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CRITICAL LOOK AT DYNAMIC SKETCHES WHEN LEARNING MATHEMATICS

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Abstract. Among neglected topics in the development of mathematics education is the frequently mentioned use of modern technology, especially hypermedia-based learning environments. This paper examines the use of computer technology for mathematics teaching on tertiary level, especially for distance learning and assessment. It focuses on dynamic sketches, which were used to offer students opportunity to recognize and represent elements of functions and binary operations within a distance education management. It seems that moving from old study culture towards a modern one brings many kinds of cognitive, emotional and social problems. Our experiences do not support the view that using interactive JAVA applets, for example, would bring special advantages without an appropriate pedagogical framework connected to reflective tutoring.

ZDM Subject Classification: D64; *AMS Subject Classification*: 00A35.

Key words and phrases: assessment, JAVA applet, conceptual knowledge, distance education, hypermedia, interactive, mathematics, procedural knowledge, tertiary education.

1. Introduction

The studies of mathematical learning processes have been important not only concerning theories of teaching and learning but when researching sustainable use of technology in education. Haapasalo & Siekkinen (2005) stress well-balanced coherence between teacher's instructional orientation and technological application, pointing out that this question is to be considered from two perspectives: the (pedagogical) structure of the topic to be learned and the use of technology. In this paper we try to highlight the difficulties when combining these aspects for mathematical topics function and binary operation, which are especially appropriate for this purpose, because they trigger an extra interesting problem: how students can develop their procedural school thinking towards abstract conceptual academic thinking[1]. In this educational task, new concepts or new procedures can be (re)invented by the students—or vice versa—the old ones can be at least implicitly applied in a new way. In order to be able to make pedagogical conclusions for comprehensive understanding of concepts, procedures, and the links between the two, it is often reasonable to start by analysing familiar concept pairs. Concept building itself can offer different kinds of problem-solving processes at its productive as well as at its reproductive stages.

[1] This problem has been recognized by the International Commission of Mathematical Instruction; `http://www.mathunion.org/Organization/ICMI` (cf. Holton 2001)

Function and binary operation seem to be difficult concepts for students at the beginning of their studies (cf. Pesonen et al. 2002). Having in mind the definition of the two concepts[2] it is quite clear that students must have full understanding of the function concept, before they can understand the concept of binary operation.

To support students to come up with their spontaneous procedural ideas and help them understand the conceptual features of the function and binary operation, two case studies searching for an appropriate balance between procedural-centred and conceptual-centred approaches to topic learning were undertaken. The first study is based upon *genetic view* (i.e. procedural knowledge is necessary for conceptual one) or *simultaneous activation view* (i.e. procedural knowledge is necessary and sufficient for conceptual one), whereas the second study utilizes is *dynamic interaction view* (i.e. conceptual knowledge is necessary for procedural one), or again the simultaneous activation view (for these views, see Kadijevich & Haapasalo 2000). The view utilized in both studies gives the learner an opportunity to simultaneously activate conceptual and procedural features of the examined topic by means of certain mental or concrete manipulations of the representatives of each type of knowledge. Whilst the papers of Pesonen et al. (2002) and Pesonen et al. (2005) give details on the interplay between conceptual and procedural knowledge in the two case studies, the report mainly concentrates on affective and technical issues concerning the interactive feature of the utilized tasks and learning environment.

2. Background

When designing our Java applets[3] we utilized the framework of the Finnish MODEM-project (see Haapasalo 2003)[4], which offers a sophisticated interplay between conceptual and procedural knowledge. Because these studies form a base work for the planning of learning environments within this framework, our aim was not yet to plan a comprehensive learning material for the mathematical concepts under our consideration. We mainly applied the basic idea of dynamic interaction principle (see above) in the identification and production phases of the mathematical concept building (see Haapasalo 2003, Ehmke et al. 2005). For the phase of *identification* (I) we gave students opportunities to train themselves in identifying concept attributes in verbal (V), symbolic (S) and graphic (G) forms. In the phase of *production* (P) they got an opportunity to produce from a given presentation of the concept another representation in a different form.

[2]A *function* denotes a rule or correspondence $f\colon A \to B$ (read: from A into B) which associates to each element x in a set A a unique value $f(x)$ in a set B. The set A is called the *domain* of f and the set of all *images* $f(x)$ is called the *range* of f. An *internal binary operation* in a set A is a two-variable function $b\colon A \times A \to A$, associating to each pair (x, y) in $A \times A$ a unique value $b(x, y)$ in the same set A. An *external binary operation* in a set A is a function $K \times A \to A$, where K is a set (of scaling elements).

[3]Hyperlinks to examples of student answer sheets are given at `http://www.joensuu.fi/mathematics/MathDistEdu/Animations2MentalModels/SavonlinnaLETTET2005/index.html`

[4]The framework of Model Construction of Didactic and Empirical Problems of Mathematics Education (MODEM) with all task types can be found in Haapasalo (2003), and even studied in detail with a CAL-program downloadable at `http://www.joensuu.fi/lenni/programs.html`

Concerning functions and especially binary operations, traditional static graphic paper-and-pencil methods become nowadays inadequate. Binary operations appear as two-variable functions, whose variables in linear algebra are usually vectors. Thus, a completely new "learning dimension" can be added by using dynamic figures, which require the learner to interact with them by dragging with the mouse or by using control buttons. These sketches, implemented for this research as interactive Java applets by means of Geometer Sketchpad and Geometria (see Pesonen 2001 and Ehmke 2001) contain text parts, figures, and geometrical elements (points, lines, rays, segments, circles and more advanced constructions) to be manipulated with mouse. Control buttons are used for showing, hiding, moving and animating. The dark dotted line in Figure 1 visualises how the image of the plane square $[-4,2]\times[-3,4]$ is traced when x and y are moved in their domains, either by mouse or by the Animate buttons.

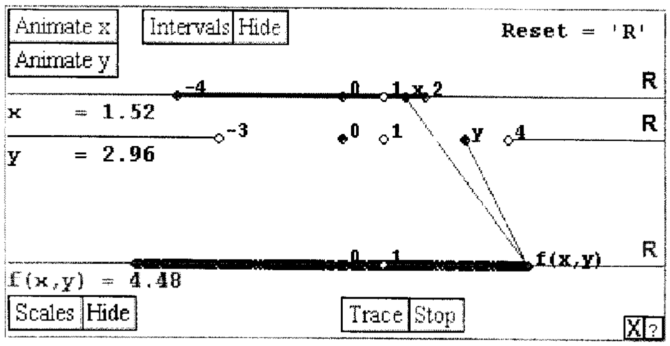

Figure 1: An example of an interactive graphical representation

Sketches were given with appropriate set of tasks (problems) by utilizing HTML pages. Three important leading principles behind the idea of using this kind of worksheet exercises were: 1) it must be possible to embed interactive figures into the worksheets, 2) the teacher must get information about what the students have done, at least receive their answers in electronic form, and 3) the work must be possible to be done outside the classroom.

3. Aims

This study addressed the following questions:

1. What kind of opinions on interactive applets do students demonstrate?
2. What kind of defects in metacognitive thinking cause problems for students' learning?
3. What kind of defects in the use of technology cause difficulties to students' learning?
4. What kinds of advantages and disadvantages appear in learning through interactive applets?

5. What kind of pedagogical tutoring would be necessary?
6. What kind of problems were caused by the use of *WebCT*[5]?

It also tried to generate a hypothesis of how far it might be possible to utilize distance learning in (tertiary) mathematics education.

4. Methods

The participants of the two studies were the students on a first course ($N = 82$) on Linear Algebra in the Mathematics Department of the University of Joensuu in Spring 2005. The outcomes of this piece of research are obtained from the data and experiences of three consecutive tests dealing with functions and binary operations. The tests were arranged using the course management system *WebCT*, and the students took the tests wherever they wanted during fixed time periods of about a week. The first test called *Functions* consisted of a worksheet with 46 questions about functions of one or two variables. A dynamic interactive figure was involved in 11 of the questions. Students' feedback of this test is in focus here because the test system was new to them, and also because two variable functions were just introduced. The second test called *Internal Binary Operations* had 33 questions concerning binary operations, and 13 of them contained an interactive sketch. The most important findings concerning the students' difficulties to utilize certain sketches containing special technical or mathematical features come from this test, which was structurally more coherent with the pedagogical framework. Some cognitive findings are represented just for considering possible explanations to these difficulties. The third test, a preliminary version called *External Binary Operations*, contained 24 tasks with 16 sketch problems. In order to provide the students with a possibility for 'learning by doing', each test could be done twice (with nearly identical content), and each student's average score was taken to be the final result. Students were encouraged to review the results—equipped with the teacher's comments—after the first trial. However, *Functions* was done twice by 18, *Internal Binary Operations* by 10 and *External Binary Operations* by 7 students only.

5. Results

5.1 Students' opinions of the tests

All three tests were closed with an open-ended feedback question, asking students to tell their opinions about the test questions. By 'opinions' we mean general attitude towards the test, evaluation of the test items and the reported difficulties. Figure 2 gives an overview on students' opinions, whilst Figure 3 illustrates them in more detail. In each test about one fifth of the students reported that the dynamic problems were suitable for learning and testing. However, more than one third complained about difficulty. This was probably caused by a new kind of representation form, which is not used in paper-and-pencil work. However, the number of

[5] Information on WEbCT is available at `http://en.wikipedia.org/wiki/WebCT`

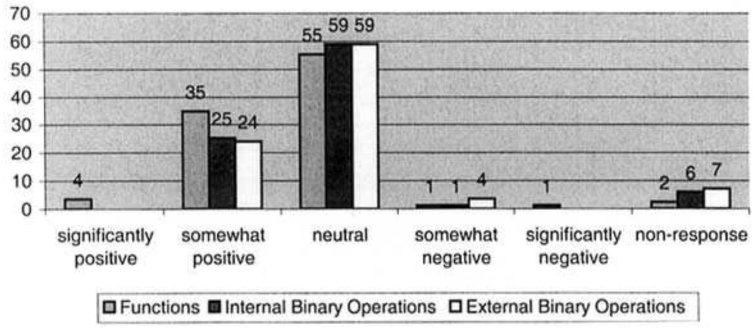

Figure 2: Percentages of students' general opinions on the tests

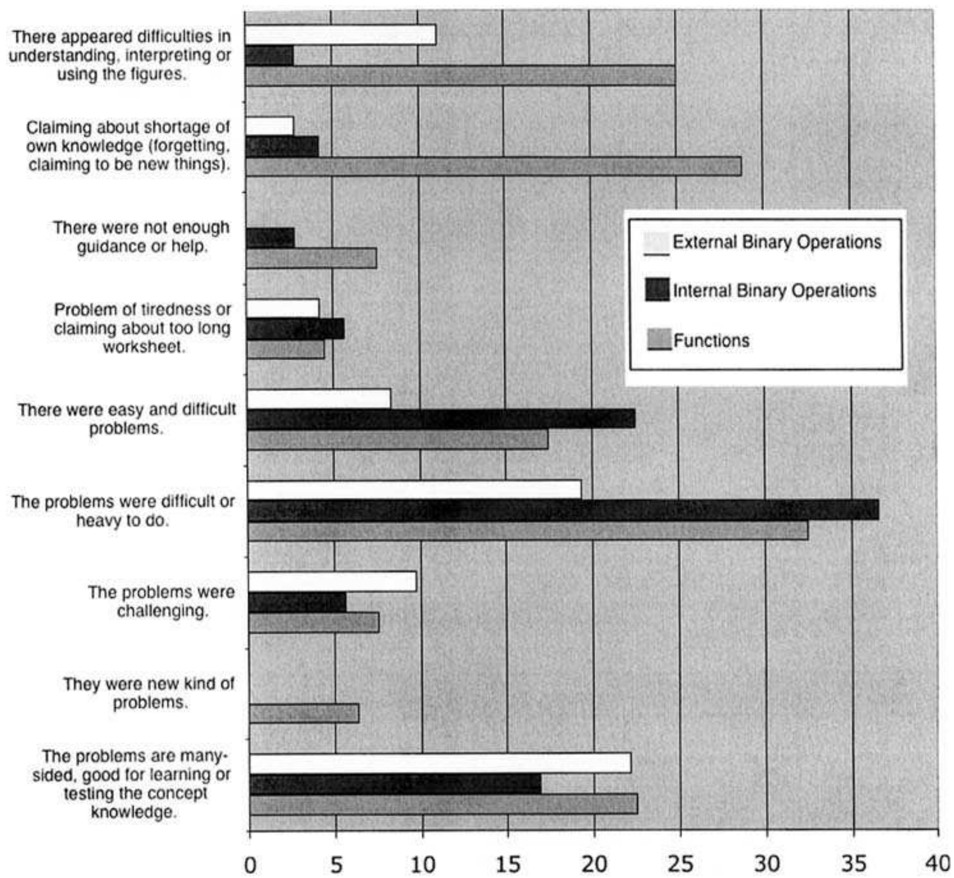

Figure 3: More detailed percentages of students' opinions on the tests

students who wanted a better guidance decreased from test to test, indicating that students became more familiar with the dynamic problems.

The first two bar triples in Figure 3 tell about special difficulties in handling dynamic sketches. In the *Functions* test 25 percent indicated difficulties in understanding, interpreting or using the interactive figures. In this test several new features of dynamic sketches (as dragging, tracing or animating) were introduced, whilst in the *Internal Binary Operations* test no new technical features were used. In the *External Binary Operations* test the applets were in a more significant role, and perhaps slightly more difficult in the mathematical sense. One could guess that this is the reason for the increasing percentage of students who complained about this. On the other hand, students' defects in conceptual knowledge seem to decrease strongly. It is obvious that when going through the Functions test nearly all students learned to handle the interactive features, and perhaps also learned the idea of two-variable functions.

We quote some typical expressions found in students' feedback in *Functions* (the percentages refer to their right answers):

- *The tasks were most suitable for testing one's mastery of the function concept.* (girl, 90%)
- *Java figures are hard to understand, how to get information out of them and what they mean. However, they are nice to do, a bit different from ordinary exercises.* (girl, 50%)
- *Especially the figure-based tasks difficult, because nothing alike was done before.* (boy, 38%)
- *Some problems easy, some not. Especially the problems concerning injection/surjections/bijections of two variable functions were not easy.* (boy, 75%)
- *The problems were difficult, since the concepts are not yet absorbed but sought. Training, training!* (girl, 34%)
- *Terrible tasks, even many of the questions are too difficult to understand.* (boy, first trial, 40%)
- *Well, it was moderately easy on the second try. Many problems were similar.* (same boy, second trial, 95%)

5.2 Defects in metacognitive thinking

Haapasalo (2003, 10) points out the importance of metacognitive thinking when using simple interactive sliders. He noticed that most students and teachers concentrate on irrelevant things when manipulating objects on the computer screen, whilst an expert's way of learning would be to change relevant elements of the problem. The same phenomena were clearly present in students work in our studies, when students changed too many objects like points or sliders on the screen at the same time, without seeing the essential problem elements. As well from students' mistakes in their answers as from our observations during students' activities in the classroom we conclude that in many cases the difficulty was not mathematical one but caused by the fact that students could not find the essential applet elements they could manipulate interactively. To avoid this kind of defect in their working it would be enough to have the very basic procedural problem solving skills (as 'change the elements of the problem', 'take a special case' in the sense of Polya (1973). A suitable example illustrating the gap between the concept entity and the different roles of the variables is the following sketch from the *External Binary Operation* test. The students had to write in a symbolic form the operation $(c, \mathbf{u}) \mapsto c * \mathbf{u}$ expressed via a dynamic sketch, see the screen shot in Figure 4.

About half of the students found a right answer like $c * \mathbf{u} = (u_1, cu_2)$. Two students told they cannot do it and 14 students gave an answer that did not contain the scalar c at all (e.g. $c * \mathbf{u} = \mathbf{u}$). Of these, 8 students gave an answer exactly or very near to $c*\mathbf{u} = (u_1, 2u_2)$. This can happen—most probably—if one does not move c on the scalar line; moving just u gives this impression.

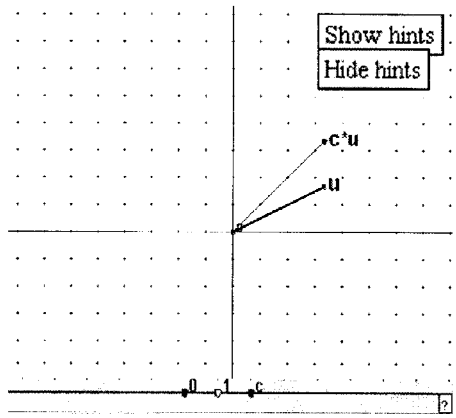

Figure 4: An external binary operation puzzle

5.3 Problems in the use of technology

WebCT uses for example Javascript in administrating the worksheet answer system. Conflict were met with when trying to use Javascript on other parts of the worksheet, like embedding the orientation module in the test upper text block. More obvious problems arise from the configuration of the equipment available for the students: the browser may not show Java applets and the mathematical notations may not be seen correctly. This we tried to avoid by leading the students into a test page before logging into *WebCT*. In spite of this, a couple of students reported problems in figures or fonts.

5.4 Advantages and disadvantages of interactive applets

The advantage of *IGR* is that students become engaged with the content and the problem setting and get a "feeling" for dependencies between the given parameters. Dynamic pictures offer new possibilities to solve problems (e.g. draw a trace or use scaling). Also an automatic response analysis, which enables immediate feedback, can support concept understanding and "learning when doing". Especially for the mathematical concept of binary operation in the plane the applets offer an advantagous possibility of a representation, which cannot be visualized by a blackboard or paper picture. On the other hand, this new kind of representation form is unfamiliar for many students (Pesonen et al. 2005). Among other disadvantages we have to mention that computer activities are time consuming, especially when getting oriented to new kind of working culture. Furthermore, the problems concerning assessment and curriculum design need a thorough reconsideration.

5.5 The need of pedagogical tutoring

It is our position that tutoring in a technology-based learning environment is to be considered from two perspectives: an appropriate pedagogical framework and tutorial measurements from the teacher's side during the working. As regards the first one, we must point out clearly that in our study, restricting only in few hours of work, students did not have opportunities to work within an appropriate pedagogical framework in whole its spirit, as within the MODEM framework, for example. It is the viable definition for the concept through students' social constructions that would have been the ideal case. We used definition tasks only for assessment in the same way as we did with identification and production tasks, applying also the theories of concept images and concept definitions (e.g. Vinner & Dreyfus 1989; Vinner 1991). The applets were from the very beginning aimed to whitewash students' naïve and stereotypic conceptions based on school mathematics, and to increase their sensitivity to use different glasses when looking at mathematical objects.

Concerning the question about what kind of help students need during their working, we stress the importance of as well metacognitive as technical tutoring. Problems illustrated in chapter 5.2 hardly can be solved by adding hints in written or spoken form on the screen but by a face-to-face tutoring. The same yields in many cases also technical problems by using the interactive materials. For avoiding data overflow on computer screen, it might be reasonable, for example, to utilize audio solutions to give hints in oral form or to highlight current problem elements by a particular sound.

5.6 Remarks on the WebCT environment

We chose the *WebCT* administration not only because it was the best available system for our purpose but because it allows automatic checking of most answer types and supports a variety of data manipulations. Although the course material for Linear Algebra included texts and paper-and-pencil sheets, we did not find any reason to embed these into the complicated *WebCT* file management system, but kept them on a www server.

In *WebCT* the test questions can be authored using plain text style or html code. The latter allows the most important features we need: mathematical formulae, static pictures and interactive Java applet figures. In addition to that, there are several advantages in using *WebCT*:

- it is easy to use, and many students know the system in advance,
- after attending the test the students can see the whole worksheet equipped with their own answers together with the correct answers and comments written by the teacher,
- a worksheet html source code can be sent automatically by email,
- a test can be usually corrected automatically, or at least by making minor revisions,
- data can be examined, manipulated and stored in many ways,

- the system can be maintained and monitored from any internet-connected workstation,
- the system is supported by the institution.

Perhaps the most evident inconvenience descends from the subject mathematics itself. Being a general system, by *WebCT* most of the special features to support a mathematically oriented system are missing, as the possibility to check answers by a computer algebra engine, for example. However, there is a modest arithmetical "calculated" problem type, which allows simple randomized problems to be used, and the system can check a numerical answer up to some tolerance level. With this feature it is possible to create problems containing for example vectors.

Our test worksheets contain, in addition to multiple choice questions, also "short answer" questions, which can, at least in principle, be corrected automatically according to a list of acceptable answers provided by the teacher. However, since the checking is done by comparing character strings, not all good answers are identified by the system.

Let us finish with a list of problems or deficiencies that we have found:

- the lack of support for (higher) mathematics,
- the system is not easy to use for the authors, e.g. navigation is complicated and running slow,
- although the problems of a test can be shown either one at a time or all at the same time as a long document, the evaluation and teacher comments cannot be seen before answering all the questions. Therefore the test system cannot be used efficiently for delivering learning material ("learning when doing", "exam as a learning tool"),
- it is not possible to correct all the answers to a certain problem manually in a row.

6. Conclusions

We now come back to our effort to generate a hypothesis of how far it might be possible to utilize distance learning in (tertiary) mathematics education. Our exemplary results show that even though interactive learning modules offer new advantages and features for teaching and learning mathematics, they do not seem to bring special advantages without an appropriate pedagogical framework. Especially the interactive graphical representation of mathematical content can be serious from the learning point of view, as reported in Sierpinska et al. (1999) and (Pesonen et al. 2005). We would like to quote the *IBMT* (*Interaction Between Mathematics and Technology*) principle by Kadijevich et al. (2004): "When using mathematics, do not forget available tool(s); when utilising tools, do not forget the underlying mathematics". In other words, mathematics cannot only direct the tool utilisation, but also help us to achieve it in a more efficient and suitable way. In such a way, mathematically-grounded "button pressing" would not compromise thinking but rather enhance it for the benefit of the learner. Even though it is well known that technology can shift mathematics teaching from paper and pencil work towards interactive learning, an adequate pedagogical theory is needed for planning and realising learning environments. While the focus of school teaching is often to reach sufficient procedural knowledge, the teaching of university mathematics aims at high conceptual understanding. For both of these, we have to be ready to handle the following dilemma: Should the student need to understand in order to be able to do, or vice versa (cf. Haapasalo 2003)? Perhaps the most promising aspect of technology-based learning (as IGR) is to utilise the principle of simultaneous activation. This allows the teacher to be freed from the worry about the order in which student's mental models develop when interpreting, transforming and modelling mathematical objects. Our examples hopefully show that more or less systematic pedagogical models connected to an appropriate use of technology can help the teacher to achieve this goal. Interactive applets can be used not only for learning but also for assessment and for increasing new kinds of complexity for the content. It is evident that even university mathematics can be learnt outside institutions by utilising web-based interactivities. Our IGR studies suggest that most students' difficulties appear in the steps of mathematising and interpreting. To validate this result, the correlation between test performance in IGR problems and in problems represented in symbolic form should be examined even more thoroughly. Furthermore, qualitative research into students' thinking processes would probably give valuable information for developing appropriate learning environments. The on-going research in the $DAAD$ project will focus on these questions.

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