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DIDACTICAL ANALYSIS OF PRIMARY GEOMETRIC CONCEPTS II

Milosav M. Marjanović

Abstract. Surveyed in historical perspective, the contents of school geometry can be sorted into the following three stages: intuitive, pre-Euclidean and Euclidean. Geometric topics usually found in primary school programs constitute the first stage of intuitive geometry. Although this geometry is related to the perception of the surrounding real world, both the coherence and order in its exposition are equally important. This paper is devoted to that didactical task.

At this first stage, pairings of the real world appearances (objects and their collections) with geometric models (iconic signs and their configurations) make the main learning procedures. These pairings are interactive, meaning that the appearances "re-animate" the models and the latter serve to the intelligible conceiving of the former. In order to discriminate between the two sides of this process of pairing, we consider two levels of intuitive geometry. When conveyors of geometric meaning are real world appearances, we speak of inherent geometry and when such conveyors are configurations of iconic signs, we speak of visual geometry. These latter signs (called here ideographs) express the meaning of geometric concepts and their deliberate use leads to the assimilation of this meaning.

Following Poincaré's views on genesis of geometric ideas, we formulate a cognitive principle stating that perception of solids in the outer space, in the way when all their physical properties are abstracted (ignored), leads to the creation of these ideas. In addition, an intelligent ignoring of extension leads, then to the creation of concepts: point, line and surface.

In the final didactical analysis, a series of basic geometrical concepts will be discussed (and this paper includes: point, line, segment, ray and straight line).

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Key words and phrases: Intuitive geometry, levels of inherent and visual geometry, principle of creation of geometric ideas, point, line segment, ray, straight line.

> *The main point I tried to get across is that geometry is not so much a branch of mathematics as a way of thinking that permeates all branches.*
>
> *Michael Atiyah*

This paper is a continuation of a previous one: "Didactical Analysis of Primary Geometric Concepts", this Teaching, vol. V, 2, pp. 99–110, 2002. Thus, it has the same title, the same author and the same objective—analysis of geometric ingredients in primary school (the first four years of schooling).

For a number of years, this author was teaching a course of didactics of mathematics at the Teacher Training Faculty, University of Belgrade, where the students are educated to be teachers in primary schools. To be admitted, these students are supposed to have completed a high school (mostly the Gymnasium). Having had math lessons all those years in school, they are also expected to be sufficiently acquainted with elementary mathematics (algebra, Euclidean geometry, trigonometry and analytical geometry). And it is our firm conviction that this amount of mathematics suffices for professional formation of these students. Moreover, if a math course is needed independently of didactics of mathematics, then it should be a *repetitorium* (a refresher course) of elementary mathematics contents (including digested basics of naïve set theory and mathematical logic). The papers [4], [5] and [12] point out lack of proficiency of primary school teachers and discuss the ways of its improvement.

Concluding again from personal experience, we find that students at educational institutions need, before everything, a thorough didactical analysis of all subject contents that they will meet in their in-service practice. Following Freudenthal's models ([7], [8]), we understand didactical analysis as the examination of a content through its decomposition into constituent parts and then, its reorganization according to the aimed didactical tasks. In this paper our considerations will be focused on primary school geometry, when all contents are at the intuitive stage.

School geometry, as it is usually taken, is a more or less simplified version of the Euclidean geometry. The historical source of this mathematical discipline is Euclid's *The Elements* and therefore that purely scientific work has been the basis for formation of school courses in geometry. In this respect, geometry stands in a total contrast to arithmetic, since the learning of the latter starts with the large intuitive ground upon which the meaning of arithmetical abstractions is derived and formal properties of numbers and operations are experienced before these properties are stated as fundamental laws of number systems. In order to suggest an analogy, let us say that a possible didactical transformation of the Peano axiomatics would resemble that of geometry. The following papers present the state of teaching and learning geometry in around last hundred years: [6] covers the period of the first years of "*L'Enseignement Mathématique*", [10] covers the turbulent period from 1950 to 1970 and [13] covers the period 2000 and later.

As it is well-known, a formal scientific exposition of an area of knowledge, and it was the case with Euclid's work, leads inevitably to the severance of ties of its content with processes of the real world that helped to generate it. When such an exposition is taken to be the starting point for didactical transformation of its content, then existing abstractions have to be made close to reality. And it is certainly better if some aspects of reality can be made to lead to creation of such abstractions. An attempt of the latter kind was the Dutch propaedeutic geometry from the first half of the 20th century (Tatjana Afanassjewa Ehrenfest, Paul Ehrenfest et al.). Also, Piaget's experimental findings and his classification of intuitive geometric concepts as being topological, projective or metric in nature have provided an important impetus for selecting and organizing geometric content in primary school. (An improving view of analysis of reality with the help of concepts is expressed in [18]).

It is quite a wide-ranging practice to start with the teaching and learning geometry in the spirit of Euclid, when children are twelve or older. In such a course, some terms stay undefined (the examples are: point, line, straight line, segment, etc.), some others are defined (broken line, circle, angle, etc.), some facts are postulated, some others proven. Such a plan of exposition resembles *The Elements* in a too much explicit way and children of that age are perplexed by all those things that gain a sense only at the stage of deductive reasoning. By drawing of geometric pictures, the involved concepts gain visual meaning but they still stay without their intuitive roots that are inherent in the objects of the surrounding real world. Thus, the main objective of primary school geometry can be expressed metaphorically as a search for meanings that have been lost. Further on, let us say that by projecting the historical perspective, the following three stages of school geometry can be outlined:

- Intuitive geometry,
- Geometry in the spirit of Thales and Pythagoras—pre-Euclidean geometry,
- Euclidean geometry.

In the sections that follow, we will be considering contents of intuitive geometry that are usually included in the programs of primary schools. Despite the fact that these contents are related to the real world appearances, the requirement for coherence and order in their exposition is as relevant as it is in the case of teaching and learning mathematics at higher levels.

A negative characteristic of primary school geometry is a wide variety of the ways how its content is selected and elaborated. Depending on the displayed tendencies (from the side of curriculum planers, textbook authors, etc.), this teaching discipline is often conceived of as heuristics which helps children to develop their perceptual and logical abilities. Without a description of the main facts and points that should be selected for elaboration, this discipline may easily be turned into a pile of various intellectual games having uncertain educational effects. The worst of all are those dull expositions full of explanations of the unexplained that are "telling" children what segments, straight lines and the other concepts are. According to our point of view, the main aim of primary school geometry is the preparation of children for the stage that comes after—pre-Euclidean geometry. This means that children have to acquire intuitive meanings of all basic concepts of pre-Euclidean geometry and to become skilled at representing them by geometric drawings. And contrary to popular belief, this elemental part of geometry has to be well organized, clearly grounded and sharply outlined.

As for the selection of geometric content at this first stage, we find that the following list of topics is quite satisfactory:

Words denoting positional relationships of the objects in the natural surroundings. Recognition of fundamental geometric shapes (triangular, circular, rectangular, of quader, cylindrical, spherical).

Point, line, surface (when flat: figure). Open and closed lines. Lines that are straight: segment, ray, straight line. Angles (acute, right, obtuse). Circle. Square and rectangle (construction of these figures). Cube and quader.

Units for measuring time, length, volume of vessels holding liquid.

Length of broken lines, area of square and rectangle, area and volume of cube and quader. (This content seems to be quite close to the one on National Curriculum for England, [11]),

Thinking of the subtleties of elaboration of this content, we find a theoretical framework that will direct our procedures in the following the ideas of Freudenthal's didactical phenomenology.

(Numbering of the sections that follow is in accordance with the one applied in our previous paper).

5. Following Freudenthal

Words label concepts but their corresponding examples provide them with meaning. A concept is always taken to be at the higher level of abstractness than its corresponding examples. The least abstract are those concepts whose all examples are observable objects of the real world. And when all examples related to a concept are visible phenomena existing in the objective space, then such a concept is said to be at the *sensory level*.

The explicit idea of geometric space had not been developed in the period of classical Greek geometry. From the time of Descartes on, the coordinate spaces have been used in the study of geometric objects and, starting with the end of 19th century, a process known as geometrization of mathematics began. Then, in the course of time, various ideas of abstract spaces have been introduced into mathematics, often having the role of a receptacle for a class of objects that are under consideration. By mentioning all these facts, we intend to make noticeable situations when concepts and their systems are related to the classes of corresponding examples. A class of examples related to the concepts of a system will be called the *underlying phenomenology* of that system. An inspiration for the use of this term we find in these Thom's general views ([19]), (left to be expressed in his own metaphorical language):

"Toute science est l'étude d'une phénoménologie ... , toute phénoménologie doit être regardée comme un spectacle visuel."

The instances for the underlying phenomenology (visual spectacles) that lie within the scope of this paper will be:

– all solid objects existing in the objective space,
– iconic representations existing in the pictorial environment, taken as the receptacle for all our graphical codifications.

These instances will determine two levels of intuitive geometry: the cases of inherent and visual geometry, respectively. (Details follow in Sections 7 and 8).

First of all, one is due to say that the Freudenthal's didactical phenomenology is a branch of didactics of mathematics as his book [9] presents it. Understanding it as a specific plan for teaching and learning mathematics, and with some freedom of personal interpretation, we could say that didactical phenomenology is a process of structuring the subject matter on the one hand and that of collecting amounts of significant information by observation of the underlying phenomenology on the other, what, then, makes a framework for connecting concepts with their corresponding phenomena. When we say that phenomena are observed, it means that they project their meaning directly. In the cases relevant to this paper, such phenomena will be scenes in the outer space and the iconic representations. Let us remark that in the latter case children are supposed to learn spontaneously the function of such representations. (In the more advanced cases, phenomena can be abstract constructions themselves as, for example, some classes of subsets in coordinate spaces, etc. Relative to the system of concepts that is under construction, such phenomena, no matter how abstract they can be, are supposed to have been conceived in the mind of the learner). Further on, concepts and their systems are learnt gradually, passing through several levels of abstractness. To express that continual process of building concepts, we have used the gerund "structuring" in the above formulation of didactical phenomenology.

6. Learning from Poincaré

It is a widely held belief that basic geometric facts are self-evident and experimentally verifiable. Such a faulty opinion is a result of confusing abstract geometric ideas with models and iconic signs representing them. This does not mean, of course, that geometric ideas are not related to the appearances existing in the real world. Quite the contrary, human beings create them in their minds being constantly in the contact with the surrounding reality. What is a fact to be accented is that the learning of geometry is not a process that develops spontaneously but the one that has to be carefully directed by a teacher. Combined of the perceiving and iconic representing of the perceived, as well as of verbal expressing of these both activities, the process develops successfully if only its mechanism is well controlled. It is particularly important that the verbal way of expressing discriminates sharply between observable things and abstract concepts of geometry. Such a skillful use of the language by a teacher will, then, easily be assimilated by children. That is why an essential side of this learning process is the understanding of the ways how geometric concepts are generated and conceived in the human mind. We find that deep philosophical considerations of the great classical mathematician Henri Poincaré (1854–1912) concerning human experience and creation of geometry could have substantial didactical implications. His profound thoughts of this sort are exposed in his book *La Science et l'Hypothèse*, ([17]). This book is at a very advanced level, thereby not being the kind of reading matter to be suggested to teachers or didacticians. Nevertheless, citation of some lines selected from that book and followed by our interpretation and intentional simplification could serve as a sort of surrogate for a wider readership.

Thus, for instance, summarizing some of his considerations, Poincaré writes:

"... les principes de la géométrie ne son pas des faits expérimentaux et qu'en

particulier le postulatum d'Euclide ne saurait être démontr par l'expérience."

Interpreting, let us say that geometric objects are abstract, thereby being the formations of human mind which cannot be identified with material objects of the real world. Neither can the assumed relationships between such objects be verified experimentally by means of such materialization. The most famous such an assumption is the Euclid's Fifth Postulate: *Let be given a straight line and one of its external points. Then, there exists one and only one straight line through the given point which is parallel to the given straight line.*

Over the course of centuries, mathematicians tried in vain to prove Euclid's Postulate that is to derive it from the other postulates that are accepted in the Euclidean geometry. When N. Lobachewsky (1793–1856) made the assumption that *an infinite number of parallels exist through a given external point*, building so a geometry equally consistent as the Euclidean is, it became evident that a proof of the Fifth Postulate is impossible in principle. And later, when G. F. Riemann (1826–1866) made another assumption that *any two straight lines intersect*, building so a consistent geometry without parallelism, this long-lasting discussion about parallel lines was definitively closed.

A visualization of the Riemannian plane is a sphere on which the shortest path between two points is the arc of the great circle joining these points. Thus a sphere with the great circles taken to be straight lines (formally they are called geodesics) is a model of a Riemannian plane.

Animating this geometric situation, we could say that imaginary, intelligent beings that would never leave a spherical surface would adopt the Riemannian geometry as being the most advantageous to them. Since we live on a somewhat rough surface of a globe and in surrounding space of it, locally we measure distances between two places imagining them as two points joined by a Euclidean straight line segment but globally, when the places are at greater distance, we measure the length of the arc of great circle on the surface of the Earth that joins them. Although, these interpretations are simple, they show that the conception of reality can be more advantageous using one or another system of geometry.

As signs of poor understanding of the nature of geometry we can take "explanations" of the type "what would happen if ... " which are often encountered in school books on geometry. Some examples of such "if" clauses are: If our drawing instruments were ideally precise ... , If the straight line were extended indefinitely ... , If the line that has been drawn becomes thinner and thinner ... , etc. In fact, straight lines are objects that are extending boundlessly and therefore, they cannot be either shortened or extended. But what can be are the drawn segments representing them. This is a typical instance of confusing concepts with their iconic representations. And as we have seen, the Euclidean Postulate is a logical assumption independent of experience; thereby no ideally precise instruments could exist and be used for its demonstration. There is no excuse for "explanations" of this kind that hinder the normal process of learning. As the relicts of the old-fashioned pedagogical improvisations, they should be banished from school books.

Deep Poincaré analysis of processes that lead to the formation of geometric ideas is based on perception of solid objects existing in the physical surroundings. Thus, he distinguishes geometric space from representative space with its triple form: visual, tactile and motor, seeing the latter to be a deformed image of the former. According to his analysis, solid objects and their displacements are not representable in the geometric space but we treat them as they were situated in that space. In this way he assumes pre-existence of geometric space as a fundamental cognitive category which is in the function of interpretation of sensory inputs.

Poincaré's analysis of the genesis of geometry is not certainly easy for a direct approach. But we are convinced that its possible translation into the language of didactics and its imbedding into teaching situations would contribute much to the elaboration of school geometry.

Now we use the Poincaré's views, expressed in the lines that follow, to formulate a simple didactical principle.

"Les mathématiciens n'étidient pas des objets, mais des relations entre les objets . . . "

"La matière ne leur importe pas, le forme seul les intéresse."

"Si donc il n'y avait pas de corps solide dans la nature, il n'y aurait pas de géométrie."

First of all, we have to understand that formation of geometric ideas starts with perception of solid objects in the outer space. According to the Gestalt psychology, perception is inseparable from interpretation and interpretation is also considered as a form of thinking and abstracting. Taken literally, abstraction is a process of selecting essential properties. But in the case of formation of fundamental concepts (numbers, shapes, etc.), it is much easier (or only possible) to indicate those properties that are inessential and by means of that process of ignoring (forgetting) such properties, the essential will be left over (without an explicit indication what it is).

Relying on the above Poincaré's views, now we can formulate the following didactical principle:

Perceiving a solid object A in the way when all its physical properties are ignored, a pure idea of a geometric object $\overline{A}$ is left behind.

This formulation of the process that leads from observing of solid objects to their geometric perception (and conception) we call the *Principle of Formation of Geometrical Ideas*. The function of the over-bar is to symbolize abstracting (or better to say, ignoring) of all physical properties of the object A.

Two objects A_1 and A_2 can be different (have different physical properties or different positions in the outer space) and two impressions in the mind $\overline{A}_1$ and $\overline{A}_2$ can be still equal. Such objects are said to have the same shape and size. In geometric space, objects that correspond to them are called congruent. Congruence of two geometric objects is often described intuitively as the possibility of their coincidence when one of them is "moved" into the position of the other one. Such moving, existing in our imagination, is an imitation of the displacement of solids in the outer space. Namely, when a solid object is displaced in the outer space, its position changes but it stays identical to itself. Congruence is the basic geometrical relation, for the properties of two congruent objects are all the same and what makes them different is their position in space. To use the idea of geometric space as the receptacle for geometric objects would be rather a complicated abstraction at this level of consideration. Instead of it, we use iconic representations of such objects, taking them to be situated in pictorial environment.

Let us remark that spontaneous meaning of the concept "shape" is quite ambiguous and when fixed mathematically, it leads to different morphological types. Let us also add that in our paper [16], that matter is discussed in a way that is approachable for a wider readership. Also, the spontaneous concept of size is quite a loose one and when formalized mathematically, it leads to different types of measures: length, area, volume, etc.

Not all shapes are of special interest. However, some objects have very characteristic shape what, with their possible use in human activities, singles them out as particularly important. Perception of such objects and the processing of the corresponding sensory inputs unite into a specific type having a stable mental representation and being denoted by a specific word in the domain of the language. And that is the way how spontaneous geometric concepts are created in the mind and expressed verbally.

Finally, speaking about ancestral experience, Poincaré says:

"… par selection naturelle notre esprit s'est adapté aux conditions du monde extéreur, qu'il a adopt la géométrie la plus avantageuse a l'espèce; ou en d'autres termes la plus commode."

"… la géométrie n'est pas vraie, elle est avantageuse."

Thinking of didactical implications of these views, we find that teachers, particularly those in primary school, should understand geometry in a broader way than from the perspective of geometric courses that they had in school. Selected topics from history of mathematics could certainly contribute to that understanding. Of particular interest could be the acquaintance with activities of our primitive ancestors who manifested a high artistic ability to make things of different shape and for different use. And an essential component of that ability was their geometric imagination which had directed such their activities. Their geometric ideas were inherent in the objects they made and such a situation with geometry continued to exist even in the period of prehistoric civilizations (East and Middle East, Ancient Egypt). Such a broad comprehension of the beginnings of geometry helps a teacher to understand ontogenetic development more completely as well as to appreciate this kind of knowledge as important heritage of human species which helps us to experience the surrounding world intelligibly.

7. Inherent geometry

Since the process of gradual building of mathematical concepts develops over a long period of time and passes through several levels of abstractness, it is necessary for a successful elaboration of the content that didactical procedures are made consistent with these levels. There is a variety of ways how such levels are conceived:

as being developmental phases (J. Piaget), as modes of thinking (van Hiele, [20]), etc. But when taken to be situated in the subject matter, where they can be formulated in the related terms, these levels of abstractness obtain the clearest meaning.

The first level of intuitive geometry that we formulate in this section is based on the fact that real world objects and their configurations are more concrete, and therefore less abstract, than geometric drawings which could be associated with them. Thus, this first level consists of all those learning situations where conveyors of meaning are objects and scenes existing in the physical space including also pictures that represent them in their absence. This level of intuitive geometry we call *inherent geometry*, expressing so the fact that the properties that are inhered in things correspond to geometric ideas.

Now, we will indicate those items of intuitive geometry that are characteristic for this level. For example, the exercises assigned to children to practice the correct use of the words denoting place and positional relationships of objects in the outer space are a part of inherent geometry activities. Selection of such exercises should be bound to those real world appearances where the observed objects and their relationships are correspondent to geometric objects and incidence relations between them. Let us consider a typical example of the sorts that are intended for children.

(a) In a picture a table is seen with a lamp hanging *above* it and a cat lying *under* it. A glass is seen *on* the table. Assignment: What is above the table and what under it? What is on the table? Children are supposed to give answers: lamp, cat and glass, respectively.

Conceived geometrically, this appearance corresponds to the configuration consisting of a plane (the table materializes it) and three points (representing the positions of lamp, cat and glass). Two points are on different sides of the plane (belong to different half spaces) and one of them is lying in the plane (belongs to the plane).

Exercises in orientation that are often practiced in preschool or early school period also belong to the inherent geometry. Doing them, children develop feeling for the position of observer in relation to objects in his/her surroundings. This feeling comes from the awareness that human body stands vertical (when in the upright position) is symmetric and has frontal and back sides. As an attempt to express this feeling more precisely, we could imagine a coordinate system being attached to our body. Its three axes and three planes determine with precision the meaning of the adverbs: *upwards* and *downwards*, *to the left* and *to the right*, *forwards* and *backwards*, as well as of the prepositions: *up* and *down*, *on the left* and *on the right*, *in front of* and *behind.*

It is worth of remark that the feeling for orientation is an inborn spatial ability, and therefore, when a scene is clearly presented to children and the questions posed by teacher well formulated, no additional explanations are needed. Let us also say that these types of assignments are not exercises in grammar but in geometry. This means that the requirement that all scenes, whether real or pictured, have to suggest an essential geometric structure.

Recognition by shape is another activity that is typical for inherent geometry. Multiple meaning of the spontaneous concept of shape requires specification of particular cases. The most important of them all, are those which make activities of comparison meaningful:

- Linear extending (the shape of segment); comparison: longer–shorter, (larger–smaller)
- Rectangular shape; comparisons: longer–shorter, wider–narrower,
- Shape of quader (parallelepiped); comparisons: longer–shorter, wider–narrower, higher–lower,
- Circular shape; comparison: larger–smaller,
- Cylindrical shape; comparisons: longer–shorter, thicker–thinner.

Let us remark that the phrases as "rectangular shape", "shape of quader", etc. are terms that are referring to the classes of concrete objects, while the terms as "rectangle", "quader", etc. are reserved to be denoting abstract geometric concepts. Thus, the mentioned phrases have to be used as wholes which should not be divided into component words and which acquire their meaning in that way. In Section 9 each of these shapes will be discussed in detail. (A book written by this author ([14]) and intended for preschool children, is full of exercises aimed at the formation of ideas of basic shapes and at the development of awareness for orientation. See also ([15])).

A difference between comparison and measuring should also be remarked on. Each measuring assumes the use of a unit and amounts of measured quantities are, then, expressed in numbers. Figuratively speaking, each measuring is a transposition of measured amounts to a scale. When a measuring requires more precision, smaller units have to be used or the results of such measuring have to be expressed in (decimal) fractions. Comparison, on the other hand, is a simpler procedure and it consists of establishing the relationship between amounts that is expressed only in words: larger, smaller or equal.

Measuring of weight of physical objects and of the volume of liquid substances also belongs to the inherent geometry. But there is, in addition, another point to be made and the fact to be aware of. As the system of natural numbers is inadequate for measuring abstract geometric objects, the topics as measuring of length, area and volume also belong to inherent geometry. Within this context, the physical objects are measured and segments, rectangles and quaders are always taken to be subdivided into the units of measure. As a sign of a serious misunderstanding of this fact are those situations when the authors of primary school books take the formula $P = a \cdot b$ as holding true for any rectangle and when they speak of the expression $a \cdot b$ as it were the product of the lengths of its sides. As a matter of fact, a and b are lengths of sides relative a chosen unit and, in addition, such a formula is applicable only to those rectangles whose sides are measured by that unit with nothing to remain.

8. Visual geometry

In proceeding further, we have to make a distinction between two kinds of iconic representation. In the case when a real world appearance (a concrete object or a scene) is represented by a picture or by a drawing then, such a graphical product will be called a *pictograph* (*pictogram*). Pictographs are used to represent real world appearances in their absence or, what is more subtle, to represent clearly their structural features, by reducing the recording of indiscriminate details. Making such drawings is a thought process which can be considered as being an intermediate step between reality and purely geometric representation. But when a drawing is used to represent an abstract geometric object or a spatial configuration of such objects, then such a graphical product will be called an *ideograph* (*ideogram*). For example, when tracing a pencil along the edge of a ruler, a drawing is made that is the ideograph representing a straight line (and not being that line itself). Changing the position of the ruler and tracing the pencil in the same way twice or more times, an ideograph is obtained which represents a spatial configuration. To make a difference between abstract ideas and the drawings that represent them, we will use the word "traced" adjectively. Thus, we will speak of "traced straight lines", "traced rays", "traced segments", etc. thinking, then, of ideographs which represent these abstract concepts. Ideographs are quite faithful copies of our mental representations, though they should never be identified with them. And no matter how fine their realization is, they still contain some amount of noise (boldness of traced lines, their color, etc.). Since ideographs express pure ideas graphically, their use is more subtle than that of pictographs. A systematic ignoring of any noise that exists in realization and use of such graphical constructs makes their cognitive function possible. And it is on at the teacher to accomplish this delicate didactical task.

Learning situations where ideographs are conveyors of meaning constitute the next level of intuitive geometry that we call *visual geometry*. Now let us consider a simple example which illustrates the transition from the inherent geometry level to the one of visual geometry.

(b) A pictograph represents two roads that cross. Children are assigned to make an ideograph which will represent this scene geometrically. Thus, they are asked to draw freehand two lines following approximately shape of roads. They are also asked to describe the scene represented by the pictograph, using the natural language. Describing the ideograph they should use the subject language: Two lines intersect at the point C. These lines represent the roads and the point C represents the crossing.

To stress the difference between the two levels of intuitive geometry, let us say again that the underlying phenomenology of the inherent geometry consists of the real world appearances and the pictographs that represent them. Altogether, it is a spectacle in the outer space. In the case of visual geometry, the underlying phenomenology consists of ideographs which make a spectacle in the pictorial environment.

In the next section and in the form of a series of short essays, we treat the development of meaning and the ways of didactical elaboration of basic concepts of intuitive geometry.

9. Basic concepts of intuitive geometry

As it is generally accepted, our perception of reality is the source of our knowledge. At a more conscious level of development, perception is inseparably followed by interpretation and, the latter is seen as the matching of sensory inputs with stable and already formed mental representations that are tamped down in the mind. Avoiding to be involved in some chicken-and-egg considerations, we assume the functioning of normally developed mechanisms that govern the thought of children of this age.

Many words found in natural languages, for instance: point (dot), line, straight, curved, round, flat, angle (corner), square, ball, etc., bear purely geometric meaning which results from a spontaneous use of the language. Thus, in this section, our analysis includes spontaneous meaning of the basic concepts of intuitive geometry, the ways how their scientific use begins and how their meaning is fixed at more formal levels of exposition (without going beyond the scope of school mathematics). Having didactical tasks in mind, we hope that this analysis would help a primary school teacher to understand genesis and the ways of didactical elaboration of basic concepts of intuitive geometry.

Several exercises that follow will be examples of interplay between two levels of intuitive geometry. On the one hand there will be natural scenes pictured or presented by drawings and on the other, drawings exposing their geometric structure. When each iconic sign of the latter type of drawings conveys the meaning of a geometric concept, such drawings will be the examples of ideographs.

A parallel procedure of using the natural language when describing pictographs and, on the other side, terms of geometry when expressing the attached ideographs has to be consistently practiced. By recognizing of this difference, the cognitive process of abstracting is greatly encouraged to develop in the mind of a child.

9.1. Point. When we turn our eyes to follow a plan flying away, it looks smaller and smaller and for a short time, before it is out of our sight, we can only be discerning it. Then, neither its shape nor even its extension in the space can be observed. Coming stimuli are so weak that our eyes register mere existence of something. At such a moment, we usually say that the plane is seen as a distant point. Many objects which are a long distance away are observed as they were very small spots and in all such cases, the coming stimuli have the same thing in common—they send a piece of information about something merely existing. And just for such objects we say that they are observable as the points in the visible space. For instance, we say that the stars are points of light in the sky.

When very minute objects are observed, the same stimulus effects are produced and then, the word "point" is also used to indicate these effects. For instance, the mark left on paper when it is touched with the point of a ballpoint pen, is an example of that kind.

The word "point" is derived from the Latin word "*punctus*", whose primary denotation was a sharp, thin end of something. This primary meaning of that word is also present in the contemporary usage of English when, for example, we speak about the point of a javelin, a pencil, a pin, a knife, etc. Do not we also describe a ballpoint pen as a pen having a small ball at its point? We use the term "vanishing point" to denote a point in the distance at which two tracks of a railway line appear to meet.

These and many other examples of the use of the word "point" are associated with the observation of something which appears to have no shape and size, even no extension in the visible space. On the basis of such visual experiences an inner representation—a mental image is formed in our mind and it is faithfully well represented iconically by a small, just visible mark as it is left when the point of a pencil has touched a piece of paper.

There exist, of course, numerous figurative meanings of the word "point" and we will confine our consideration to the case when the word denotes a place. For instance, one could say that London will be his or her starting point for a journey in Europe. A town may also be named as a point of intersection of two roads. On the map of an international airline company, we can see big world cities represented by points. These are instances of intelligent ignoring of the extension of objects in situations when only their mutual positions are something that matters.

The ignoring that we have just mentioned is an excellent example of abstracting which, when deliberately elaborated, will be at the reach of the youngest pupils. Let us consider a few of many possible exercises of that kind.

(c) A garden is seen in a picture. Daddy and Mummy are sitting on a bench, while Tom and Suzy are playing around a small pond. To this picture a simplified drawing is attached, consisting of the frame of the picture which represents the area of the garden and with a circle inside it representing the pond. Talking with children the teacher explains what the attached drawing represents. Then, the children are asked to draw dots representing by them positions of each person in the picture.

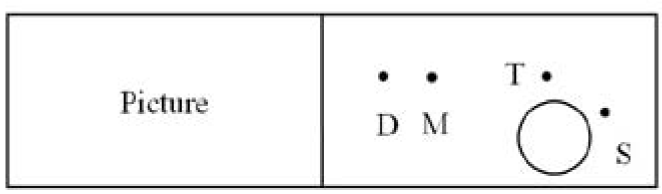

Fig. 12

Colored dots may be used to correspond to the colors of shirts of Daddy and Tom and of blouses of Mummy and Suzy. If children can already write, it is still better that they use the initial letters of the names of these persons to denote with them the points.

(d) A picture represents three boys standing aligned, with their shirts being in different colors. The attached drawing consists of three circlets joined by a thin, straight line going through their centers.

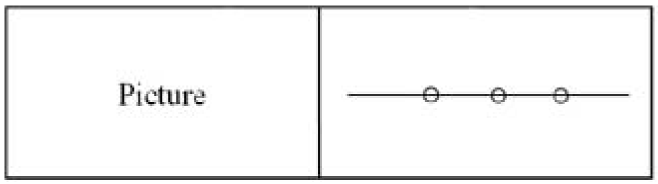

Fig. 13

Now, children are asked to color the circlets according to the colors of the shirts. Several questions can be asked related to both, the picture and the drawing. For example: Who is in the middle, who on the right (left) of him? Who is standing first, second, third when looking from left to right (and in the reversed direction) and who is at the left (right) end? Similar questions should be asked replacing the boys with the circlets (dots).

By attaching drawings, we represent the underlying mathematical structure of the pictured situations. Being free from the unnecessary details the drawings represent what is essential and such their use is on the best way to abstractions. Pictures represent the real world objects and scenes in the way they could have been seen, thereby being pictographs. These two simplified drawings are also pictographs. But by drawing dots, the first step into visual geometry is made and whenever they are drawn to represent the idea of point, they should be considered as ideographs. Such a case will be in those situations where dots are used to indicate the position of objects whose extension in the space is ignored.

Let us explain why the attached drawing in the exercises (c) should be considered as a pictograph. The circle that is traced in it represents possibly a wall or a fence around the pond and, in the context of this exercise, it does not represent the idea of a circle. In the similar way, the straight line that is traced in the attached drawing of exercise (d) stands to suggest the idea of linear arrangement and not that of a straight line. If dotted, such a line would serve that purpose equally well.

By drawing two or more dots (on the same sheet of paper) children begin to accept spontaneously the fact that different points exist. To say that two points have the same shape and size is, of course, a logical nonsense and such congruence would have no intuitive ground, either. The concept of point is a purely intellectual creation and its meaning results from the way how it is used and represented. A dot is an ideograph which represents a point located in the space. Two points differ having different locations and it is the only difference that should be pointed out and in which they are not like each other. Only on that basis the congruence of any two points is acceptable. Moreover, differently located points are examples of the concept "the point" and we refer to such an example using the indefinite article that is by saying "a point".

At the end, let us just mention that in the contemporary Euclidean geometry all geometric objects are considered as being built of points. At more advanced levels, where various sets are taken to carry geometric structures, the elements of such sets are also called points.

9.2. Line. The word "*line*" is derived from Latin "*linea*" meaning thread and "*linea*" was derived from "*linum*" meaning flax. There exist many other things in the natural surroundings which impress an inner representation upon human mind corresponding to the idea of line. Things as hairs, thin strings and wires, etc. look like they had no thickness and as they were extending only in one direction. The same impression is formed when we are looking from some distance at stretching ropes, cables, etc. The outer edges of visually experienced objects, the contours dividing parts of their surface are further examples of appearances that produce the same kind of stimuli. An excellent example of those corresponding to the idea of line is a drawing made when the pin of a pencil moves freely and continuously on the surface of a sheet of paper. Just such drawings are taken for ideographs that represent abstract idea of a line. The brain organizes this sort of visual inputs, the sameness of which gives the meaning to the spontaneous concept of line.

In many situations, some objects are intentionally viewed as they were extending only in one direction, while their real extension in the space is ignored as something unessential. Examples are numerous: ropes, wires, railway tracks, etc. as well as in specific context, roads, rivers, etc. When we buy a piece of rope, the shopkeeper uses a tape to measure its length and its thickness is not measured, it is taken as quality of the rope. Roads are often viewed as lines connecting different sites, rivers as boundary lines between regions, etc. Of course, the context is something that always matters and, for example, a person standing on one bank of Danube ready to swim to the opposite bank, will not certainly conceive the river as a line. But this intelligent ignoring is just the right thing to be used in linking reality with iconic representations of lines. Let us consider a number of exercises that are intended for first graders.

(e) Description of the picture (Fig. 14): A road is seen, going from a country house, through a small grove to a village school building. A tractor and a fountain are off the road. In the attached drawing, a line represents the road following approximately its shape (or children are asked to draw that line).

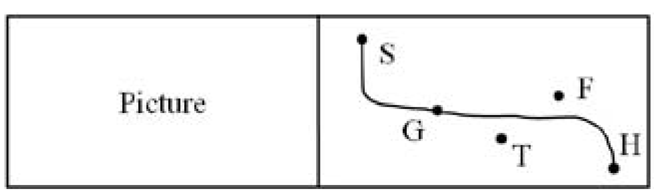

Fig. 14

Then the children are asked to draw dots representing the objects seen in the picture and to denote them, using the initial letters of the names of these objects. To complete such an exercise, description of the natural scene should run parallel with the expression of geometric configuration that represents it. In the former case natural language is used and in the latter the language of the subject matter.

Description of the scene:	Expression of the configuration:
The school building is at the end of road.	The point S is at the end of the line (is the end point of the line)
The road goes through the grove.	The line goes through the point G. (The point G belongs to the line.)
The fountain and the tractor are off the road.	The points F and T do not belong to the line.
The road joins the school building and the house.	The line joins the points S and H.

Here the attached drawing exposes the geometric structure of a real world scene. Since the drawing consists of iconic signs representing geometric concepts (the line and the points), it is the example of an ideograph. Let us also say that at this level we never consider lines as consisting of points. Instead of it, the incidence relation: a point belongs (or does not belong) to a line determines the relationship of such objects (as it was the case in classical Greek geometry).

(f) Picture 1 (Fig. 15): A school building is seen surrounded by a high fence. The gate in the fence is open. A tall conifer grows in the school's yard and a plane tree out of it.

Picture 2: Everything the same, except that the gate is closed.

In the attached frames, children are assigned to draw freehand a line representing the fence (denoting it by "f") and two points representing the two trees (denoting them by "C" and "P").

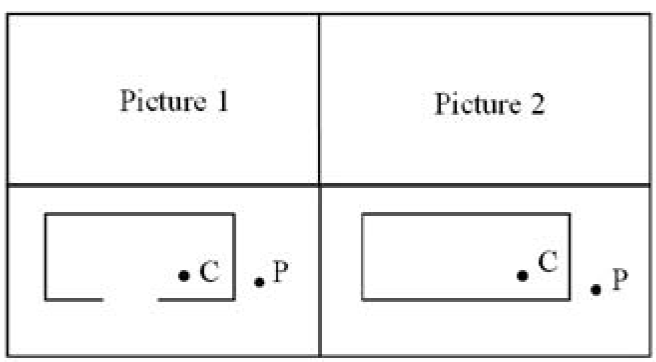

Fig. 15

Then, in the former case they also draw a line representing the path along which Johnny can go from the plane tree to the conifer.

Description:	Expression:
	Case 1
Johnny can go from the plane tree to the conifer without jumping over the fence.	A line can be drawn from the point P to the point C without crossing the line f.
	Case 2
Johnny cannot go from the plane tree to the conifer without jumping over the fence.	A line cannot be drawn from the point P to the point C without crossing the line f.

The teacher can use this (or a similar example) to say that a line as the one in the former case is called *open* and that in the latter case *closed*.

End points of open lines are sharp visual discriminates upon which they are distinguished from closed lines. The irregular shape of some closed lines makes their property to separate the plane into two regions less visible. To exhibit this property clearly, a number of exercises (as (g) below) have also to be done.

Fig. 16

(g) Use a yellow pencil and be coloring the white area of each drawing, starting from upper left corner (Fig. 16). While coloring be never crossing the given line. If a part of the area stays white, use a green pencil to color it as well. See how it has been done in the case of first two drawings. (In the conditions of a classroom, the teacher can formulate this assignment in an even more direct manner).

(h) Use letters to denote the ends of open lines. Then, be coloring as you did it in the previous example (Fig. 17).

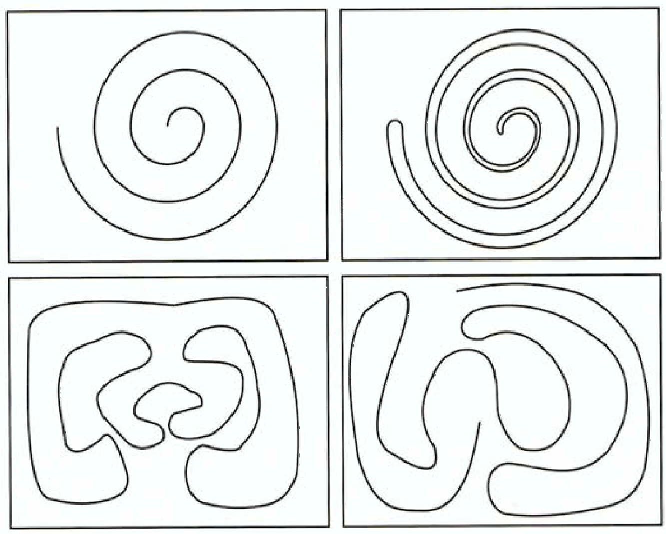

Fig. 17

In general, the relationship point-line is of two kinds: point belongs to line or it does not. In the case of a closed line such relationship is of three kinds: point is *in*, *on* or *out* of the line. Using these kinds of relationship, several interesting exercises can be composed as, for example, the one that follows.

(i)

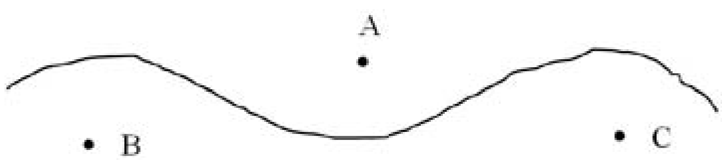

Fig. 18

Extend the given line to be closed and so that:

- Point A is in and the other two out,
- Point B is in and the other two out,
- Point B is on and the other two out,
- Point B is on and the other two in, etc.

In a book for children, a copy of the above drawing should be attached to each of these requirements.

Now let us turn our attention to the related mathematical formalization. The idea of line, as being the path of a moving point, is dominant in science and the terms "line" and "curve" are alternatively used. Taking the interval $[0,1]$ to represent the running time, expressed mathematically, a path in $\mathbf{R}^3$ (in $\mathbf{R}^2$) is a system of three (two) equations (in fact, functions in t)

$$x = x(t),\ y = y(t),\ z = z(t) \qquad (x = x(t),\ y = y(t))$$

which determine the position of the moving point

$$M(x(t), y(t), z(t)) \qquad (M(x(t), y(t)),$$

at each moment t.

To ensure the moving of the point M be free from jumps, the 19th century mathematicians imposed the condition of continuity on these three (two) functions and a curve (line) was defined to be the set of all points M obtained when t varies from 0 to 1. But when Italian mathematician G. Peano discovered so called space filling curves, it was a shocking surprise which went against the intuition of scientists of that time. Namely, Peano found a system of two continuous functions $x = x(t)$, $y = y(t)$ such that the moving point M goes through each point of a square. (In the case of three functions, the moving point M goes through each point of a cube). To escape this unpleasant situation, it was natural to impose further conditions upon the system of functions in the definition of a line. As a matter of fact, different conditions can be imposed, leading to the different concepts of line that are more suitable for the use in specific areas of science and mathematics.

We will not wade through this matter any longer, but we will formulate two conditions which restrict the class of lines in $\mathbf{R}^2$ that are also relevant in primary school geometry:

(i) for each $t, t' \in [0,1]$, $t \neq t'$ implies $M(x(t), y(t)) \neq M(x(t'), y(t'))$,

(ii) for each $t, t' \in [0,1)$, $t \neq t'$ implies $M(x(t), y(t)) \neq M(x(t'), y(t'))$ and $M(x(0), y(0)) = M(x(1), y(1))$.

Adding the condition (i) to the continuity of functions $x = x(t)$, $y = y(t)$, the mathematical definition of *topological arc* in $\mathbf{R}^2$ is obtained as being the set of all points M when t varies from 0 to 1. And adding (ii) instead, such a set is *topological circle*. In primary school mathematics, these two kinds of lines are called *open* and *closed* lines respectively.

Let us end this subsection with a number of general remarks. When a solid object is displaced in the outer space, it stays identical to itself. Such displacements lead to the idea of congruent geometric objects. When we imagine such a moving of a traced line (in the pictorial environment) the idea about its congruent copies arises. All these copies are corresponding examples of the concept "*the line of given shape and size*". On the other hand, the lines of a given shape and size are examples corresponding to the general concept "the line" which, in the case of primary school geometric content, comprises only of open and closed lines.

9.3. Lines that are straight. First of all we start here with the description of a number of phenomena in Nature that convey the sense of straightness. Outlines of trunks of many trees look straight as well as the stems of many plants. Paths of falling drops of rain look like straight lines. On the other hand, stretched ropes or cords look straight. We also walk from one position to another going straight. This feeling for the shortest (straight) path is something inborn and, as it is noticed everyday, animals also have a well developed instinct for straight paths along which they run. Our streets, our homes, furniture in them, abound with objects which sharply project their straight outer edges. Thus much, indeed, that it would not be an exaggeration to say that straightness is quite symbolic for civilization.

The curved comes as a negation of the straight and both are fundamentally present in the mind of a preschool child so as that any explanation of their meaning has no purpose. Thereby, in the process of learning, efforts should be aimed at the transformation of spontaneous meaning of these ideas to their scientific use.

In primary school geometry, children learn about the following three sorts of lines that are straight: the straight line (in the narrower sense), the ray and the straight line segment (often shortened to segment). What discriminates them sharply are their topological properties: *to have no*, *one* or *two end points*, respectively. Such characteristics are marked by drawing the dots that represent them in thick type. Since each of these sorts of lines is preserved under projections, they are projective concepts. But when learnt, they are covered in the order that is reversed to the one above. The reason for it is the fact that endless extension is a quite subtle idealization for an immediate acquiring.

Drawing of lines that are straight (in fact, ideographs that represent them) requires the use of a device as a ruler or a straight edge of cardboard or plastic, better if without any scale. The first exercises should be simply technical and intended for forming the skill in the use of such a device.

9.3.1. Straight line segment. First we will be searching for the intuitive roots of this concept involved in the activity of comparison of lengths of physical objects. The phenomenological situation relates to those objects that are naturally conceived as being extending in the outer space between their two ends. The spontaneous meaning of the word "length" denotes the amount of such extension and two objects, when compared, one of them is *longer* (*shorter*) than the other one or they are *equally long*. The purest possible idea of such objects is represented by (straight line) segments. Thus, in the inherent geometry activities, objects whose shape resembles segments should be compared first. (Examples are stretched ropes, pencils, sticks, etc.).

When objects are compared, they should be laid down to be parallel to each other and adjusted so that two of their ends come one below other. Such a configuration of the objects makes easier their direct comparison. More difficult exercises are those when children have to "move" objects in their thoughts imagining such a configuration. Let us observe that two objects, as for example, a pencil and a helix are, the former may be longer than the latter. In this case, both of them preserve their solid form and the extensions between their ends are compared. But when these objects are conceived as two lines, the helix may be much longer than the pencil. In the latter case, the idea of length is different from that in the former case. There is no danger that children would confuse these two different meanings of the word "length". Performing the activities of comparison, they do not even use that word. Such confusion may arise in heads of some poorly qualified teachers and authors of primary school books.

The words "end" and "extension" have their spontaneous meaning established by the standard use in everyday speech. These meanings have to be whetted by the conception of such a fundamental geometric idea as the segment is. And that task achieves in the interplay between those who know—the teachers and those who learn—the pupils, starting with the activities of intelligent conception of reality.

Let us also observe that linear extension between two ends of objects is a fundamental idea upon which the concept of quantity is based. And should not we think of each procedure of measuring as of the transposition of amounts to a scale which materializes the idea of segment?

The characteristic property of segments to be the shortest among all open lines having the same end points has to be related to some models consisting of real things. It does not mean that this property is proved in that way, but rather it shows that geometry is concordant with reality.

(j) Description of the model: Ends of two ropes are joined together and fixed to a board. The red (shorter) rope is stretched tight and the blue (longer) one hangs relaxed.

First children draw two dots representing the ends of the ropes and they denote them by letters. Then, they use their straight edges to draw the line that joins these two points and which represents the red rope. The blue rope is represented by a freehand drawing of a line which approximately follows its shape. When this drawing is completed:

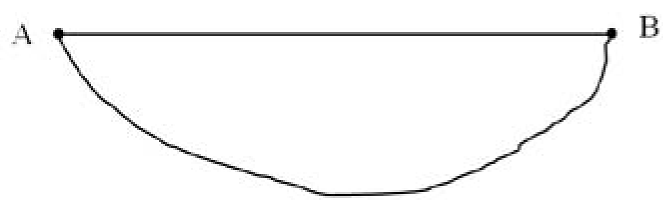

Fig. 19

the teacher can ask several related questions: Which line represents which rope? Which rope looks straight and which one curved? Which line is straight and which one curved?

Then, the right ends of ropes should be released and when stretched tight, they are compared in length, being the red rope evidently shorter. Now the question "Which line is shorter?" has its right place. In addition, weaving a story, the teacher can help children imagine some other real world situations which could be represented by this and same drawing. Examples are: two roads joining two places, a curved river and a straight road from one bridge over the river to the other, etc. Such reversing of thoughts, going from the abstract to the concrete is doubtlessly instructive.

To compare the lengths of any two lines having the same end points would be an idea without any sense at the level of intuitive geometry. But the fact that segment is shorter than any other line having the same end points with it is a fundamental fact having its place and role already at the level of this geometry.

The idea of straight line segment enters children's activities as a line that joins two points and that is drawn by the use of a straight edge. This is not, of course, a definition but rather a description of the way of representation.

When things are compared in length, their linear extension is the only property that matters or, in other words, then we reduce them, in our thoughts, to line segments. Less noise, easier the comparison and when two segments are compared, the whole information will be their extension in the space, what makes such comparison the clearest. That case is also a situation when it is seen how the abstraction takes effect.

In the exercise that follows the technique of using straight edge for comparison of segments is demonstrated.

Fig. 20

(k) When two segments are in special position, as it is illustrated in Fig. 20, the segment AB is said to be shorter (for the segment BC) than the segment AC. (A symmetric expression of this relationship is: the segment AC is longer (for the segment BC) than the segment AB).

To compare two segments in general position, they have to be carried over to the straight edge. In Fig. 21, it is illustrated how a segment is carried over to a straight edge: the left edge levels with the end point A, while the mark B' is drawn so to level with the point B.

When two segments AB and CD have been carried over to a straight edge and when marks stand as it is illustrated in the Fig. 22, then the segment AB is said to *be shorter* (for the segment $B'D'$) *than* the segment CD. (Symmetrically: the segment CD *is longer* (for the segment $B'D'$) *than* the segment AB). When two

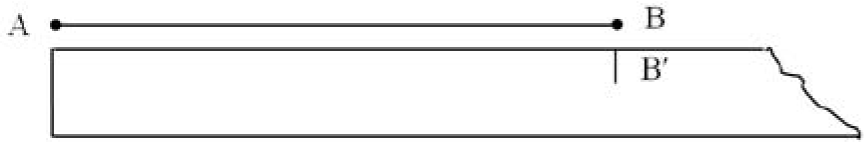

Fig. 21

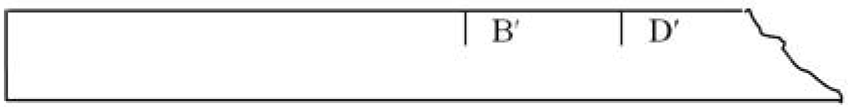

Fig. 22

marks B' and D' coincide, the segments are said to be *congruent* or to *have the same lengths*.

Let us notice that the concept of length is implicit in the expression "have the same lengths" and this expression, as a whole, denotes a relationship between segments (not between their lengths). This way of comparison is the reason why the straight edge should be without any scale and why the word "congruent" should be preferable to the phrase "have the same lengths".

Finally, let us remark that the common property of all mutually congruent segments is their length. Thereby, two segments may differ in length (one of them being longer or shorter than the other one) or else, when they have the same lengths, then their positions in the space are different. Taken in this sense, the length of a segment is a magnitude, not a number. Such a conception of length also has its place in the intuitive geometry, but not at this early stage when other didactical tasks are in the front plan.

9.3.2. Ray. Since rays and straight lines are geometric concepts which extend endlessly, the former in one direction and the latter in two directions, some realistic reasons have to be employed to give the meaning to this extension. Physical objects that are straight (paths, roads, streets, etc.) and whose one end or both ends are out of sight or should be naturally ignored; make the intuitive basis upon which this meaning is established. Technically, it means that the corresponding ideographs have to be used to represent such object. The next exercise is an instance of that procedure.

(1) Description of the drawing: A boy stands beside a ball that is put in a position designated by the point A. The straight line traced from the point A goes between two players from the opposite side. Thus the line represents the direction in which the boy is going to kick the ball. To reduce unnecessary noise, all three boys and the ball should be represented by very simple, symbolic drawings. A story should also be woven concerning the traced line and the point A as being its only

end. And a reason for it has to be given. For instance, the teacher can say that the point at which the ball will stop is not indicated and that such a drawing is an illustration of a pass that is good. To the traced line in this and other similar examples the word "*ray*" is attached and children are left to assimilate its meaning spontaneously.

Under the dynamic circumstances of a classroom, the teacher may be intentionally shortening and extending traced lines, stressing that such modification is inessential and that all such lines represent equally well the same rays. But we have to say that it is a serious mistake to speak about shortening or extending of a ray, confusing so a concept with its ideograph. It is still better if children are assigned to do some exercises, where they themselves are motivated to extend the traced lines. An example of that kind is the following exercise.

(m) A line is drawn representing a ray. Several points are scattered around it (Fig. 23). Children are assigned to find which points belong to the given ray and which do not. (Using their straight edges to extend the traced line, they will find that the points A, D, and E belong to the given ray).

Fig. 23

9.3.3. Straight line. A possibly good idea could be to consider first the straight line as being consisted of two rays with the coinciding end points. Let us sketch that idea elaborating the exercise that follows.

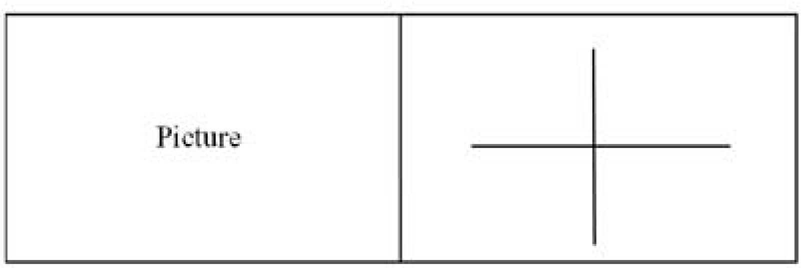

Fig. 24

(n) Description of picture: Two straight, crossing roads are seen. On the one of them a car is seen. In the attached drawing (Fig. 24), two crossing perpendicular lines are given, representing these roads.

Children are asked to draw two points denoting them by letters so that the points represent the car and the crossing of the roads. When completed the drawing looks as follows:

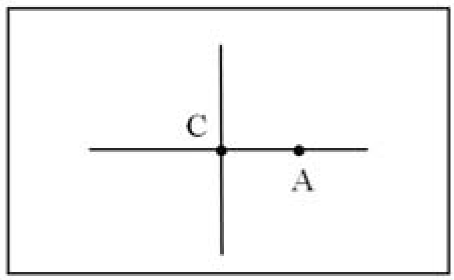

Fig. 25

Now the teacher may put several related questions and requirements.

Natural environment:	Pictorial environment:
How many tracks going from the crossing can you see?	How many rays with the end point C are there?
When at the crossing, in how many ways the car can continue to run?	Show the lines representing each of such paths and tell in which case the line is straight and in which broken.
Show the road along which the car can run straight ahead.	Show the line representing that road. It consists of two rays in two ways. Tell how you see it.
How many straight roads going through the crossing can you see?	How many straight lines through the point C are there?

The word "straight line" has already been used and it will be gathering its precise meaning doing a number of similar examples.

To establish the difference between the concept of a straight line and its iconic sign, as well as to make the idea of it to develop as an object that extends endlessly on both sides, some exercises as the following one should also be done.

(o) In Fig. 26 a straight line is represented with the points scattered around it.

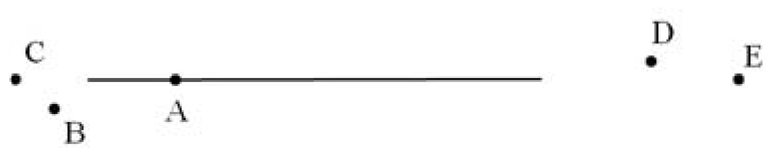

Fig. 26

Children are assigned to find which of the given points belong to this straight line. (Answer: A, C and E). In this case as in all others, we consider just a few examples that serve to sketch activities and procedures through which the concepts under consideration start to be used and therefore to gain meaning. Many subtle details needed for a complete elaboration are omitted here, but a good textbook should always contain them.

Let us end with the remark that two straight lines differ only by having different positions in the space, thereby being always congruent. For that reason, by using definite and indefinite articles in the terms "the line" and "a line", a general concept and its corresponding examples are denoted, respectively.

(to be continued)

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Mathematical Institute, Kneza Mihaila 35/IV, Belgrade, Serbia

E-mail: `milomar@beotel.yu`