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DEVELOPING A PROOF-WRITING TOOL FOR NOVICE LYCEUM GEOMETRY STUDENTS

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Abstract. Geometry students, while moving from junior high-school to Lyceum, gradually cease to deal with practical topics and have to confront with other topics of theoretical nature. Worldwide research has shown that Lyceum students have great difficulty in writing formal proofs in geometry, particularly when traditional teaching methods are employed in the teaching of geometry. In this paper, we focus on the findings of a research project, which is part of a wider research that aims at investigating the ability of 15 years-old novice geometry students to write formal proofs. In particular, we use data from a random sample of schools in Athens from students attending the first year of Lyceum. Our research findings evidence that, while attempting to write formal geometry proofs, students who had employed a tool, called "Reasoning Control Matrix for the Proving Process" (RECOMPP) had significantly improved their ability in writing formal geometry proofs than those who had not employed this tool.

ZDM Subject Classification: G43; *AMS Subject Classification*: 00A35.

Key words and phrases: Proof; Geometry; Van Hiele model.

Introduction

Apart from being a core subject of secondary education, geometry has also numerous applications in everyday life, holding a central role in many other sciences, as well as in the arts (Mammana & Villiani, 1998). Proof is probably the most important tool used in geometry (Hanna, 2000; Martin & Harel, 1989; Leron, 1985; Recio & Godino, 2001; Senk, 1985; Usiskin, 1982). But after all, what is a proof? In daily life the term proof is linked to confirmation, through situations of validation and decision, to the products arising from the followed proving practices, and to argumentation being accepted by a community or a person. The diverse meanings of proof are identified by the terms explanation, argumentation, and demonstration. However, in all circumstances, a common idea exists; that of justifying or validating a proposition by providing justifications or arguments (Godino & Recio, 1997).

According to Martin & Harel (1989), "people think of proof in their everyday life as what persuades them", but Euclid in his Elements makes us believe that proof represents something more powerful for geometry; a rational, deductive reasoning based on inalterable axioms, definitions and theorems. Proofs are described as arguments that consist of logically strict deductions of hypotheses results (NCTM, 2000).

However, if we pose the question: "What is a proof?" to either a student or a teacher, it is almost certain that their answers will vary significantly, partly because their interpretation of the question will also be considerably different. In what follows, we focus on three factors that justify this phenomenon. Firstly, we elaborate on the fact that the term proof means many different things for students, therefore their interpretation of the meaning of proof may be different from that of teachers. At the same time, the interpretation of the meaning of proof may vary from teacher to teacher (Tall, 1989). Secondly, we concentrate on the fact that the term proof varies from context to context, for instance, when referring to daily life issues, empirical sciences, or professional mathematics (Recio & Godino, 2001). Thirdly, we focus on the fact that, according to the van Hiele model, those who are engaged in the learning-teaching process usually function at different levels of thinking (Senk, 1985, 1989). This phenomenon will be hereinafter referred to as "van Hiele gap". As a result, although teaching students to write proofs has been an important goal of the geometry curricula for the secondary schools on an international level, contemporary students, in fact, rank doing proofs in geometry among the least important, most disliked and most difficult of school mathematics topics. In particular, students have difficulty not only when they reproduce the proofs of theorems taken from their textbooks, but also, when they prove simple propositions of Euclidean Geometry (Weber, 2003). Data collected by the research project "Understanding the Proof", held in UK by Healy and Hoyles (1998), and also by the research project "Cognitive Development and Achievement in Secondary School Geometry" (CDASSG), held in USA by Usiskin (1982) and Senk (1985), confirm that writing proofs is difficult for most students.

However, research conducted by Senk (1985) bare evidence that students, even after having participated in an introductory teaching of proof, could neither realize the necessity of deductive proofs, nor distinguish the various kinds of mathematical reasoning. Furthermore, many undergraduate students don't know what a proof constitutes (Recio & Godino, 2001), and they are even unable to confirm the validity of a proof (Selden & Selden, 2003).

Furthermore, when a mathematical proposition, subject to proof, has a visually evident character, or it is given ready-made without requiring from students to discover it themselves, or when students are already aware of its validity, then they feel there is no need to prove it (Hadas, Hershkowitz, & Schwarz, 2000). Last but not least, according to van Hiele theory, students begin to realize the importance of deductive proof methods only after they have attained the fourth level of geometric thinking (van Hiele, 1986).

Difficulties in understanding geometry and Van Hiele's levels of geometric thinking

Based on the work of Piaget (1928), research related to the understanding of geometry has stressed on the methodical nature of deductive thinking and on the difficulties students have when they try to think in that way in the context of geometry (Freudenthal, 1973; van Hiele, 1959).

Moreover, research related to the understanding of geometric concepts by students has evidenced that students have difficulties in the definition and recognition of geometric shapes and in the use of deductive thinking in geometry (Pyshkalo

1968; Burger, 1982; APU, 1982; Hart, 1981). For instance, research findings from the UK revealed that 85% of 11-years old students recognized successfully a regular hexagon. However, when the same 11-years old students were shown non-regular pentagons or non-regular hexagons, then their success rate decreased dramatically, ranging from 25% to 43% depending on the complexity of the question asked (APU, 1982).

Naturally, the problem of difficulties in the understanding of geometry is not an unprecedented phenomenon. On the contrary, it has already been mentioned from the ancient times. This is at least evident by the answer of Euclid to the king of Egypt, Ptolemeus I, when the king asked Euclid to explain to him in an easier way his Elements. According to Proclus, Euclid gave Ptolemeus the following answer: "There is no king road for geometry" (Heath, 1921). However, we consider the didactic method one follows when teaching geometry as a crucial factor towards the progress and achievement of a student in geometry. Clements and Battista (1992) state that one of the main reasons responsible for the difficulties students have in geometry may well be the methods employed by the teachers when teaching geometry topics. Besides, Cemen (1987) underlines that difficulties in mathematics—and in geometry as part of mathematics—appear not only due to the accumulative and chain nature of knowledge, but also due to the way these topics are taught.

In 1969, the Dutch researchers Pierre Marie van Hiele and his wife, Dina van Hiele-Geldof noticed the difficulties students had in geometry and postulated a model of learning geometry that has attracted considerable interest among researchers (Hoffer, 1983; Burgher & Shaughnessy, 1986). As a descriptor, the van Hiele model posits the existence of five discrete levels of thinking, called "van Hiele levels", along which students progress while learning Euclidean geometry. Hoffer (1981) named these levels: recognition, analysis, ordering, deduction, and rigor.

According to this theory, students move sequentially along these levels of thinking without skipping a certain level, i.e. if a student is at the third level, he/she must first have attained the first two levels. However, not all students pass through the five levels in the same way. Furthermore, according to Senk (1985), the van Hiele model states that two persons reasoning at different levels may not understand each other. In particular, a student who has attained only level n will not understand the thinking of level $n+1$ or higher, which means that a "van Hiele gap" exists. Moreover, in her research that refers to the relationship between van Hiele levels and the achievement of students in writing proofs, Senk (1989) notes that the predictive validity of the van Hiele model is supported, while the hypothesis, that only those students who think in levels 4 can write formal geometry proofs, is not supported. However, Senk (1989) notices that,

> " ... a student, who starts a high school geometry course unable to recognize common plane geometric figures, has little chance of learning to write geometry proofs later in the year. Another one student, who starts the year able to recognize common geometric figures, but unable to describe properties of those figures, is likely to be able to do some simple standard geometry proofs by the end of the school year; but such a student

has less than one chance in three of mastering proof writing. In contrast, a student who is able to recognize geometric figures by sight and to describe their properties has at least a 50-50 chance of mastering proof writing by the end of the year. The same study demonstrated that, although there is no individual van Hiele level that ensures future success in proof writing, Level 2 appears to be the critical level. By measuring the students' level of geometric thought, we ascertained, in our research, that the majority of the students, who comprised our sample, were performing at Level 2 according to the van Hiele levels."

Approach to proof for novice first-year Lyceum geometry students

The inefficiency to teach the notion of "proof" is almost global (Hadas et al., 2000). As a subject, Euclidean geometry is taught under a theoretical framework, during the first two years of Lyceum in Greece. High-school students usually count and calculate, based on specific situations, whilst seldom make use of abstract procedures. Thus, first-year Lyceum students, who move from specific procedures to more abstract ones, are not familiar with the role of axioms, definitions, and theorems. Instead, they have to cope with the concept of proof in a purely theoretical context. Mason's (1997) research findings corroborate the same assumption. In Masons research, participants were talented students of 6th and 12th grade. Furthermore, the researcher mentioned that those talented students did not know how to write an acceptable formal proof. Therefore, we believe that students first contact with the concept of "proof" should be realized in an appropriate way. In other words, we should give students some help, guiding them to complete the proving procedure. After having taken into account that our students had no experience in writing proofs, we decided to study only direct proofs in our research, because it was very possible for them to get confused. In what follows, when using the term "proof", we refer only to a "direct proof". It is because students make now their first steps towards writing a geometric proof that we consider them "novices". Also, they are not yet familiarized with the proving procedure, and they cannot successfully write a proof.

A proof consists of simple justifications and partial proofs. By integrating these justifications and partial proofs into a set, we take a completed proof. For the sake of clarity let consider the following example comprising of two propositions:

Proposition (P-1). *If the line segments AB and AC are equal then the triangle ABC is an isosceles triangle* (see Fig. 1).

Proposition (P-2). *The exterior base angles of an isosceles triangle ABC are equal* (see Fig. 1).

Since $AB = AC$, by saying, for P-1, that the triangle ABC is isosceles then we have fully reasoned why the triangle ABC is isosceles. On the contrary, by saying that the exterior base angles of an isosceles triangle ABC are equal, because they are supplementary to the angles B and C, respectively, we have not fully reasoned

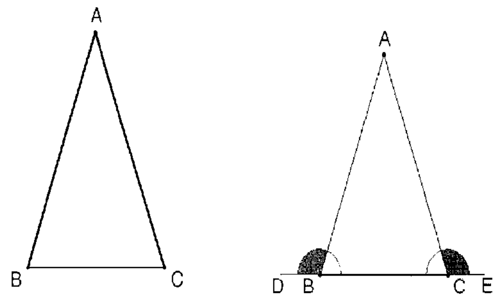


Fig. 1

our claim. We must also prove why the angles B and C are equal. To discriminate the proof of equality of the angles B and C from the complete proof of proposition P-2, we shall call "partial proof" the proof of equality of the angles B and C. As a result, a proof consists in justifications and partial proofs. In the following tables, we summarize the justifications and the partial proofs of P-1 and P-2 during the proving procedure:

<table>
<tr><th colspan="3">Proving procedure of proposition P-1</th></tr>
<tr><td colspan="3">Proof of P-1</td></tr>
<tr><td rowspan="2">(P-1):
The triangle ABC is isosceles</td><td colspan="2">Simple reasoning (Full reasoning)</td></tr>
<tr><td>because</td><td>AB=AC</td></tr>
</table>

<table>
<tr><th colspan="5">Proving procedure of proposition P-2</th></tr>
<tr><td rowspan="4">(P-2):
The exterior base angles of an isosceles triangle ABC are equal</td><td colspan="4">Proof of P-2</td></tr>
<tr><td>Not full reasoning</td><td colspan="3"></td></tr>
<tr><td rowspan="2">because</td><td rowspan="2">because they are supplementary to the angles B and C, that are equal</td><td colspan="2">Partial proof</td></tr>
<tr><td>because</td><td>the base angles of the isosceles triangle ABC are equal</td></tr>
</table>

Let, ABC be an isosceles triangle, where $AB = AC$, and M, N are the midpoints of the sides AB, AC, respectively. We extend the base BC of the triangle from both sides by equal segments BD, CE and we ask students to prove that $DM = EN$ (see Fig. 2).

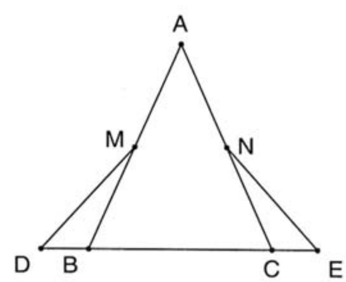


Fig. 2

During the process of proving this proposition, we must prove that the triangles MBD and NCE are equal. Therefore, we need to prove that the angles $\angle MBD$ and $\angle NCE$ are equal, i.e. $\angle MBD = \angle NCE$.

In fact, this represents a partial proof. According to Criterion of Equal Triangles, we must also prove that the line segments BD and CE are equal, i.e.

$BD = CE$, which is part of the hypotheses of the proposition and therefore constitutes a simple reasoning. Furthermore, partial proofs require simple justifications and other partial proofs and so on. In general, proofs are analyzed into simple justifications and partial proofs. In their turn, partial proofs are also analyzed into simple justifications. Thereby, if we want the proving procedure to continue smoothly, in a step-by-step mode, the above mentioned justifications demand from the students to produce some reasoning.

Novice students do not know how to begin to write a proof. What they usually do is only drawing a sketch of the problem. They find it quite difficult to discriminate the hypotheses of the problem from the conclusion. Even if they succeed in discriminating the hypotheses from the conclusion of the problem, they cannot suspect to collect them all somewhere so as to have them constantly available for use, when needed. In many problems students are asked to draw a line or a circle etc. so as to facilitate the proving procedure. Students have to invent this line or circle by themselves, because it does not exist in the sketch. Admittedly, such problems are too difficult for novice students to solve.

In our research, we tried to make students achieve something like this. Thus, we coined a matrix, dubbed "Reasoning Control Matrix for the Proving Process" (RECOMPP). The advantage RECOMPP offers, when employed by novice lyceum geometry students in their proving of a proposition is that it can help them produce and control their reasoning in a more effective way so as to successfully write the given proof. We should mention here that although the layout of RECOMPP is predefined, and must be filled in a specific way, each time a different content is required to be written, that is, each time students have to produce different reasoning to fill RECOMPP. Practically, this requires a substantial amount of reasoning by students' side. Therefore, we see that RECOMPP is not a compass for students. Instead, it serves as a reasoning production tool.

During our instruction of the proof, we employed the cognitive apprenticeship instructional model, in which we incorporated the van Hiele phases of instruction, since Van Hiele proposes that: " … the ideas that have been used here have a place in every method of teaching" (Van Hiele, 1986; p. 177). We intentionally used that method because the class participants were novice solvers, and cognitive apprenticeship is an instructional design model whose goal is to make students' thinking processes visible to both the students and the teacher (Collins, Brown, & Holum, 1991). Moreover, by observing an expert's thinking processes and the way that person practices his/her skills, students can learn how to become themselves experts in proofs (Collins, Brown, & Newman, 1989). Besides the established teaching methods the model provides i.e. modelling, scaffolding, articulation, reflection etc., while employing the cognitive apprenticeship instructional design model, we also had the opportunity to encourage our students and to help them develop their self-efficacy in order to successfully write proofs. This is the reason why we intentionally informed our students, that we also had difficulties in writing some proofs. In this way, we convinced our students that also experts are susceptible to stumble, to fail or to postpone a problems solution for a later time. In other words, we helped novice students realize that neither themselves are the only students who find proofs difficult, nor their difficulty in proofs is a sign of inefficiency.

Reasoning control matrix for the proving process (RECOMPP)

RECOMPP is a reusable matrix pattern that helps students produce reasoning production. Its layout and its filling technique are predefined. More analytically, it consists of six discrete sections and its layout consists of rows, columns, and cells that may contain figures, hypotheses or conclusions, proofs, and partial proofs. Furthermore, a student, when filling RECOMPP, follows two basic rules: that of horizontal transit, and that of transfer. We will describe these rules in more details later in this article. RECOMPP can be used in every chapter of geometry content because it is a reusable pattern of reasoning production.

The layout of RECOMPP

As shown in Figure 3, RECOMPP is a modular, reusable matrix consisting of six discrete sections. In what follows, there is a detailed description of each one of the sections:

- **Section 1** is where the formulation of the problem is given. Here, in a textbox, the student can read the full description of the problem, before he/she moves on to the proving procedure.
- **Section 2** is where the hypotheses, and the conclusions of the problem must be written. Here, the student is given a table (consisted of two rows and two columns), where he/she must write down, in two separate lines, the hypotheses, and the conclusions of the problem, respectively. Of course, students must first have read the description of the problem, that can be found in Section 1, very carefully, before he/she is able to find, discriminate, and record the hypotheses.
- **Section 3** is where the sketch of the problem must be prepared by the student. Here, based on the description of the problem he/she read in Section 1, and according to the hypotheses, and the conclusions that he/she wrote down in Section 2, the student progresses to draw the sketch of the problem in a blank field. Students will use this as a visual aid to formulate the proof.
- **Section 4** is where the teacher may offer scaffolding to student. Here, in order to offer students contextual, and just-in-time help, the teacher can provide a list of hints to them.
- **Section 5** is where the student is motivated to reason, collect, and write those statements and relationships among the elements of the sketch prepared before that will lead him/her to the successful writing of the proof. Here, the student is given a table (consisting of just two columns and several rows). The student must write a statement e.g. “Statement A”, that needs to be proved, in this table, in the first column, labelled “To prove that … ”. The student must write a statement e.g. “Statement B”, that is necessary in order to prove “Statement A”, in the second column, labelled “It is required to prove that”.

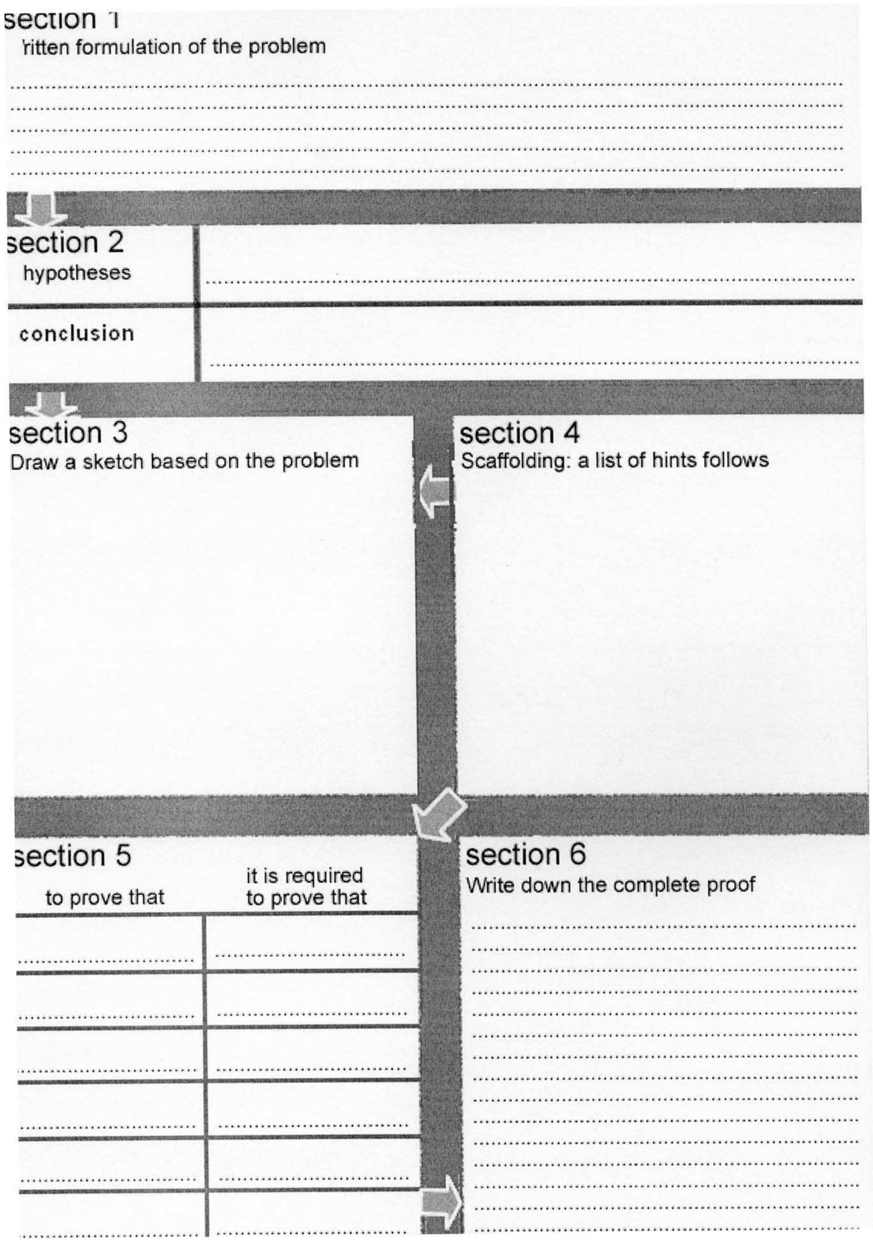
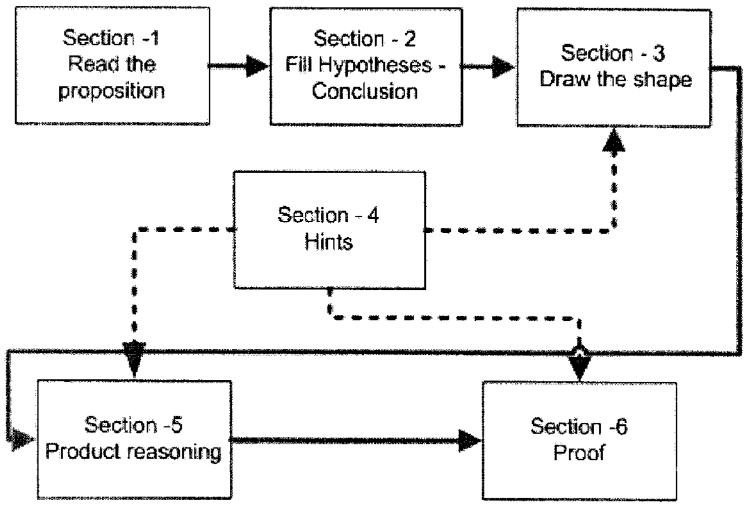


Fig. 3

Student must set some goals, in order to move horizontally and proceed from the left column to the right one. Thus, the left column represents the earliest stage of a student's reasoning and the right column represents the latest stage of a students reasoning.

- **Section 6** is where the proof must be written by the student. Here, in a textfield, the student must write the proof.



Flow chart that represents the flow of students' actions

Significance of RECOMPP section 5

Section 5 of RECOMPP must be filled according to the following three rules (see Fig. 4):

(a) The rule of horizontal move from left to right, i.e. the student first fills the left column, labelled "to prove that", and then continues to fill the right one, labelled "it is required to prove that". The significant contribution of this rule of horizontal move from left to right is that it demands from the student to produce reasoning. This process is repeated in every row of the RECOMPP.

(b) The rule of how to fill the first cell of the left column, labelled "to prove that" According to this rule, the student must always fill the first cell of the left column with the conclusion from Section 2. This is especially important for the student, as it indicates to him/her where to start from in the proving procedure.

(c) The rule of reassignment of produced reasoning. According to this rule, the content of the right column in each row (produced reasoning), is reassigned to the left column of the below row.

As shown in Fig. 4, each move from the left to right represents a new *step in student's reasoning*, and each reassignment becomes a new creative causation for the student to think. The right column is not one-way defined, but it usually contains more than one propositions P_i, $i = 1, 2, \ldots$, among which the student must select the most appropriate to reassign to the below row. The difficulty students have when it comes to selecting among the most appropriate proposition, provides them with the opportunity to try and therefore become more experienced.

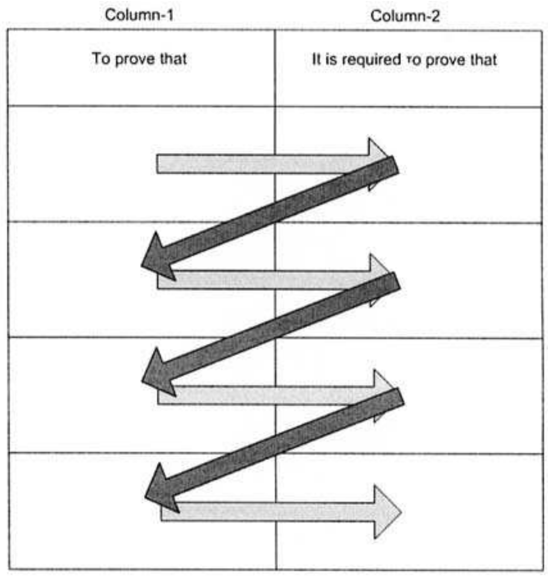


Fig. 4

We mention that the following are required in every step of the proving procedure:

- The conclusion should never be written in the right column.
- To check P_i that are in the right column, labelled “It is required to prove that”, e.g., if P_i exists in the given of the exercise, then the proving procedure ends.
- To connect P_i with hypotheses, in order to select the most appropriate P_i.
- To reassign the most appropriate selection from the right column of a row to the left column of the below row.

In this way, a sequence comprised of the steps of reasoning is being created. As shown in Fig. 5, the sequence of reasoning steps ends either to a simple reasoning, or to a partial proof.

Taking into account students' van Hiele gap, and also that the majority of students were assigned to van Hiele level 2, we would like our students to be able to clearly distinguish the hypotheses of the problem from the conclusion. Thereby, by asking each time students to fill Section 2 (see Fig. 3), we discussed several problems. In other words, they should write down thoroughly the hypotheses and the conclusions of the problem. Based on Hoffers work (1981), we stressed the importance of drawing correctly the geometric sketch.

Therefore, before students make use of RECOMPP, we recommended them to follow some guidelines. These guidelines are listed below:

1. *Draw the geometric sketch. Check, whether you drew the sketch correctly.*
2. *Try if you can tell the problem, with the aid only of the figure, i.e. without reading it from Section 1.*

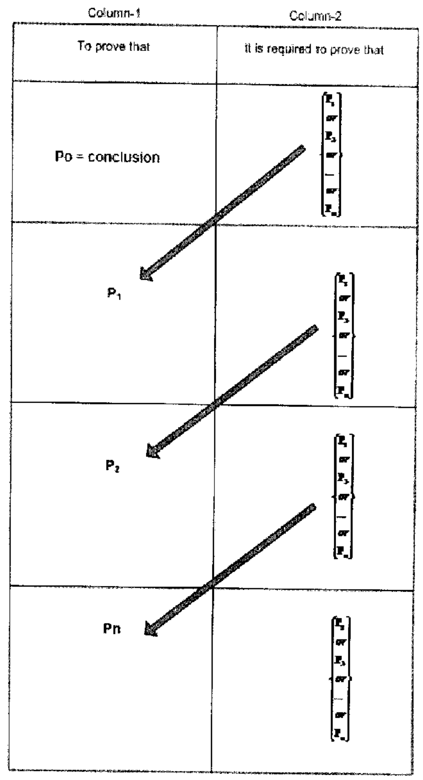


Fig. 5

3. *Translate the hypotheses and the conclusions of the problem in Section 2 into relationships.*
4. *Fill Section 2 with the hypotheses and the conclusion.*
5. *Recognize the role of each point and each segment in the geometric sketch.*

A research was conducted to test the following hypothesis: "If students are taught geometry, utilizing the RECOMPP in their learning process, then significantly better results can be achieved in their ability to write successfully proofs, if compared with the results of traditional instructional methods".

Method

Participants and procedure

We used a random sample of schools to participate in our research. Then, 2 schools (6th Lyceum of Peristeri and 8th Lyceum of Peristeri) were randomly selected from the initial sample, qualifying in the criterion of not having a statistically significant difference in their grades in the first semester in geometry score ($t_{87} = 1.44$, $p > .05$) between the experimental group ($t_{87} = 1.44$, $p > .05$)

(1 first-year class from each one of the 2 high schools—mean score of class in geometry $= 15.98$) and the control group (1 first-year class from each one of the 2 high schools—mean score of class in geometry $= 15.22$). The final sample consisted of 89 students. The control group consisted of 44 persons (1 class from each one of the 2 schools), of which 18 (40.9%) were males, and 26 (59.1%) were females. The experimental group (1 class from each one of the 2 schools), consisted of 45 persons, of which 16 (35.6%) were males, and 29 (64.4%) were females. The van Hiele level of geometric thinking of the classes participated in the research was also assessed, and we discovered that all of the students were assigned to level 2. Thereby, no van Hiele gap existed between the students.

Those students, who initially belonged to both groups, took a diagnostic pre-test, consisting of proof-writing evaluation exercises. After this, students, who belonged to the experimental group, participated in geometry classes in which we employed the RECOMPP for the following 3 months. Meanwhile, using this time a more traditional instructional method, we taught students, who belonged to the control group, the same geometry content. Next, we gave to students, who belonged to all of the classes, a post-test that consisted of proof-writing evaluation exercises. Each correct answer was awarded with 5 points, giving a total sum of 20 points for the whole test.

Instruments

Van Hiele test. Based on students' performance on the van Hiele geometry test, we assessed their van Hiele level of geometric thought. This 35-minutes duration test, consisted of 5 subtests, each containing 5 multiple-choice questions. These questions referred to each level separately. If a student answered correctly 4 out of 5 items in a subtest, then he/she was considered to have attained the specific level. According to the test results, all of the students were assigned to the second van Hiele level of geometric thought.

Proof-writing evaluation exercises. Two pairs of exercises, similar to those in a textbook used in classroom, were given successively during the pre-test and the post-test. (Totally, 4 exercises per test). The first exercise from each pair corresponded to a simple proof and the second one corresponded to a complex proof, that was either an extension or a slight modification of the first exercise. We asked students to prove one other result, whose proof was based on the result of the first exercise.

The pre-test comprised of the following exercises:

Exercise 1. In an isosceles triangle ABC, with $AB = AC$, the points M, N lie on the line segments AB, and AC, respectively, such that M is the mid-point of AB, and N is the mid-point of AC. We equally extend the base BC of the triangle by the line segments BD, CE, such that $BD = CE$. Prove that $DM = EN$ (see Fig. 6).

Exercise 2. In an isosceles triangle ABC, with $AB = AC$, the points M, N lie on the line segments AB, and AC, respectively, such that M is the mid-point of AB, and N is the mid-point of AC. We extend the base BC of the triangle

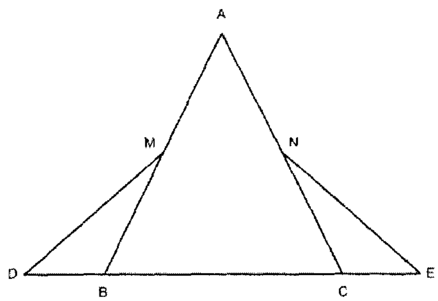


Fig. 6

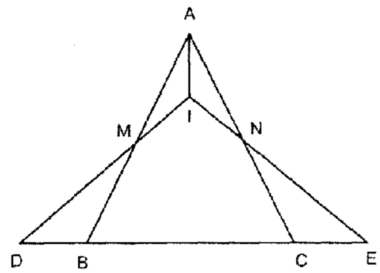


Fig. 7

through B and C, respectively, to points D and E, so that $BD = CE$. Segments DM, and EN intersect at point I. Prove that segment AI is the bisector of angle A (see Fig. 7).

EXERCISE 3. Prove that the common external tangents BC and $B'C'$, of two externally tangential circles K, and O, are equal (see Fig. 8).

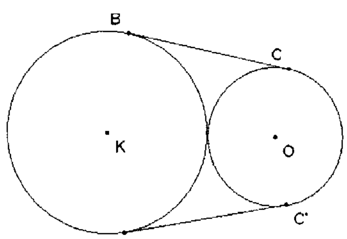


Fig. 8

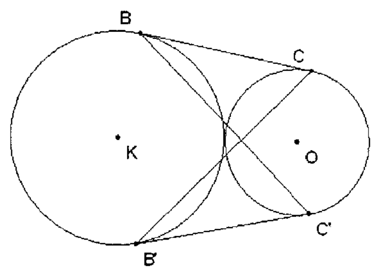


Fig. 9

EXERCISE 4. Given BC and $B'C'$ are common external tangents of two externally tangential circles K and O, prove that $B'C = BC'$ (see Fig. 9).

The post-test comprised of the following exercises:

EXERCISE 1. In parallelogram $ABCD$, the points M, N lie on the line segments AB and CD, respectively, such that M is the midpoint of AB, and N is the midpoint of CD. Prove that $ANCM$ is a parallelogram (see Fig. 10).

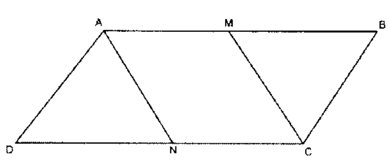


Fig. 10

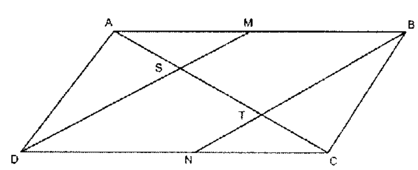


Fig. 11

Exercise 2. In parallelogram $ABCD$, the points M, N lie on the line segments AB and CD, respectively, such that M is the midpoint of AB, and N is the midpoint of CD. Prove that the diagonal AC is trisected by the segments MD and NB (see Fig. 11).

Exercise 3. In square $ABCD$, the point E lies on the segment AB. From point A we draw a perpendicular to DE, which intersects BC at point Z. Prove that $DE = AZ$ (see Fig. 12).

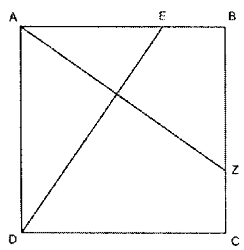


Fig. 12

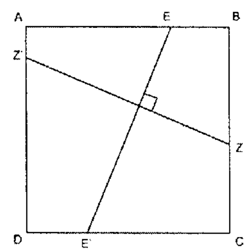


Fig. 13

Exercise 4. In square $ABCD$, two vertically intersected line segments intersect sides AB, BC, CD, and DA of the given square at points E, Z, E', and Z', respectively. Prove that $EE' = ZZ'$ (see Fig. 13).

Reasoning control matrix for the proving process: RECOMPP was employed for proving propositions assigned to students, which also required a theoretical documentation.

Results

A 2×2 mixed repeated measures ANOVA with one within-subjects factor—time (time1 = pre, time2 = post measures) and one between-subjects factor—group (control versus experimental group) was conducted to evaluate the effect of group and time, on students' performance. The ANOVA yielded a significant pre-post main effect ($F_{1,87} = 146.931$, $p < .001$; partial eta squared $= .628$, observed power $= 1.00$), and a significant group main effect ($F_{1,87} = 16.03$, $p < .001$; partial eta squared $= .156$, observed power $= .977$). The interaction between the pre-post measures and the group was found to be significant ($F_{1,87} = 92.70$, $p < .001$; partial eta squared $= .516$, observed power $= 1.00$).

Also, we conducted paired samples T-test to follow up the significant interaction. Table 1 displays the pre/post-measures scores, and change scores for the experimental and control groups.

As shown in Table 1, the research results revealed that children of the experimental group significantly improved their performance ($t_{43} = -14.395$, $p < .05$) in comparison to students of the control group who did not make progress in statistically significant terms ($t_{44} = -1.986$, $p > .05$). Also, an independent samples

	N	TIME1 $M (SD)$	TIME2 $M (SD)$	MEAN DIF. (SD)	df	T	P VALUE
Experimental group	44	10.34 (2.60)	13.64 (2.66)	-3.30 (1.52)	43	-14.395	.0005
Control group	45	9.51 (2.89)	9.89 (2.98)	-.38 (1.34)	44	-1.986	.064

Table 1 Pre (TIME 1)/Post (TIME 2)-Measures scores and change scores for experimental and control group

T-test conducted after the pre-test revealed the non significant statistical difference, in terms of performance, between the two groups ($t_{87} = 1.42$, $p > .05$).

Figure 14 illustrates the pattern of interaction between pre (time1)/ post (time2) measures, and groups (control and experimental).

Profile plot

Additionally, we conducted a similar ANOVA to evaluate the effect of sex (boys versus girls) and time (pre versus post measures) on students' performance. The ANOVA yielded a significant pre-post main effect ($F_{1,87} = 64.54$, $p < .001$; partial eta squared $= .426$ and observed power $= 1$), and a non significant sex main effect ($F_{1,87} = .05$, $p > .05$). The interaction between the pre-post measures and the sex was found to be non significant ($F_{1,87} = .171$, $p > .05$).

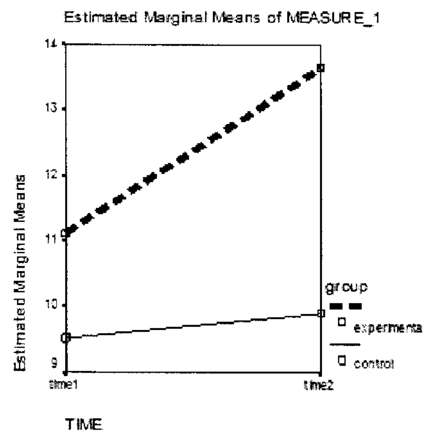


Fig. 14

Also, a two-way repeated measures ANOVA with one within-subjects factor [time (pre versus post measures)] and two between-subjects factors [group (control versus experimental group) and sex (boys versus girls)] was conducted to evaluate the effect of group, sex and time on students' performance. Apart from time, all the other effects of that ANOVA were found to be non significant.

Discussion

In order to move on to the next stage of the proving procedure, students' first contact with formal proofs should convince them, not only about the necessity to prove a geometrical proposition, but also about the necessity to produce reasoning. In every step of the proving process students should be able to understand the need for reasoning. Instructors should realize how important is to either offer some help to those students, who write a typical proof for first time or to encourage their students to write formal geometry proofs. RECOMPP is designed in such a way that stresses students' need to reason in every stage of the proving procedure, while it simultaneously puts in order their thinking. The role of hypotheses-results cell is very important, because it assists students to distinguish the hypotheses from the questions of an exercise. As a result, it assists them to define what they need to prove. At the very beginning of their solution many students declare their difficulty, i.e. they ignorance on how to begin the proving procedure.

An important characteristic of RECOMPP is that it serves as a guidance tool for the students to begin their proving procedure. At the same time, students who are not experienced in writing formal proofs, are given the opportunity to develop a reasoning production skill that is necessary during the process of writing a formal geometry proof. The application of RECOMPP into the cognitive skills requires the externalization of some processes that are often implicit. Besides, it encourages students to reflect on their action. In this way, it promotes the cultivation and development of generic control strategies and diagnostic skills. It also urges students to justify their reasoning provided that they already employ some forms of control strategies.

As a reusable pattern of reasoning production, RECOMPP enables its transfer to other learning situations. It is worthwhile to mention that during the official examinations conducted at the end of every year the students who had been taught to employ RECOMPP in their proving procedure, used it at their initiatives to solve the exercises of the examinations. Moreover, they answered the examinations' exercises using the framework of RECOMPP.

Finally, the research results give rise to some important questions for further research: "Does the improvement in proof-writing observed with the employment of RECOMPP lead to an overall improvement of students' performance in the subject of Euclidean Geometry?" Furthermore, given that Senk (1989) found a positive correlation between van Hiele levels and proof-writing, the following question rises: "If we slightly modify RECOMPP can it help to reduce van Hiele gap among students?"

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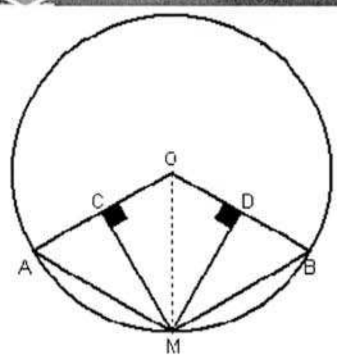
Appendix

On a circle (O,r) take the points A, M, B. so, as AM = MB. If MC is the distance from the point M to OA and MD is the distance from the point M to OB prove that MC and MD are equal.

hypotheses	$AM = MB$ $MC \perp OA$, $MD \perp OB$
conclusion	$MC = MD$



Hint: Draw the line segment OM

to prove that	it is required to prove that
$MC = MD$	OM is a bisector $\widehat{AOB}$
OM is a bisector $\widehat{AOB}$	$\widehat{AOM} = \widehat{MOB}$
$\widehat{AOM} = \widehat{MOB}$	$\widehat{AM} = \widehat{MB}$
$\widehat{AM} = \widehat{MB}$	AM=MB
AM=MB	

AM=MB is true by hypothesis

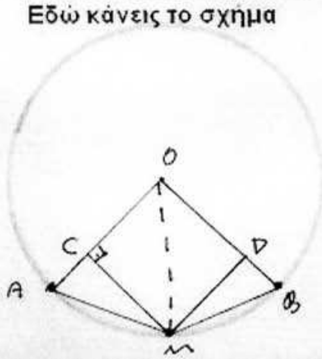
Τάξη ......Α......

Όνομα ......Μαρία Ζ. ......

Δίνονται κατά σειρά τα σημεία Α, Μ και Β πάνω σε ένα κύκλο(O, r) Αν οι χορδές ΑΜ και ΜΒ είναι ίσες να αποδείξετε ότι οι αποστάσεις MC, MD του σημείου Μ από τις ακτίνες ΟΑ και ΟΒ αντίστοιχα είναι ίσες.

Υ	$AM = MB$ $MC \perp OA$ $MD \perp OB$
Σ	$MC = MD$

Εδώ κάνεις το σχήμα



Υπομνήσεις - βοήθεια

Να φέρετε την ΟΜ

Για να δείξω ότι...	Αρκεί να δείξω ότι
$MC = MD$	ΟΜ διχοτομ. $\widehat{AOB}$
ΟΜ διχοτόμος $\widehat{AOB}$	$\overset{\frown}{AM} = \overset{\frown}{MB}$
$\overset{\frown}{AM} = \overset{\frown}{MB}$	$AM = MB$

ΑΠΟΔΕΙΞΗ

ΑΜ = ΜΒ από την υπόθεση της άσκησης.

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