THE TEACHING OF MATHEMATICS
2007, Vol. X, 2, pp. 75–86

DIDACTICAL ANALYSIS OF PRIMARY GEOMETRIC CONCEPTS III

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Abstract. This paper contains essays on angle, circle and rectangle aimed to illuminate these concepts at all levels of their gradual building. Angles, when conceived of as magnitudes expressing the degree up to which directions of two rays differ, are compared using an instrument called comparator—protractor without graduation on it.

Avoiding the formalism of differential calculus, we approach the concept of curvature of lines in an elementary way. Speaking of the average rate of the change of direction of tangent happens to be sufficient to see circles as the lines being equally curved everywhere and the straight lines as being nowhere curved.

Rectangular shape (an inherent geometry concept) and rectangle (a visual geometry concept) are treated with due attention to detail. In particular, the role of this shape has been related to everyday comparison in length and width.

When comparison is concerned, such an activity is clearly seen as the relationship between magnitudes of the same nature.

ZDM Subject Classification: G22; *AMS Subject Classification*: 00A35.

Key words and phrases: Angle, circle, rectangle, angles as magnitudes, curvature of lines, comparison in length and width.

9.4. Angle. The spontaneous meaning of the English words "angle" and "corner" is not equivalent in their entirety of usage but it is certainly in the area when they denote the meeting place of two converging lines with the space between them. This area includes those situations that are physically present and observable and when, the use of the word "corner" may be more preferable. Thus, this word is used in description of scenes in the outer space while "angle" is a scientific term used to denote a geometric configuration. Let us also note that the form of the English word "angle" is a derivative from its Latin etymon "*angulus*". In the period of intuitive geometry, the meaning of the concept of angle is instilled in childs mind by an active use of its ideograph.

Figuratively speaking, an angle is the opening formed by two rays having the same end point. Then the size of angle is taken to be the amount of such opening. As a matter of fact, this is the initial idea of angle present at this early stage of studying geometry. Later on, however, when trigonometry and analytical geometry are studied, this idea resumes a different and more general meaning. Namely, then, an angle is conceived of as the amount of rotation of a ray about its end point. Then, also, rotations going counterclockwise are counted as being positive and those going clockwise as being negative. These two ideas of angle are easily accorded, when in the former case, one side of angle is taken to be initial and the other one to be terminal. Such an angle is called directed and we can think of a directed angle as being the result of the rotation which moves the initial side until it falls on the terminal one. Therefore, there is nothing anxious to feel, if the meaning of a geometric concept evolves together with its use in different areas of mathematics.

It turns out not to be useful for the purposes of elementary geometry to regard rays and lines as angles. This means that "zero angles" and "straight angles" should not be included as special cases of angle at this initial level. It is even less reasonable to include in the consideration (and generally, in the whole content of Euclidean geometry) the case of reflex angles (i.e., those angles exceeding 180 degrees). For a justification of these arguments, the Moise-Downs' book [21] can be seen.

Now we focus our attention on the first steps in didactical elaboration of this theme. Starting with the idea of angle as a pair of rays with coinciding end points, first the relationships "a point is inside an angle" and "a point is outside an angle" should be a matter of exercises that are assigned to children. Let us also remark that the speaking of the interior and the exterior of angles is a premature act remaining as a relic from the period of "New Math". Not only that these two sets are infinite but their elements are abstract geometric points, what altogether means that such a conception is out of reach of children of this age. After all, if some reasons exist to speak of this sets, it is much better to call them interior and exterior regions.

At this level, the meaning that has a full sense is the conception of angle as a "broken" straight line and there does not exist any visible reason to attach the interior region as an additional part of this geometric concept (though, the practice of doing it is often encountered). Conceived of as a line, an angle separates the plane into two regions and for a pair "point-angle", the following three relationships are possible: a point belongs to the angle, a point is inside the angle and a point is outside the angle. Pictographs (instead of verbal definitions) are used to communicate these meanings. Accordingly, the teacher should use a picture as the following one is:

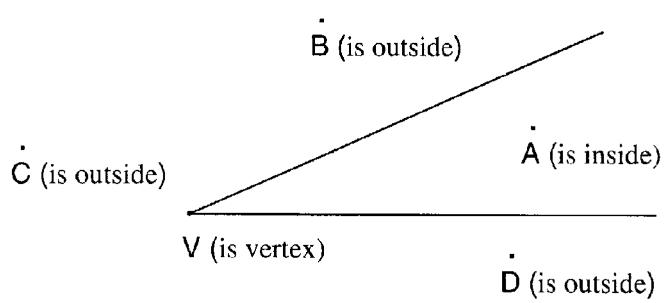


Fig. 27

and he/she should say that such a pair of rays having the same end point is called an *angle*. From the same picture children see which points will be considered to be outside the angle and which ones to be inside it. The vertex as being the common end point of the two rays is particularly a noticeable point.

Property of separation of the plane by a line is topological in character and children easily accept the inside of an angle as the part of the plane that is situated between its sides. Even when they are trained to extend traced lines representing two rays, many of them still have a slight hesitation to decide if a point, being inside the angle but not between the two traced sides, is in fact inside or outside that angle. Because of it, children should be assigned to do a number of exercises as the two following ones are.

(p) Find which of the given points are inside of the given angle and which outside of it.

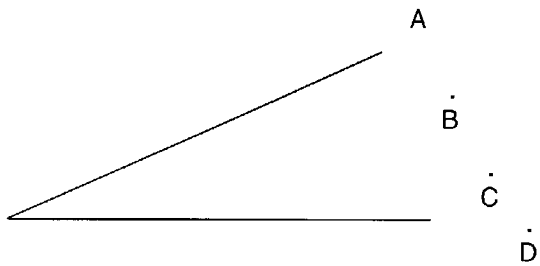


Fig. 28

(Answer: B, C are inside; A, D outside).

(q) Look at the given picture and say which pairs of points can be joined by a line not intersecting sides of the given angle:

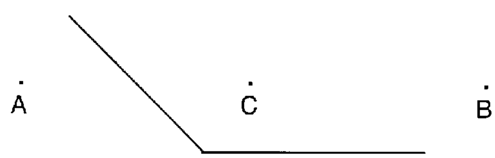


Fig. 29

(Answer: A, C and A, B cannot, B, C can). In order to help children, a suggestion to extend the traced sides of the angle, could be given. In addition, the examples of angles to be given should also include the cases of right and obtuse angles (without any attempt to classify them at this stage of elaboration).

In the frame of intuitive geometry, angles are geometric objects which are magnitudes expressing the degree up to which directions of two rays differ. Therefore, activities of comparison should come before those of measuring. Now we focus our attention to that kind of activities. To compare angles, children should have an instrument being a semicircular piece of clear plastic or cardboard (and the teacher may help them to make some by themselves). To be functional, the instrument has to have a dot mark at the center and a bold radial line drawn along its straight edge. (See Fig. 30).

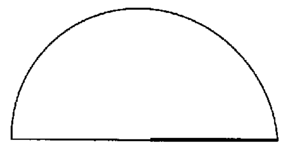


Fig. 30

Being without degrees on it, such an instrument is better for practicing this activity of comparison than a usual protractor would be.

Now we describe the way how this simple instrument—comparator is laid down over an angle and is adjusted to it. When the dot coincides with the vertex of the angle, radial line falls along one side of the angle and the part of the other traced side is covered by the comparator, then such an adjustment is correct. For example, in the case of the angle given in Fig. 31:

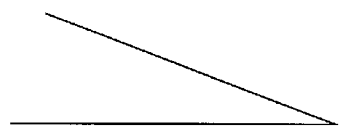
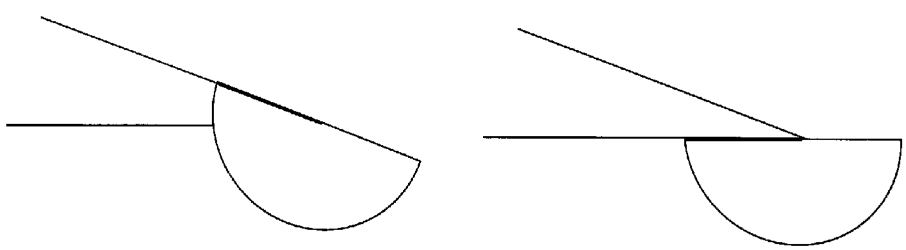


Fig. 31

the right and wrong adjustments are shown in Fig. 32:



(Rightly adjusted) (Wrongly adjusted)

Fig. 32

Let us notice that by right adjusting of comparator, one of two sides of an angle is selected as being initial one. Thus, the activity of comparison develops implicitly the delicate sense for directed angles that will certainly be useful in future learning.

When the vertex of an angle is the point A and the points B and C belong to its different sides, then the angle is denoted by $\angle BAC$ or $\angle CAB$. The name of the vertex always appears between the names of the other two points. But when it is plain what the sides are supposed to be, the angle may be denoted by writing simply $\angle A$.

When two angles are in special position, as it is shown in Fig. 33:

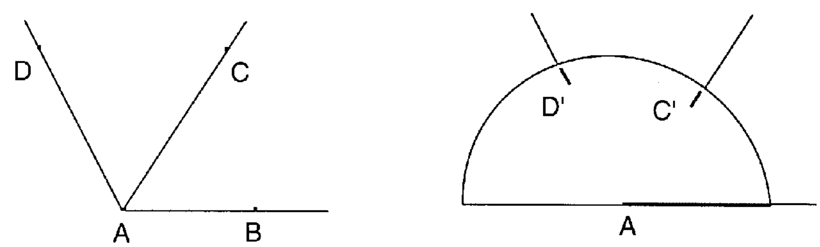


Fig. 33

then $\angle BAD$ is larger than $\angle BAC$ or reading it symmetrically, $\angle BAC$ is smaller than $\angle BAD$. (Two marks D' and C' drawn on the comparator also register this relationship).

After the right adjustment, an angle is carried over to comparator by drawing a mark being a short extension of the partially covered side of the angle towards its vertex. (See Fig. 34).

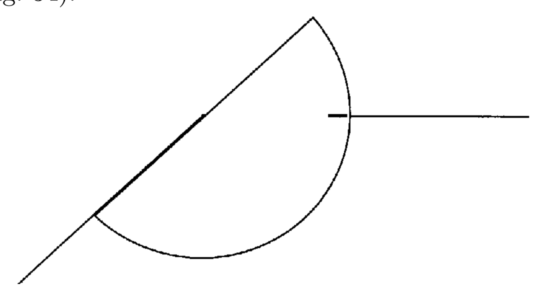


Fig. 34

To compare two given angles (Fig. 35):

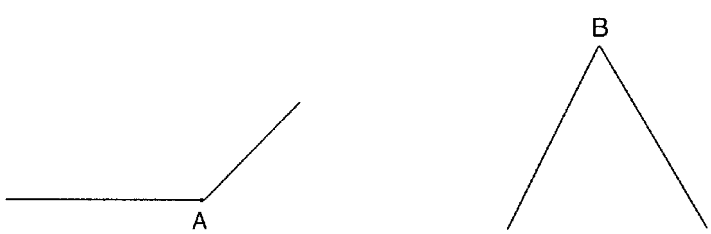


Fig. 35

they both are carried over to comparator and the corresponding marks are denoted by the letters that distinguish them (Fig. 36).

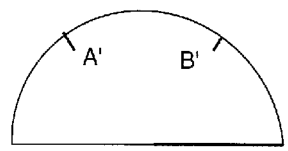


Fig. 36

When the marks stand as it is shown in Fig. 36, then the angle $\angle A$ is said to be *larger than* the angle $\angle B$ or, reading symmetrically, the angle $\angle B$ is *smaller than* the angle $\angle A$. When two such marks coincide, the angles are said to be *congruent* or *equal* and such two angles differ only by their position in the plane. (Let us note that it is quite convenient to denote a point and its displacement by the same letter, attaching a prime in the latter case).

For a more extensive elaboration of this theme, comparison of angles may be more refined and then, it supposes the activity of carrying over one of them to the position of the other one, as the following figure illustrates it:

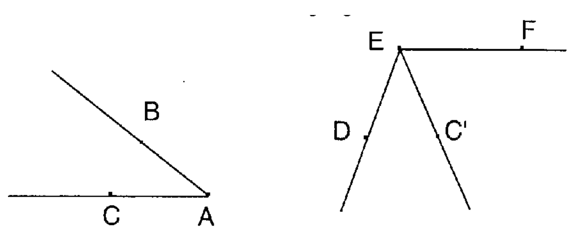


Fig. 37

In this case, it is said that $\angle DEF$, ($\angle BAC$) is *larger* (*smaller*) than $\angle BAC$, ($\angle DEF$) *for the angle* $\angle C'EF$.

At this level of learning, a good comprehension of the meaning of the concept of angle is particularly essential for the teachers, who often confuse this meaning with everything else they have learnt about it. This confusion could also be caused by a too early use of protractor, when an angle and its measure in degrees are not clearly seen as being ideas of quite different kinds. Moreover, on the basis of empirical research and without any subtleties of didactical analysis, the conclusion that the concept of angle is difficult for children to acquire is often drawn. But, as matter of course, what is difficult for those who transmit knowledge will be difficult for children as well.

9.5. Circle. The idea of circle also comes as a result of spontaneous abstracting as an irresistible urge of human beings to select some essential simple components from complex percepts, upon which the sheer forms are created in the mind.

Wherever we stand and look around, we form an illusional model of the surrounding world as being bounded by a huge circle upon which the heavens lean. The full moon is seen at night as a bright circular disk in the sky, crowns of many sorts of flowers are of circular shape, not to mention a large variety of industrial objects having that shape. Every day many circular objects come into our view and an inner representation develops being the basis for the spontaneous idea of a circle.

The word "circle" is a derivation from the Latin word *circulus* being the diminutive of *circus*—meaning a ring. For children a circle is, first of all, a recognizable shape either being a circular disc or the line that encloses it. Now we focus our attention on those characteristic properties of the circle which determine it as a unique geometric object. Mathematically, a circle is a closed line consisting of points equally distant from a point inside it, called the center of circle. Drawing a circle by means of compass is the best way of a concrete demonstration of this property.

Among all open lines (topological arcs) with the same pair of end points, the straight line segment has the extremal property of having the shortest length. Among all closed lines (topological circles) having the same length, the circle has the extremal property of enclosing the region of largest area. This characterizing property of the circle is known as Steiner's theorem (after the Swiss mathematician Jacob Steiner, (1796–1863)) and it can be proved in a quite elementary way.

When we stretch tight a piece of wire, it becomes straight. Bending that piece it becomes curved and bending more it becomes more curved. When we say that a road is curved, then it can be a way of expressing our impression of the shape of that road when it is in our view. But more often we express in that way the efforts we had driving a car along such a road. When at wheel, we put more efforts to control the direction driving along a more curved road. In all such cases we express our impressions, not having an objective criterion what the curvature of such objects should be. Now we intend to base such an impression on a more rational ground.

Representing a road by a curved line and a moving car on it, by a point, then at each moment the tendency of the car is to continue to move along the tangent to the curve at the point which marks the position of that car (Fig. 38).

Fig. 38

Now it is reasonable to take as the curvature of that line the amount of variation of the direction of tangent relative to the length of the line. More precisely, given two lines of the same length, as they are presented in Fig. 39, the sizes of the angles α and α' are the measures of their curvatures.

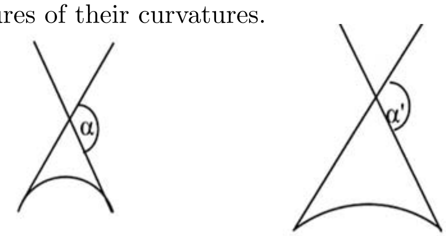


Fig. 39

If a line is such that for each its two arcs of equal length, the corresponding angles as the measures of their curvature are also equal, then that line is said *to be of constant curvature*. For instance, taking a circle and its two arcs AB and $A'B'$ equal in length, the corresponding angles α and α' are also equal (Fig. 40).

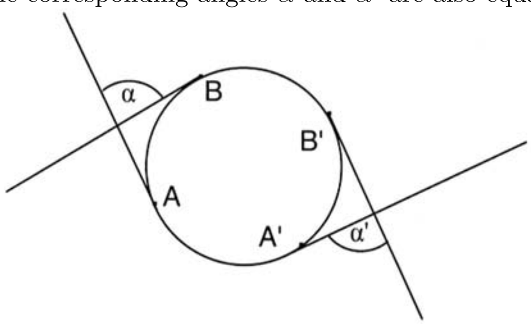


Fig. 40

As we see it, the circles are lines of constant curvature. This is also a characterizing property of circles in the class of all closed curves. Namely, it can be proved that a closed line of constant curvature is necessarily a circle.

Now, let us consider two circles having different radii, as they are shown in Fig. 41:

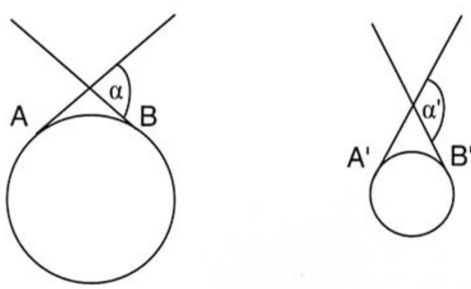


Fig. 41

For two arcs AB and $A'B'$ equal in length, the angle α' attached to the smaller circle is larger than the angle α attached to the bigger circle. Hence, smaller the radius of a circle larger is its curvature.

Although the property of a line to be curved is projective in character, the measure of curvature is a metric property as the example of two circles with different radii evidently demonstrate it.

Let us notice that the tangent of a straight line coincides with that line at each of its points. Thus, for each straight arc, the corresponding angle, as the measure of variation of direction of tangent, is zero. Hence, the straight lines are also of constant curvature which is zero along any of their arcs.

Summarizing, we see that circles and straight lines possess specific properties which make them exceptional geometric objects. Their ideas, at the level of inherent geometry, are related to the objects playing a prominent role in a rational observation of surrounding physical world and in everyday practice.

Let us also remark that the curvature of a line is a number attached to each of its points. Exact definition of this number requires a good knowledge of Calculus, whereby it stands beyond the usual mathematical preparation of an elementary school teacher. Our elementary approach of using the average rate of change of the direction of tangent suffices to see circles as the lines being equally curved everywhere and the straight lines as the sort of lines being nowhere curved.

9.6. Rectangle. Surfaces of still water, faces of some regular rock formations are some of natural phenomena that look flat. Floors and walls of our homes, walls of buildings in our streets and many other objects in surroundings of our dwelling places also convey the sense of flatness. On the other hand, sheets of various materials, fabrics, plates and many other flat industrial products are perceived as models of two dimensional geometric shapes. Their third dimension is ignored and, when taken into account, then the thickness of such objects is considered as their quality rather than their extension in the space. By the way, the amounts of such materials are always expressed in square measures. Visual inputs that come from such objects form an intuitive ground upon which the idea of flat surfaces generates in the mind. Among them, rectangular shape is particularly important because the edge lines of the objects of this shape are either parallel or perpendicular.

Kindergartners further their mental development playing with the pieces of plastic, wood or another material which are shaped as triangles, circles, squares, rectangles, etc. When these geometric shapes are the only such things in the play, it often happens that to the question “if an apple has the shape”, children answer “no”. Thus, it is also reasonable to speak about the shape of some physical objects (apple, cup, boot, finger, etc.) in preschool instructions, though these shapes are of no interest for geometry. Nevertheless, a number of exercises, as the one that follows, could be included in the books for first graders.

(r) Color the attached labels according to the shape you recognize:

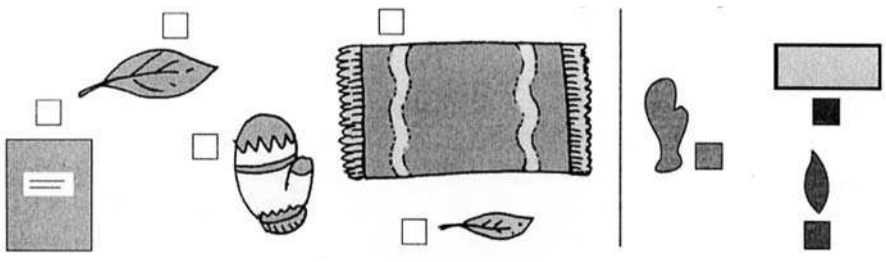


Fig. 42

(Via dialogue, the teacher induces his/her pupils to use terms "shape of rectangle (rectangular shape)", "shape of glove", "shape of leaf", etc. Let us notice that this is the level of inherent geometry and the picture in the upper–right corner of Fig. 42 does not represent a rectangle but rather one of its colored models).

Recognizing by shape and classifying material objects (models), children form the idea of rectangle at the level of inherent geometry. Later, their activities of drawing and iconic representing develop further this idea. A difference has to be made between a rectangle as a flat object and its rectangular boundary as a line. In the former case referent models are rectangular plates and in the latter rectangular frames. For the same reason, pictographs representing such plates should be shaded or colored. These tiny details are stressed because they should be observed and applied in practice.

The role of rectangular shape is particularly significant for the activities of comparison in length and width. For that reason, this shape is a fundamental visual concept which gives sense to these activities. Another detail that deserves our attention is the fact that length and width are relative concepts depending on specific positions of observer and the observed object. Such positions are illustrated in Fig. 43:

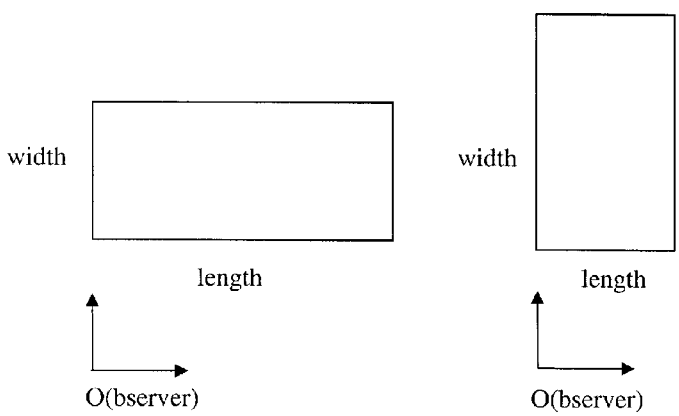


Fig. 43

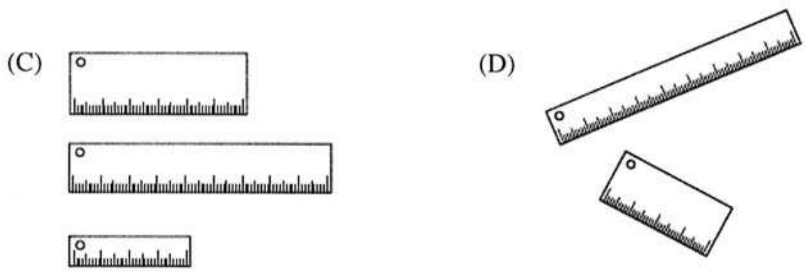
where the extension of a rectangular object in the direction of the observer's right hand is its length and that in the direction of his/her left hand is its width. (Two arrows represent two hands and the way how they are held out. The same picture illustrates the fact that the rotation of a rectangle for the right angle interchanges the meaning of length and width. Let us also note that in the case when one of these extensions is dominant, it is usually considered to be length.

Now a typical exercise of comparison to be assigned to first graders is the following one:

(s)

The rulers are equally ... (wide) and the second is ... (shorter)

The rulers are equally ... (long) and the first is ... (narrower)



Which ruler is longest, widest, shortest, narrowest?

The first ruler is ... (longer) and the second ... (wider)

Fig. 44

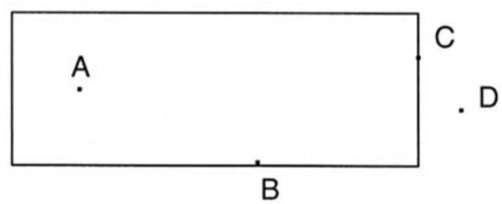
Using all these comparison terms through a number of similar exercises, children assimilate easily their precise meaning. There is no reason to force children to use terms "length" and "width". Let us also remark that, at this level, these terms do not denote measures (expressed in numbers) but magnitudes that are amounts of extension of edges of rectangular objects.

From the activities of recognizing and naming, performed at the level of inherent geometry, the concept of *rectangle* arises in visual geometry as a figure of rectangular shape. For the sake of purity, we speak of rectangular shapes in inherent geometry, while in visual geometry, rectangles start to be already treated as abstract geometric objects. To help this process of abstraction develop, in the latter case, rectangles are represented by purely geometric drawings without any shading or coloring.

It is worth noticing that in intuitive geometry (and throughout the whole period of school geometry), the general topological concepts of line, surface and solid are spontaneously used. Their meaning develops with the enlargement of geometric content but they stay to be general, undefined terms that contribute to the coherence of this content. And as d'Alembert (Jean Le Ronde d'Alembert, 1717–1783, French mathematician and encyclopedist) had pointed it out far ago, general concepts are often simpler and easier to be acquired. This is particularly true for these three geometric concepts.

Since the same iconic sign is used to denote a rectangle and its boundary, a contextual cue is needed to appreciate this difference, as the following exercise (aimed to third graders) demonstrates it.

(t) Which points belong to: (i) the rectangle, (ii) its boundary?



(i) ______________ (A, B, C) (ii) ________________ (B, C)

Fig. 45

Let us also notice that boundary is again a topological concept that is easily comprehended by children. (The term "circumference" is used to denote both, boundary of a figure and its length. Usage of terms "rectangle" (for the figure) and "rectangular line" (for its boundary) might be convenient to avoid a possible confusion).

At the end, let us remark that the word "rectangle" has been derived from Latin "*rectangulum*" (neut. of *rectangulus*, meaning right-angled). Defined as a quadrilateral (a projective concept) having all its angles right, rectangle is a Euclidean concept. Although taken to be a shape separated from rectangle at the sensory level, square is a concept having nothing very specific to be treated as an autonomous didactical unit.

Let us also say that teachers should make a distinction between numbers and magnitudes. The way how contemporary school books are written, this distinction becomes even more hidden. H-G. Steiner's paper [22] is an excellent theoretical ground to view the role of positive rational numbers as operators acting on domains of quantities (magnitudes). Details of this sort we leave for another occasion.

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