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INTERACTIVE LEARNING AND TEACHING OF LINEAR ALGEBRA BY WEB TECHNOLOGIES: SOME EXAMPLES

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Abstract. Supposing that students have successfully passed lectures and assessments in linear algebra in traditional way, it is suggested to introduce block-teaching by adequate software tools. These tools would, in 2D and 3D, enable students to establish and to confirm their knowledge by use of numerical, symbolical and visual representations of previously accepted concepts. At the same time they would explore actively and interactively, either individually or in a group, in order to adapt these activities to their level of knowledge and their learning style. This is a modern methodological concept of presentation and acceptance of mathematical knowledge in linear algebra, concerning the systems of linear equations emphasizing the discussion of solutions and analysis of special system cases, using determinants or matrix algebra system. Contribution of this paper is a presentation of methodological elements and specific examples with a group of corresponding questions, which could be efficiently applied to this teaching unit, based on suggested software tools.

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1. The teaching "approach to student"-method in a traditional Linear Algebra class

The teaching of linear algebra has always been a challenge for teachers of mathematics, because it is extremely important that students become introduced into complex and abstract mathematical system of linear algebra, and learn concepts which can be successfully applied later in other mathematical topics. It is necessary that teachers better understand how students learn, and recognize and allow that the appropriate content, methods and context could be different in different environments [3].

The traditional approach that "only talking is teaching" is not acceptable anymore and it is not sufficient, because it completely ignores the cognitive level and degree of development of each individual student. The role of teacher is to assist students in this construction process to acquire knowledge, but it is quite clear that simply talking and showing, no matter how qualitative it may be, probably will not significantly improve their learning of such abstract topics. It is recommended that students acquire knowledge by themselves.

As mathematicians, we are aware of the significant interconnections of different ideas and concepts, which is difficult to recognize and understand. We should not forget that understanding of these kinds of interconnections develops through active and hard exploration of mathematical topics (through missteps, false generalizations, incomplete and inconsistent conceptions, ...) through permanent discovering of new interconnections and relations.

Thus, primary role of a teacher is to try to move students to take an active part during the class concerning important and difficult concepts, either through the form of individual opinion or through the form of group discussions. Even when the lectures have been supported by powerful technology, it is possible that students are still passive observers, and we know that passive students are rarely successful in qualitative learning.

It is not easy to suggest teaching methods, especially in comparison to traditional lectures, which would be effective and would actively engage students and generate stimulating learning. Some of very interesting questions for considering linear algebra teachers are [11]:

- What is optimal to teach at the first course and what should be a student's previous knowledge?
- Which degree of abstraction should the teaching aim to?
- Which facts should be used in the proofs?
- How much time should be spent on some lesson topics?
- What is the measure of qualitative understanding of basic principles and conceptual learning in linear algebra?
- Whether, when, and how to use technology?

There are several different roles that technology can play in instructions: from eliminating computational drudgery in realistic applications to providing environments for active exploration of the properties of mathematical structures and objects, or to getting a variety of experience using different software tools. Sometimes teachers can use adequate software tools for some topics to facilitate calculation of some examples and then they can direct discussion to analyzing results (for example, whether the results are as expected and how to verify them?).

Some teachers have lectures in electronic form for all the topics in their linear algebra courses. They hold lectures in computer laboratories and students work on these lessons individually or in groups, while the teacher is present and assist as much as necessary, what is an effective collaborative learning method [3].

Classroom voting through multiple choice questions or true/false questions is a powerful technique which can easily be incorporated into a traditional class. These techniques prevent students to be inert and passive listeners and require their active participation permanently, creating a more effective learning environment. The students vote on the correct answer for the following type of questions: "Who thinks that this idea will be correct? Who doesn't?", either by holding up a hand or using an electronic clicker device, and then teacher can guide or direct the class through a discussion toward the concepts involved. The discussions are usually very live, and typically result in correct responses from a large majority of the students. It is a practice that some of the students, who understood correctly, publicly explain their answer [5].

Self-evaluation and self-regulation are very meaningful parameters of a student's work. Also, students learn best in a non-stressful environment in psychological, emotional, and physical sense. However, some research suggests that a neutral environment, such as computer environment, is best for older, highly-motivated students. The neutral environment allow students to have a choice in selecting tasks and activities whenever possible, or allow them to participate in group work, group discussions, especially in cooperative learning, which successfully increases motivation of students.

2. An integration of technology in Linear Algebra classes

Each individual's learning style can be defined as the way how that person best absorbs and processes information. Most people, when they need to learn something new, have a consistent tendency to use one of their senses more than the others, especially sense of sight, sense of hearing, or sense of touching. People differ in the way how they approach a learning task. Generally, the understanding how students learn is very important in the process of teaching and learning mathematics [4].

It is necessary to consider the following questions:

- Why do some students learn more mathematics than other students in the same class?
- What can linear algebra teachers do to enrich or replace traditional lecturing in order to improve meaningful learning?
- What contribution of technology could be in the fields of experimenting, observation, and discussing?
- How many reasons are there, for yes or no, for example for use of graphics calculators for computing matrix inversion or for solving linear systems?

ICTs and 'Computer Aided Teaching' have become an important part of life today, and are widely used to improve teaching and learning techniques. The main forms in teaching mathematics with computers are:

1. Web teaching and learning systems

   Using ICTs many universities have established virtual education and distance learning systems in the field of mathematics, with all new possibilities for students.

2. Mathematics software packages

   Many mathematics software packages have been developed (for example Maple, MATLAB, Mathematica), which have very powerful, numerous functions, such as:

   - Instantaneous numerical and symbolic calculations;
   - Data collecting, analysis, exploration, and visualization;
   - Modelling, simulation, and prototyping;
   - Presentation graphics and animation in $2D$ and $3D$;
   - Application development.

The goal is to use technology to provide an environment for active exploration of mathematical structures through multiple representations, or to show students some aspects of mathematics that are not possible with pen and paper.

Many teachers agree that after mastering basic hand calculation techniques use of the machine arithmetic is very helpful and preferable. In most real world problems, the large dimensions of matrices make hand calculation completely inadequate. Calculators and software enable student to be free to concentrate on what the computations mean, and when and why to perform them. It is especially useful to face them with unexpected results and special cases in work.

In linear algebra, the theory plays an essential role in computations and a computer can be used to motivate learning of theory and to reinforce the concepts. The students can concentrate on ideas instead of trying to get the arithmetic right in the solution of some linear system. Additionally, in examples of solving linear system, computers can stimulate the students' geometric intuition through interesting visualizations in $2D$ or $3D$, or computers make it possible to ask questions involving theoretical issues that are arithmetically too complicated in the traditional way [10].

Some teachers worry that using computers in mathematics will turn students into "button pushers" without thinking and that only hand matrix multiplications provides understanding of final results. But, it is evident that the possibility of rapid and effective solving of a large number of examples, certainly contribute to better understanding.

3. Interactive examples in teaching and learning linear algebra with computers

Although my teaching methods are still traditional (textbook, lecture notes, lectures, assignments, final exam), my plan is to try to make some changes in the way of work in order to make my teaching courses more effective and to improve the learning outcomes of my students. I have noticed that many students do not like the course of linear algebra, because it is very abstract and contains too many new concepts, or some of them are not able to make connections with real applications. Thus, the learning outcomes of students are poor, and some fail the final examination.

This short paper provides a few examples of computer uses of algebra software in my classroom. This software has changed situation in my classes, and opened up new opportunities. Students have learned new techniques and have been able to use more advanced materials and options, and they have also been able to use a variety of representations. Most important, students have felt more capable and familiar with all types of problems in linear algebra.

Using examples and questions given below, the teacher will guide students to full understanding of the meaning and nature of solutions of linear systems of equations. These examples illustrate low-order systems to assist students in personalizing important concepts about systems of linear equations by "discovery method". Low-order examples are suitable methodological models for overcoming visual limitations of examples of higher-order, because the real-world problems often involve systems of thousands of equations in thousands of unknowns.

The teacher should define the terms "consistent, inconsistent, homogeneous", "one solution, no solution, infinite solutions", and precise definitions of "linear equation, linear system, nonlinear system" and "solution" should be given. Also, the teacher should strongly encourage geometric visualizations through the lesson plan. Various software has been used to enhance my lectures of linear algebra.

There is a gallery of dynamically-generated basic equation plotting examples on address [6]. Student can start with these examples and explore different variations of the results. Also on this address, there is a very helpful tool for plotting multiple equations and solving systems of linear equations in 2D and 3D. With only one click, student can get all the graphs and points of intersections in a visual and interactive way.

Example 1. Consider the following system of equations whose graphs are presented (for example, $y = x + 2$; $y = x^2$).

Q1. Based on the nature of their graphs, what types of equations are represented, linear or nonlinear? Why?

Q2. Does the system have any solutions? How many of them?

Q3. What do you mean when you say that there is a solution?

Q4. Can you locate the solutions? Give a geometric interpretation of the solutions. How would you write down these solutions explicitly?

Q5. The x-intercept is where the graph crosses the x-axis. With that in mind, what value is y always going to have at the x-intercept? If the x-intercept is where the graph crosses the x-axis, where do you think the graph crosses for the y-intercept.

Q6. How would you check to see if any point (x_0, y_0) lies on both lines?

Example 2. Consider the system in Example 1 together with one additional equation (for example, $y = x + 2$; $y = x^2$; $y = -x + 3$). Ask the students to sketch the graph of the third equation in the coordinate system containing the graphs of the first two equations.

Q1. Do you think that the new system of 3 equations in 2 unknowns has any solutions? How many solutions does this system of three equations have? Describe the location of the solutions (if any). How can you check your answer?

There is a large window Java Applet on address [7] that helps students to explore the solutions of systems of linear equations, giving a complete picture on solving systems of equations with the existing algebraic methods (elimination, Cramer's rule, ...) through detailed explanations. Student can use interactive tutorial (with or without Java Applet) for next type of questions:

Example 3. Consider a family of equations which take the form $ax + by = c$. For different values of a, b and c, the graphs of these equations are lines in a two-dimensional coordinate system. Let the students choose any real numbers $a_1, b_1, c_1, a_2, b_2, c_2$ to define a system of two equations in two unknowns:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

These graphs can be easily generated using any standard graphics calculator.

Q1. Can you change the numbers in the second equation so that the resulting system has

a) no solution?

b) exactly two solutions?

c) infinitely many solutions?

d) Set the constants $a_1, b_1, c_1, a_2, b_2, c_2$ so that the two lines have a point of intersection (x_0, y_0), and check that the ordered pair (x_0, y_0) is the solution of the system by any algebraic method.

e) If you do get one (infinite) solution for your final answer, is this system consistent or inconsistent? Would the equations be dependent or independent?

Q2. Student can set a_1, b_1, a_2, b_2 for values such that $a_1b_2 - a_2b_1 = 0$, and consider the next questions:

- How are the two lines positioned with respect to each other?
- How many solutions the system has?
- What is the relationship between the slopes of the two lines?

Q3. Student can set all 6 constants to values such that $a_1/a_2 = b_1/b_2 = c_1/c_2$ and explain how many solutions the system has in that case.

Q4. The teacher should now add the following third equation to the system $(a + a')x + (b + b')y = (c + c')$, and then continue with the questions below:

Ask the student to graph this third equation and then ask them to analyze if this new system of three equations in two unknowns has any solutions. Where it came from?

Q5. Ask the students to select numbers of the third equation randomly and to graph it along with the first two. Normally such a random selection should generate an inconsistent system with a large probability, whose lines bound a triangular region. Does this new system have a solution and whether they have a solution or not?

Q6. For homogeneous linear systems, the teacher should focus on the following types of questions: Why is there always at least one solution (geometry)? When are there other non-zero solutions of a homogeneous system?

Q7. Consider the system

$$2x + 3y = 9$$
$$y = -\frac{2}{3}x + a$$

If $a = -6$, is the system consistent or inconsistent? Explain your answer.

There are two online calculators and solvers on address [8] for systems of 2 by 2 and 3 by 3 linear equations, which use Cramer's rule to solve the system. This tool can be used to check the solutions of 2 by 2 and 3 by 3 linear systems solved by hand. It can also be used, efficiently, to explore a system of equations. Student can enter the coefficients of systems as real numbers and press "Enter". For example, the solutions of linear equations with three variables and three equations can be displayed. It is easy to see that every equation represents a plane in three-dimensional space and a common intersection point of the three planes represents a solution of the equations. So the solution of the system can have three different cases:

- unique solution, when the three planes have a unique common intersection point,
- no solution, when the three planes have no common intersection point, and
- infinitely many solutions, when the three planes have infinitely many common intersection points.

There is an application for solving linear systems on address [9]. If a student chooses matrix method, a linear system written in the matrix form $A \cdot \overrightarrow{X} = B$ is solved, where $\overrightarrow{X}$ is the vector of variables, coefficients matrix A is corresponding to coefficients of system, and the column of constants B is corresponding to free coefficients of the system. With this application, student will not have much difficulty in finding the determinant or inverse matrix of 4×4 type or higher!

The most exciting of these changes in the learning of linear algebra is the way that software opens up questions to alternative solutions. For example, finding an inverse matrix is a standard task when student learn matrices for the first time. Since software automatizes the finding of the inverse, one possibility would be a parallel finding of inverse matrix by hand. For example, find x so that the a 2×2 matrix (with one element being x) does not have an inverse matrix. Students must be sure that, if the products of the diagonals are equal, the matrix does not have an inverse matrix. But, some students could solve the task, taking into account that they can not divide by zero, what is certainly more applicable and important concept.

Additional information can be find in the books [14] and [15].

4. Conclusion

The essence of teaching is to help students to learn the material that they need and want to learn—since different students want to learn different material in different way, we should expect to change the way of our teaching. The integration of ICTs and theoretical mathematics is natural in linear algebra, so that students can use their experience with linear algebra as a starting point for seeking similar integration in other mathematical areas and understanding of mathematics. Students have learned new techniques and have been able to model and evaluate a situation that was challenging, interesting, and real. Technology brings to students and teachers the opportunity to individualize learning—to generate illustrative examples, to follow interesting topics to the desired depth, to choose their own problems and appropriate tools for solving them. Information technologies have transformed the workplace for teaching-learning of mathematics, but not yet the curriculums of mathematics. Methodologically various teaching-learning models, which involve mathematical software, need to be developed in future.

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