THE TEACHING OF MATHEMATICS
2007, Vol. X, 2, pp. 107–108

A PROOF OF CHEBYSHEV'S INEQUALITY

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Abstract. Interrelating inequalities by proving that one of them is a specific case of others, makes their proofs transparent and often easier. Thus, we derive here Chebyshev's inequality from two inequalities related to convex combinations (and also having some interest in themselves).

ZDM Subject Classification: I34; *AMS Subject Classification*: 00A35.

Key words and phrases: Relation of majorization, convex combinations, Chebyshev's inequality.

This short note is just an addendum to the paper [1].

Recall that, for decreasing sequences $a = (\alpha_i)_{i=1}^n$ and $b = (\beta_i)_{i=1}^n$ of real numbers, a is said to *majorize* b, what is denoted by $a \succ b$, if the terms of these sequences satisfy the following two conditions:

1° $\alpha_1 + \alpha_2 + \cdots + \alpha_i \geqslant \beta_1 + \beta_2 + \cdots + \beta_i$, for each $i \in \{1, 2, \dots, n-1\}$;

2° $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_n$.

First, we prove the following two lemmas.

LEMMA 1. *Let $a = (\alpha_i)_{i=1}^n$, $b = (\beta_i)_{i=1}^n$ and $x = (x_i)_{i=1}^n$ be three decreasing sequences of real numbers, such that $a \succ b$. Then the following inequality holds:*

$$\sum_{i=1}^n \alpha_i x_i \geqslant \sum_{i=1}^n \beta_i x_i.$$

Proof. Denote $A_i = \sum\limits_{j=1}^i \alpha_j$, $B_i = \sum\limits_{j=1}^i \beta_j$, for $i \in \{1, 2, \dots, n\}$, and put $A_0 = B_0 = 0$. Then we have

$$\begin{aligned}\sum_{i=1}^n \alpha_i x_i - \sum_{i=1}^n \beta_i x_i &= \sum_{i=1}^n (\alpha_i - \beta_i) x_i = \sum_{i=1}^n (A_i - A_{i-1} - B_i + B_{i-1}) x_i \\ &= \sum_{i=1}^n (A_i - B_i) x_i - \sum_{i=1}^n (A_{i-1} - B_{i-1}) x_i \\ &= \sum_{i=1}^{n-1} (A_i - B_i) x_i - \sum_{i=0}^{n-1} (A_i - B_i) x_{i+1} \\ &= \sum_{i=1}^{n-1} (A_i - B_i)(x_i - x_{i+1}) \geqslant 0,\end{aligned}$$

being $A_i - B_i \geqslant 0$ and $x_i - x_{i+1} \geqslant 0$ for each $i \in \{1, 2, \dots, n-1\}$. ∎

In particular, when a and b are decreasing, $a \succ b$ and $\sum\limits_{i=1}^{n} \alpha_i = \sum\limits_{i=1}^{n} \beta_i = 1$, the above inequality holds for convex combinations of points $x_1, x_2, \dots, x_n$.

LEMMA 2. *Let $a = (\alpha_i)_{i=1}^n$ be a decreasing sequence of nonnegative numbers such that $\sum\limits_{i=1}^{n} \alpha_i = 1$ and let $b = (\beta_i)_{i=1}^n$ with $\beta_i = \dfrac{1}{i}$. Then $a \succ b$.*

Proof. It is clear that $\alpha_1 \geqslant \dfrac{1}{n}$, and let us suppose that

$$\alpha_1 + \dots + \alpha_i \geqslant \frac{i}{n},$$

for some i. It is not true that $\alpha_1 + \dots + \alpha_{i+1} < \dfrac{i+1}{n}$, what would immediately imply $\alpha_{i+1} < \dfrac{1}{n}$. From $\sum\limits_{i=1}^{n} \alpha_i = 1$ it follows that $\alpha_{i+2} + \dots + \alpha_n = 1 - (\alpha_1 + \dots + \alpha_{i+1}) > \dfrac{n-(i+1)}{n}$, and so $\alpha_{i+2} > \dfrac{1}{n}$, which would contradict the fact that a is decreasing. Hence, $\alpha_1 + \dots + \alpha_{i+1} \geqslant \dfrac{i+1}{n}$ and the inductive proof that $\sum\limits_{j=1}^{i} \alpha_j \geqslant \sum\limits_{j=1}^{i} \beta_j$ for each $i \in \{1, 2, \dots, n-1\}$ is completed. ■

Now, we derive the well-known Chebyshev's inequality

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \leqslant n \sum_{i=1}^{n} a_i b_i.$$

for decreasing sequences $(a_i)_{i=1}^n$, $(b_i)_{i=1}^n$ of real numbers.

Proof. Without loss of generality, we can assume that the terms a_i and b_i of the given sequences are nonnegative (if, for example, some of a_i's or b_i's were negative, we would apply the procedure that follows to the terms $a_i' = a_i - a_n \geqslant 0$ and $b_i' = b_i - b_n \geqslant 0$).

Denote $A = \sum\limits_{i=1}^{n} a_i$, $\alpha_i = \dfrac{a_i}{A}$ and $\beta_i = \dfrac{1}{n}$. Then the sequences (α_i) i (β_i) are decreasing and, by Lemma 2, $(\alpha_i) \succ (\beta_i)$ holds true. Applying Lemma 1 to the sequences (α_i), (β_i) and taking $x_i = b_i$, we obtain

$$\sum_{i=1}^{n} \frac{a_i}{A} \cdot b_i \geqslant \sum_{i=1}^{n} \frac{1}{n} \cdot b_i,$$

i.e., $n \sum\limits_{i=1}^{n} a_i b_i \geqslant \left(\sum\limits_{i=1}^{n} a_i\right)\left(\sum\limits_{i=1}^{n} b_i\right)$. ■

REFERENCES

[1] Z. Kadelburg, D. Đukić, M. Lukić, I. Matić: *Inequalities of Karamata, Schur and Muirhead, and some applications*, The Teaching of Mathematics, Vol. VIII, 1 (2005), 31–45.

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