Thom's sharp criticism turned against premature generalizations and doctrinaire ways of dealing with the universal schemes which govern the mathematical thought in learning processes.

In this half of the century, the cognitive psychology has been intensively developed, being focused on the study of reasoning, thinking and problem-solving. Evolving from the Gestalt concept of the form, the idea of cognitive scheme (or structure) appeared and now it is generally assumed that no pure intuition exists without being operated and shaped from inside by certain cognitive schemes.

Since the time of Wertheimer's "The Area of the Parallelogram" [9], many interesting papers which touch the mathematical contents have been written by psychologists. Selecting to treat some isolated items and providing a very subtle analysis of delicate details and differences, the psychologists usually do not consider these items as being embedded into the internal integrity of the subject matter. And such considerations should be the acts leaned upon the visage of mathematical contents as a set of topics with the uniting links between them and, therefore, they can be drawn by the subject experts. Easily we agree with Freudenthal that a science of mathematial education does not exist yet but the correlated efforts of subject specialists and psycholigists may lead to a sound, up-to-date art of mathematics teaching.

3. A schematic representation of concepts

Toute science est l'étude d'une phénoménologie. ... toute phénoménologie doit être regardée comme un "spectacle" visual.
René Thom

The existing theory of instruction still lacks a generally accepted language, what often hinders communication of meaning. To avoid any confusion, we shall explain the way we use some terms which appear in various domains of thought and which shall be used throughout this series of papers with an unchabged meaning. That usage seems to us most favorable for the purposes we have in mind and, out of it, we have no other pretension. Our first concern here is to expose and interpret a graphical scheme which represents concepts in a general sense.

A psychologist is often inclined to say that the concepts exist in our minds, a formal logician would say that they exist in language and a dialectician takes them to exist in the relation between man and nature. Reacting to it, and not quite joking, we would say that everybody is right. But, first, let us take a historical look.

Speaking about the act of conception, Plato says: "God has created the archetype of the table after which a joiner makes a simulacrum (embodiment) and a painter makes a simulacrum (image) of that simulacrum." This is a typical Platonic idea which is in accordance with "the immortal soul having been born many times and having seen all things that exist". Following Aristotle, in classical logic, the concept is a dyad consisting of `genus proximum` – class of similar objects (of thoughts) and of `differentia specifica` – specification of that similarity (by predicating or naming).

Rationalism of the Enlightenment philosophy emphasizes that the sources of all knowledge are sense experience and reflection. And today, as a benefit of modern psychological analysis, we have a quite clear outline of this rationalism.

An up-to-date narrative about the concept of table usually starts with the explanation how a print on `tabula rasa` is formed. First of all some concrete tables must exist in the child's surroundings. Call them – the related examples. Seeing these objects at different distances and angles, the incoming sense-data are varying but still some sameness, as a result of processing, stays in the child's mind. Call that inner presentation – the related mental image. When formed, this image helps him/her to recognize and to sort further the similar objects. Living in a society, the child hears the word "table" pronounced in connection with such objects. Call again that word – the related name.

Some tables are made of wood, some of metal etc., some are white, some yellow etc. These are the properties that some of the objects have, and some do not. Such properties not shared by all these objects shall be collectively called – *noise*. We add here that ignoring of the noise is, often, a more realistic aspect of abstraction.

Summarizing the essentials, we can say that the concept of table is a triad which consists of the three related components: class of examples, mental image and name.

Proceeding similarly, the concept of, say, number three consists of a class of examples being all three element sets, a mental image, with a normal inclination of each of us, to form a picture in the mind of three points and a name which is the word "three" spoken or written, as well as the symbol "3" as an abridged substitute for that word. Once again, out of specific details, the same triad appears.

The nature of elements of the mentioned sets, together with all properties related to their possible arrangements represent the noise. Moreover, number three, says the definition of cardinal numbers, is the common property of all these sets.

Note that speaking about the concept of number three, we have used the following terms "three point sets", "picture of three points" combined with that concept. No reason to see in it a vicious circle, since no attempt to define anything is present. After all, some fundamental concepts define best themselves, by existing. As for the described mental image, note that H. Bergson [2] says that every clear idea of number implies a visual image in space. When considering the question if the intuition of space accompanies every idea of number, he writes:

> "Anyone can answer this question by reviewing the various forms which the idea of number has assumed for him since his childhood. It will be seen that we began by imagining e.g. a row of balls, that these

balls afterwards became points and, finally, this image itself disappeared leaving behind it, as we say, nothing but *abstract* number."

From the foregoing examples, and many others, the scope of a triad, which represents concepts in general sense, is seen. The component called *mental image* belongs to mental imagery and serves as a mental code of a concept. Being a language codification, *name* belongs to a language. In case of primary concepts, examples belong to the natural environment, but, often, they are also concepts of relatively lower degree of abstractness. Accordingly, we can say that *examples* belong to lower level of abstraction.

In Fig. 1, a visual representation of this triad is given.

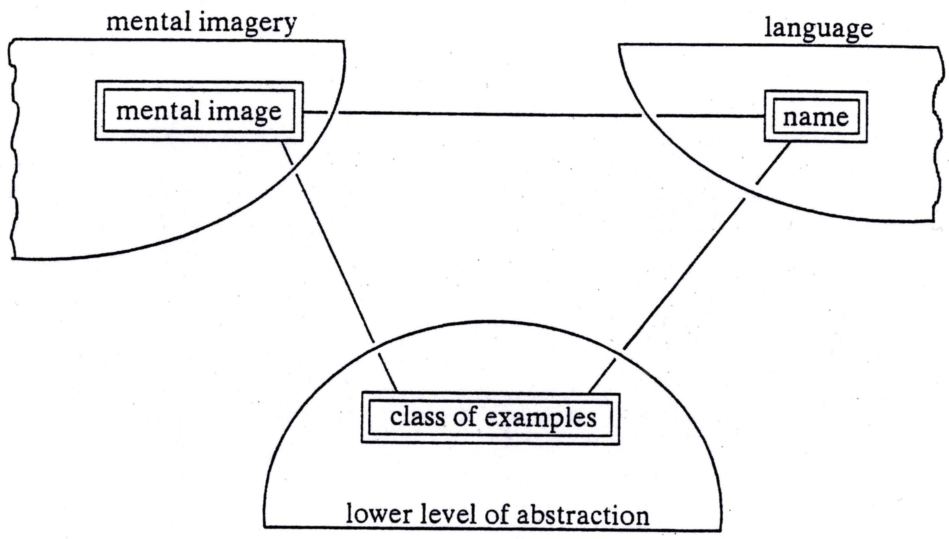


Fig. 1

The lines connecting three boxes serve to exhibit the interdependence of the components in the sense that each may cause the appearence of the others (say, seeing or imagining a table we may use the corresponding word and vice versa). Of course, a very subtle psychological analysis of the function of these lines exists, but we are here on the course of a phenomenological approach and, therefore, we consider a concept as a ready-made product. This does not mean that we shall not consider the ways how some specific concepts are created, fragment by fragment, but we shall not do it in general. For example, the fifth and sixth chapters of L. S. Vygotsky's book [8] give such a general and inspiring approach. We also use examples instead of their successive past experiences (what, again, could be more convenient when the process of learning is emphasized and what is, for example, done in R. Skemp's book [6]). Note also that we use the term "class of examples" instead of logically not quite correct "set of examples" which may cause some trouble in others than this context.

Addendum 1.

In set theory, when A is a set, it is postulated that $A \notin A$.

Now consider if the term "the set of all singletons" is causing trouble (and the singletons (one point sets) are examples related to the concept of number one). Assume that such a set exists and denote it by ϑ. Then, the singleton $\{\vartheta\}$ belongs to ϑ, i.e. $\{\vartheta\} \in \vartheta$. On the other side, $\{\{1\}\} \notin \{1\}$ and $A = \{1\}$ is a set having the property $\{A\} \notin A$. Thus, the following subset of ϑ

$$D = \{ \{A\} \mid \{A\} \notin A \}$$

is non-empty. Let $A = D$.

(I) If $\{D\} \in D$, then the defining property of D is satisfied and $\{D\} \notin D$. A contradiction!

(II) If $\{D\} \notin D$, then the defining property is not satisfied and $\{D\} \in D$. Again a contradiction!

Evidently we imitate here the well-known Russell's paradox.

Examples of the concept of natural number are the numbers 1, 2, 3, ... and examples of the concept of figure are: triangle, square, rhombus, ... So, we see that "be an example of" serves for comparison of some concepts.

Generally, when a concept Q is an example of the concept P, then P is said to be *more abstract* (or *of higher order*) than Q or, by reversing, Q is said to be *less abstract* (or*of lower order*) than P. At the same time, all examples of Q are also examples of P (and so this comparison is a transitive relation in the class of all concepts).

In classical logic `conceptus summum` is the most abstract of all concepts belonging to a given class. An example is the concept of set relative to the class of all concepts of classical mathematics.

Addendum 2.

Let us mention that class, category, functor etc. are not the concepts of classical mathematics.

On the other hand, no idea of `conceptus infimum` exists and the real world objects are never considered to be concepts. But it is still reasonable to think of the concepts which would be at the lowest level of abstraction. Such are the concepts whose all examples are real world objects and then we say that they are *at sensory level*. Their examples are: table, spoon, motor car, ... and we do not include words which name some qualities as: green, hot, heavy, etc. It is an attempt to be less extensive but more effective.

When Q is a lower order concept than P, then sometimes a definition can be used to fix the extent of Q within the class of examples of P. For instance, a rhombus is a parallelogram with all sides congruent or a parallelogram is a quadrilateral in which each pair of opposite sides is parallel. Taking it generally, a *definition* is a sentence of the form: Q is $P +$ `differentia specifica`. This difference is an expression of the characterizing properties of Q within the class of examples of P.

Let us add at the end of this paragraph that we shall be using the terms "*cognitive* (or *mental*) *scheme*" and "*structure*" to mean a system of mutually related mental images and such a system of concepts, respectively.

In fact, many concepts are just systems of individual concepts named collectively. Take examples from mathematics, as polygon, conic section, algebraic equation etc. Then, it is more effective to consider them as mathematical structures giving to this term a larger meaning than the fundamental mathematical structures have. Accordingly, in this case, the mental images are also blends of mental images related to the individual concepts which are integrated by the same name. And yet, it remains to say that structure is an easy term for more strict *conceptual structure*.

4. Signs – significant and signifying

The following classification of signs is due to Ch. S. Peirce (1839–1914):

(I) *indices* – beings or objects associated with the referent and necessary for its existence (e.g. smoke is an index of fire);

(II) *icons* – graphical representations more or less faithful to the referent;

(III) *symbols* – conventionally accepted signs not necessarily bearing any resemblance with the referent.

On the other side, dealing with the ways of capturing experience in memory, J. S. Bruner describes three modes of representation:

(I) *enactive* – when past events are represented through appropriate motor responce (and what is typical for infants at sensorimotor stage);

(II) *iconic* – which takes a step away from the concrete and physical to the realm of mental imagery;

(III) *symbolic* – happening whenever operating with conventional signs occurs.

We underline also that, according to Bruner, each of two latter modes depends on the one, that preceeds it, and each of three is requiring a great deal of practice before the transition to the next can occur. (See, for example, [5]).

Concepts are learned and, in that process, signs play an important role, both as carriers of their meaning and as landmarks. When realized graphically, the signs are recognized by shape and shapes themselves are concepts, says R. Arnheim in his excellent book "Visual Thinking" [1]. Let us quote a couple of passages from this book.

> "In the perception of shape lie the beginnings of concept formation. Whereas the optical image projected upon the retina is a mechanically complete recording of its physical counterpart, the corresponding visual

percept is not. The perception of shape is the grasping of structural features found in, or imposed upon, the stimulus material."

...

"Perception consists of fitting the stimulus material with templates of relatively simple shape which I call visual concepts or visual categories."

These templates, or maybe it is better to say, their projections in space, materialized as graphical representations are the roots of some fundamental mathematical concepts. Of course, something similar could not be said in general and a mental image is unlikely to be more or less a faithful replica of some visible thing. As a mind's account of things, such an image summarizes them by representing only their important characteristics.

Not grasping at an effective comprising of all concepts, we continue to elaborate further the schematic representation from the preceeding section. If in case of some concepts as for example, those formed on man's emotive responses such representation could be not more than a "graphical" metaphor, for the concepts which are in the ways of our considerations, it comes as a matter of course.

Drawings of real world objects done, say on paper with pencil or pen, tend to transform their shapes into recognizable pictures. Looking at them, the objects seem like being stamped down, but when needed, their lost dimension still can be perceived. Imagining them, we form pictures in mind, often accompanied spontaneously with a piece of background which also helps the rise of the third dimension. The realm in which, by imagining, we see things is called the inner space. Is it the merit of this space in seeing two and perceiving three dimensions? We can only ask such a question.

Meaningful drawings are pictorial signs which suggest the part of reality they stand for. For instance, three balls may be represented pictorially by three circles and that representation steps towards the abstract idea (mental image) of number three. When three men are represented by three points, such a step on the way to abstraction is even longer.

A pigeon we see looking through the window is a "materialization" of the concept (idea) of pigeon or, as also said, it is a real world example. A pigeon seen in a picture is not an example of that concept, now its pictorial sign is seen.

In the opposite direction, the abstract idea of number three may be represented drawing three balls and its "materialization" are three balls on the floor. In the same direction, acting from memory, a drawing of pigeon is done.

We see the double role of icons in representing the concrete and physical on one side and the mental, on the other side. In the latter case, an icon shall also be called *ideograph*.

A picture may be realized in many ways, on paper, blackboard, canvas etc. and we call the whole extent of pictorial representation *pictorial environment* or, as a tribute to Komensky, `orbis pictus`.

In relating a concept, its icon and its possible real world example (model), let us use the following general scheme to summarize the already said.

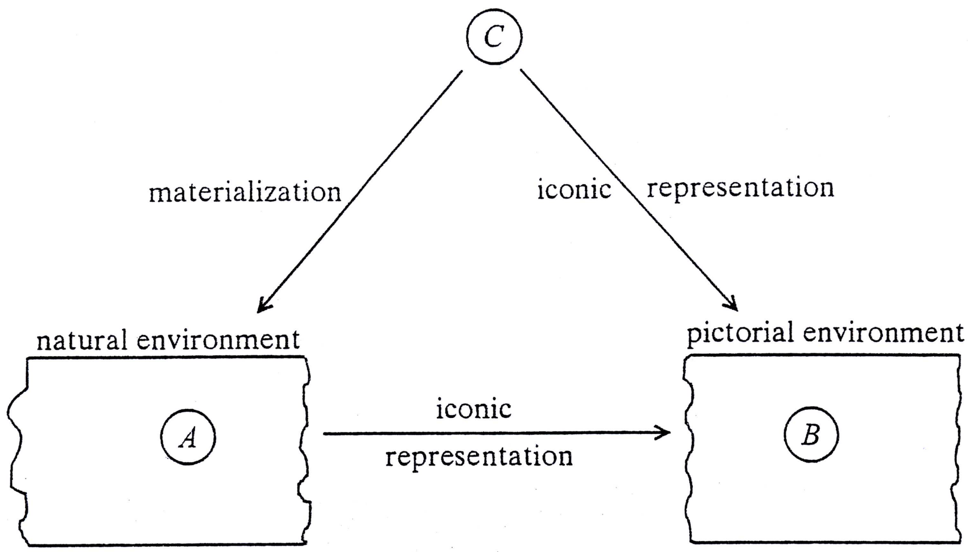


Fig. 2

To illustrate it by an example, take:

A to be the set of three balls on the floor,

B to be the set of three points drawn on a piece of paper,

C to be the concept of number three.

Then,

A is a materialization of C,

B is an icon of A,

B is an icon (ideograph) of C.

As it is everybody's experience, mental images are easily disturbed and drawings that can be expected to relate well to them, serve for their stabilization. Think only of function of drawings in geometry.

A physical object is equal to itself in each instant and independently of its place in the outer space. On the contrary, a pictorial representation has to be considered as a snap-shot of a scene from either outer or inner space. A "leaf" used in that representation plays an important role as a precedent model leading to the conception of geometrical plane. Call it *pictorial leaf*. To strenghten the mentioned role, as well as to avoid some possible discrepances, pictorial representation has to comply to the following rule: *two icons occupying different places on pictorial leaf are different* (and the same is true for two sets of points in plane).

Pictorial signs whose shape makes them distinct and recognizable but whose meaning is established by convention are called (written) symbols. Examples of such symbols are numerals, letters, words, mathematical signs, syntactical signs etc. Devised for human vision their shapes are usually very simple but it is still hard to describe their structural features and to explain the way they are recognized.

Following the same template, two persons can write the same letter with a varied style and size and still such two different signs have the same meaning which does not depend on the places that they occupy, either.

Out of recognition, the morphological type of symbols is of no significance what contrasts them with icons which are the significant signs. Devised to signify, symbols stand for concepts denoting them. For example, the same word appearing many times at different places always has the same meaning and, on the basis of that meaning all such denotations are equal.

Call the frame of symbolic representation *syntactic environment* and a "leaf" used in such representation *syntactic leaf.* Again, we shall express a rule: *two symbols occupying different places on syntactic leaf are equal, meaning equally.*

The following graphical scheme illustrates two representations of a concept C.

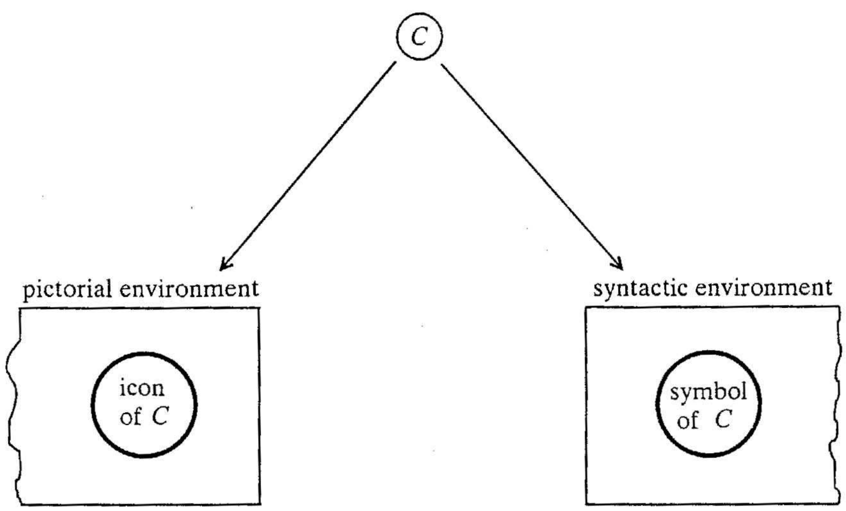


Fig. 3

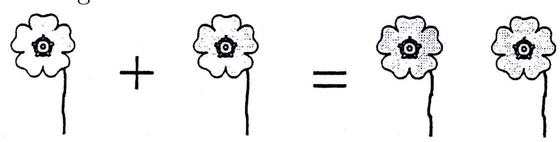
and, according to the two stated rules, these leaves are different (though, practically, their contents are normally seen on parts of the same page, blackboard, etc.).

5. Some often encountered discrepancies

The first step in constituting the set concept is the acquirement of sense of the element as a unity of existence. Some natural aggregations of objects or beings which are alike and just somewhat distinct from one another serve to it best. Even a preschooler has the meaning of the terms as "flock of sheep", "flight of birds", "box of pencils" etc. already formed. The elements of these natural sets preserve their identity independently of the change of their position in such aggregations and, usually, there exists no inconsistency in the treatment of such examples.

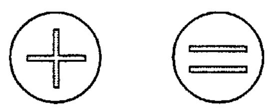
It is entirely different with the examples from pictorial environment which are sometimes quite confusing.

Pick first a "barbaric" equality from a textbook, where pictorial and syntactic signs are seen mixed together.



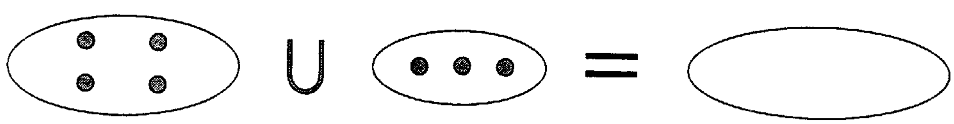
It is not the question of good taste which makes the equality bad, but the fact that numbers, and never flowers, are summed up. For example, a picture of Suzy holding a flower and Mark running and bringing another one drawn on the left half of page and one more picture of Suzy holding two flowers on the right side, would be a much better illustration. Expecting teacher to interpolate a story telling what was going on, the pure equality: $1 + 1 = 2$ should follow.

If the idea of the author was to compose the sentence "One flower and one flower are two flowers" by using pictures instead of written words, then the stickers



(read "and", "are") would do the job much better. And again, such an idea would not be very appealing.

As the next blunder, consider the following set equation



An expected solution would be seven circlets drawn in the enclosed space on the right-hand side of the equation. On the basis of a strange logic (maybe, it is better to say, of no logic) each of the given circlets is equal to the only one drawn on the right-hand side and, at the same time, is different from all others!

It will be no improvement if the circlets are replaced by seven mutually distinct geometrical figures. Solving the equation as before, then each figure on the left-hand side is equated with the one of the same shape on the right-hand side. Now, different positions of congruent figures do not make them distinct and, therefore, they all would merge into a single one. On the basis of such equating, the derived absurdities could be quite shocking. For instance, there are no two distinct points in the plane, no two distinct circles with equal diameters exist, and so on.

Blindness of the authors of similar nonsensical equations is easily explained. Having two heaps of four and three counters, we have two sets in natural environment. Gathering the counters together, a third heap is formed which is the union of the two sets. This "barbaric" equation is evidently designed to suggest that activity. It is not only a poor means to do that, but, what is even worse, it violates the efforts to cultivate the child's right procedure of iconic representation.

Say at the end, that the right way to find the union of two sets given in pictorial environment is to consider their elements to comprise a third one and, if denoting, to encircle them.

According to the definition, two sets A and B are equal if they contain exactly the same elements. Then, every element of A is an element of B, and every element of B is an element of A. When equating sets, pairs of elements are equated and the basis on which this is done is something that is out of the set theory. Whenever such a basis is vague, then considerable discrepancy between the ways two sets are compared could exist.

This set equality

$$\{\bigcirc, \square, \triangle\} = \{\triangle, \bigcirc, \square\}$$

is often found in textbooks. Talking before a large group of elementary school teachers I directed them the question if these two sets:

$$\{\bigcirc, \square, \triangle\}, \qquad \{\triangle, \bigcirc, \square\}$$

were equal. "Yes", was their answer. And what about these two:

$$\{\bigcirc, \square, \triangle\}, \qquad \{\triangle, \circ, \square\}$$

was my following question. "They are different, because two circles are different", was their answer. And now:

$$\{\bigcirc, \square, \triangle\}, \qquad \{\triangledown, \bigcirc, \square\}$$

I asked. "Different again", they said.

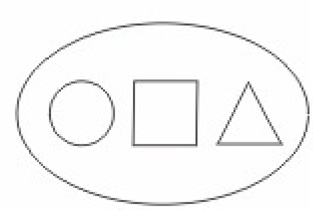
In the first two cases, congruence was evidently the basis of equating and in the third, they left it and they probably understood the elements as a kind of conventional signs. But what do they stand for?

In fact, a set can be equal to itself and that is all. When equating, we do it with different notations of the same set. Thus, when examples are from natural or pictorial environment, then no reason exists to consider set equalities. Such equalities arise when the elements of sets are denoted by syntactic signs and then, two signs, signifying the same, are taken to be the same.

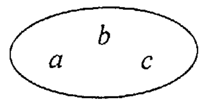
When elements of sets are geometrical figures, then a notation as:

$$\{\bigcirc, \square, \triangle\}$$

is "barbaric" and the following



is the right one. In the like manner, when elements are syntactic signs, a notation as:



is also "barbaric" and $\{a, b, c\}$ is the right one.

Many similar "barbarisms", being mixtrures of signs from two different environments, are often found in school books.

6. Sets and counting

6.1. At the first stage. In Section 3, when comparing concepts with respect to their abstractness, we emphasized the set concept as being the most abstract among all concepts of classical mathematics. To make that idea clearer, say that each classical mathematical object can be considered to consist of elements (which, in geometry, are usually called points) and these elements, taken to exist together, form a set. Then, this set is considered to be endowed with structure which, speaking freely, organizes its elements. Thus, a mathematical object can ever be viewed as a set endowed with structure.

Whenever some characteristic attributes of a concept are removed, we arrive at a more abstract one. Removing the structure of a mathematical object that is forgetting the organization of its elements, nothing but an amorphous set stays. The last emphatic expression underlines the abstract conception of a set, when:

element is understood as a unit of existence (everything else as its possible property being ignored),

set is understood as a mere existing together of elements (and everything else, possibly derived from the ways of their existence, is ignored).

Addendum 3.

A mathematician can think of many examples of "forgetting": assign to a group G the set $\overline{G}$ of its elements, forgetting the multiplication and hence the group structure; assign to a topological space X the set $\overline{X}$ of its elements, forgetting the family of open sets and hence the topological structure and so on.

Recall also that a forgetful functor assigns to each object A of a concrete category the set $\overline{A}$ of its elements, forgetting so the structure with which A is endowed (and to each morphism f the same function $\overline{f}$ regarded as a mapping of sets).

Thus we see that the concept of set is poorer in content but broader in range than any other concept of classical mathetmatics. Accordingly, the related examples exist at all levels of abstraction and a full meaning of that concept can be developed only when the acquanitance with the examples at all levels is accomplished (and this practically means, after years and years of studying). Therefore, to fix the *first stage of learning* in the spirit of our phenomenological approach, means to fiz *extent of related examples.*

In case of the set concept, that extent consists of *examples in natural and pictorial environments.* Use the following scheme to express it graphically

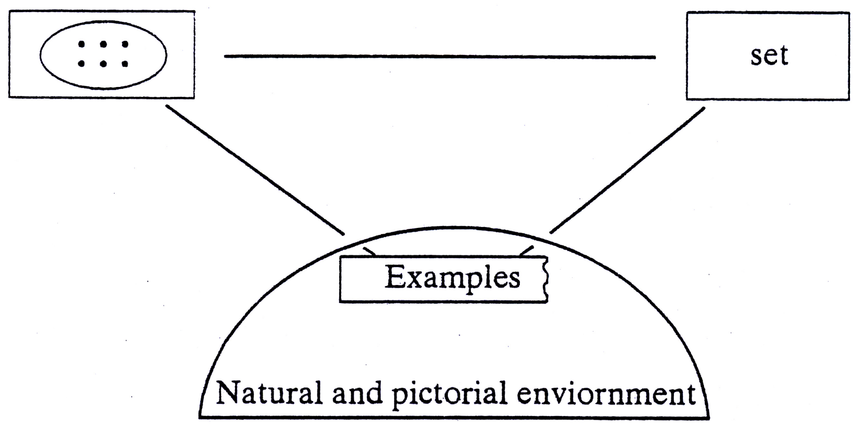


Fig. 4

As a matter of course, all examples are finite sets (of small cardinality).

6.2. Number in foreground – set in background. System of natural numbers is the biggest thema of elementary school mathematics. Starting with the specific concepts of numbers 1, 2, 3, ... building and extending their small blocks, learning how to denote them positionally and developing a feeling for spontaneous induction, after years of learning, a general idea of natural number forms. And as a very start of this thema, we see counting.

Counting begins with the ability of a child to name numbers orderly. When these names are attached to objects of a collection, then counting is not only a mechanical operation, but the spoken words start to assume some meaning, reflecting so an important feature of the collection. Respecting the order of the involved operations, first a collection is singled out and then its objects are counted. This means that the idea of set precedes that of number. Thus, it is not only fashionable but also natural, to introduce young children in the primary grades to some set notions.

Freudenthal's Chapter XV, "Sets and Functions" [3] illustrates many examples of "false set theory" in schools. In the period of New Maths, sets in primary grades were a nuisance troubling equally children and teachers, and the following pretensions were easily visible:

(I) to group together in a set any kind of odd things,

(II) to base the general meaning of natural number on any kind of one-to-one correspondences (while primitive tribesman from prehistory was celebrated as the first one-to-one corresponder),

(III) to represent formally any kind of related mental operations (causing so a stumbling confusion).

At early stage of cognitive development the mind does deal with numbers in their natural dependence on sets of objects in the child's surroundings. The way how the idea of number is formed in mind, starting with a concrete set of objects A, the *principle of invariance of number* states, directing these two forgettings

(I) of the nature of elements of A,

(II) of the arrangements of elements of A,

and leaving behind an abstract number $\overline{\overline{A}}$.

This cognitive (and not mathematical) principle is due to G. Cantor and his notation with two dashes "$\overline{\overline{A}}$" suggests two above forgettings. Principles (laws) of arithmetic can also be viewed as specific forms of this principle, what we will explain later on, in a suitable context.

In order to start counting, a collection of objects is selected first. Then, its elements are conceived as being alike, forgetting their differences and considering each as the unit of counting. Fortunately this ability is developed early and spontaneously and a preschooler usually has no problem with it. And finally, through the second forgetting, objects, in an arbitrary way, are corresponded to number names. Many words in the natural languages, as English "flock", "flight", "brood", "shoal", "bundle" etc. mean exactly the same as the word "set" does, except that they specify the nature of grouped elements. Or to be more precise, *today teaching of mathematics in schools creates that universal meaning of words "set" and "element" which has not been spontaneously formed in any natural language.*

A deliberate usage of the terms "set" and "element" within the contents of arithmetic is the best way to introduce them and to let children assimilate their meaning. Exercises in the next subsection serve to show how it can be done when combined with counting.

6.3. Sets in picture-form. The first steps in arithmetic have ever been related to sets, presented as groups of discrete things (beads, counters etc.), no matter if the word "set" has or has not been explicitly used. Sketching a series of exercises, our aim here is to suggest how to use the words "set" and "element" in situations which arise naturally and when a more universal meaning has to be expressed. All sets we deal with will be represented by such pictures which the mind focuses on momentarily as meaningful collections. We will also follow a stereotype, avoiding dialogues and writting the words which are expected to be elicited replies from pupils, in spaced out letters.

6.3.1. Here we start with examples involving natural usage of words naming specific sets and their elements.

(a) A small flock of sheep is seen in the picture. Some are white, some black.

We see a f l o c k o f s h e e p.

How many white: s i x (children count).

How many black: t h r e e.

How many all together: n i n e (children count).

(b) Two basketball teams are seen, each on its half of the lawn.

How many teams do we see: t w o.

How many players in each: f i v e (children count).

How many all together: t e n (children count)

etc.

In exercises of this type, the same word names elements of sets and those of their subsets.

6.3.2. Now a more general word denotes elements of a set and more specific ones those of subsets.

(a) A group of children is seen in the school-yard.

We see a group of c h i l d r e n.

Three are b o y s and f o u r g i r l s.

All together we see: s e v e n p u p i l s (children count; being induced to use word "pupils" by a teacher's woven into story).

(b) Table acts as a gatherer and, in this role, is a natural space (place) holder. Pencils and erasers are seen on the table.

How many pencils do we see: f i v e (children count).

How many erasers: f o u r.

How many objects are there on the table: n i n e (children count).

We see a group of nine o b j e c t s (t h i n g s) on the table.

etc.

6.3.3. Examples in which the usage of words "set" and "elements" is under compulsion of good sense for language.

(a) Pencils and apples are seen on a table.

How many pencils: f o u r.

How many apples: t w o.

Together, the pencils and the apples make a set. This set has s i x elements (children count and the teacher forces usage of the word "element").

(b) First the picture is described: three men and their three cars make a s e t. This set has s i x e l e m e n t s.

etc.

6.3.4. Examples with an intentional forcing of the usage of words "set" and "element" as possible substitutes for many other more specific words.

(a) A tray and a basket are acting as gatherers. Five cakes are seen on the tray and four pears in the basket.

In this picture we see t w o s e t s.

The elements of first set are called c a k e s and of the second p e a r s.

All together we see n i n e e l e m e n t s.

(b) A bicycle park and two groups of boys and girls are seen standing separately.

How many sets can you see: t h r e e.

Name their elements: b i c y c l e s, b o y s, g i r l s.

etc.

All types of these exercises can be done at the very beginning when children cannot read yet. A continuous usage of words "set", "element" should exist throughout all arithmetic topics adding later the word "subset" as well. Of course, the frequency of usage of these words, instead of those more specifying, is mostly a matter of good taste and no overusage is anything good.

With regard to arithmetic, this little piece of set-theoretical language helps description of the schemes upon which formal operations are based and is the way how the invariance of unity of counting is expressed.

On the other side, using a single word to name members of a set, we operate with a *rhetorical variable*. Using "element" we do it with the most universal. This usage develops the subtle meaning of variables found in natural language connecting them with their referents in everyday's surroundings of the child.

The following step when we shall be treating sets again is postponed for much later. At that stage, the elements will be conventional signs and then, the first set-theoretical notations will be used.

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