

DIVISION – A SYSTEMATIC SEARCH FOR TRUE DIGITS

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Abstract. For the sake of an easier understanding of the procedure of division, first this operation is conceived as continual subtraction. Taking into account place values of groups of digits of dividend, while proceeding with successive subtraction of divisor from such groups a slow, and in the real time feasible method of division is established. Based on this method, it is shown how a long division (quotients having two or more digits) is reduced to the short divisions (quotients are one-digit). In the case of short division, the first (second) guide number of divisor is defined which rounds down the corresponding dividend. Increasing by one the first (second) digit of divisor, it becomes rounded up and by dividing these guide numbers by the numbers that are increased by one, two methods of division are obtained, both having their advantages and disadvantages. In Section 6, a combined method is presented that should be at the top of practice in division.

All three of these methods are clearly defined algorithms which are neat (free from trial and error correcting and erasing) and they successively produce true digits.

This paper is intended for teachers, and it has a form containing all necessary details and examples that are within their reach.

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1. Introduction

Discovered in the remote past, school arithmetic facts are, so to say, eternal truths. Fixed once for ever, these facts cannot be changed but what we can change are the methods of their didactical transposition. And it is quite surprising that after so many generations of teachers, there still exists room for further improvement of these methods.

We call decimalization the process and the results of transforming arithmetic expressions into their unique decimal notations. Being the main cases of decimalization, transformations of sums, differences, products and quotients are topics which comprise a great deal of arithmetic course for fourth graders.

The skill of doing long calculations has definitively lost its practical value that it once had and a pocket calculator greatly surpasses such our abilities. Now the focus is turned on a better understanding of calculating algorithms and their skilful performing at least up to the degree which supports this understanding. Due to the distinctive developmental features that these algorithms possess, they have their prominent place on school curricula and in no way they should be neglected topics.

A particularly difficult topic is the division by several-digit divisors. It is a common practice to illustrate this complicated process with a few examples and then to expect children to carry on from there. As a result of such an unsatisfactory elaboration, many educated people (among them many teachers) have a poor skill in performing and no adequate understanding of division. For many children, estimation of quotient digits by “trial and error” method is a road to confusion and discouragement. Since the first guess is often wrong, some teachers tell children to work neatly and to have an extra sheet of paper for practicing this “trial and error” method. This kind of arising uncertainty in performing division stands in a marked contrast with directly feasible algorithms for addition, subtraction and multiplication. All in all, division is the most tortuous theme of primary school mathematics.

In the first decades of 20th century, there had been much ink spilled over division. (See the book: Guy M. Wilson, *Teaching New Arithmetic*, Mc Graw-Hill Book Company, Inc., New York, 1951). Even though several procedures of estimation of quotient digits were then

suggested, none of them had a clear perception of a general method free from trial and error correcting, erasing and starting over again.

Here, we propose a way of successive determination of the true quotient digits which will be based on two well-defined algorithms. To give a narrative description of this procedure, we could say that the trial digits are good when not exceeding the true digit, and being closer to it, better they are. And as it will be seen later, all written forms following this determination are neat and they contain details in plain sight.

This paper is aimed at a full understanding of the process of decimalization of quotients and, therefore, it may be of a large interest for teachers. That is why we shall use a lot of examples in order to make this process clearly seen as gradual improving of continued subtraction.

2. Division as continual subtraction

The procedure of division that is the easiest to be understood is the continual subtraction of the divisor. For instance, to calculate the value of the quotient $3864:84$, we have to be subtracting continually the same number 84 until we reach the remainder 0.

Denoting elliptically this lengthy procedure by:

$$\begin{array}{r}
 3864:84 \\
 \underline{84} \\
 3780 \\
 \underline{84} \\
 3696 \\
 \dots \\
 84 \\
 \underline{84} \\
 0
 \end{array}$$

we compress in that way 46 repeated subtractions of the number 84. Thus the result of such a “division” is the number 46.

Following the same procedure, in the case of division $3864:46$, we would have 84 repeated subtractions of the same number 46.

In the case of divisions $3865:84$ and $3887:84$, there would be 46 repeated subtractions in each case, with the remainders 1 and 23 respectively. Hence, we can write

$$3864:84 = 46, \quad 386\underline{5}:84 = 46, \quad 388\underline{7}:84 = 46 \\
 \qquad \qquad \qquad 1 \qquad \qquad \qquad 23$$

or, rewriting these relations in the inverse form, we have

$$3864 = 84 \cdot 46, \quad 3865 = 84 \cdot 46 + 1, \quad 3887 = 84 \cdot 46 + 23$$

Being too lengthy, this procedure cannot, of course, be taken as being a method of calculation of quotients. It is just a first step and the procedure that we describe next reduces considerably the number of repeated subtractions. What makes the difference is that we will use place values of groups of digits, while decomposing dividend into sums that will be divided successively.

Let us take again as an example the same quotient $3864:84$. On the left-hand half of the page we shall expose how this algorithm runs and on the right-hand we shall describe in words the running steps.

a) $3864:84 = 11$

$$\begin{array}{r}
 \underline{84} \quad 11 \\
 302 \quad 11 \\
 \underline{84} \quad 11 \\
 218 \quad 1 \\
 \underline{84} \quad \underline{1} \\
 134 \quad 46 \\
 \underline{84} \\
 504 \\
 \underline{84} \\
 420 \\
 \underline{84} \\
 336 \\
 \underline{84} \\
 252 \\
 \underline{84} \\
 168 \\
 \underline{84} \\
 84 \\
 \underline{84} \\
 0
 \end{array}$$

We divide 386 tens by 84, subtracting successively 84, starting with 386. We get 4 tens and 50 tens remain. Subtracting each time, we write 1 in ten column.

We bring down 4 units, continuing to divide 504 units by 84. Following each subtraction by writing 1 in the unit column, we get 6 units.

Altogether, it is 46 units.

Let us notice that this “intermediate” method shortens considerably the number of subtractions and, in the example under consideration, instead of 46 we have only $4 + 6 (=10)$ of them.

For the sake of a further explanation of this method, let us consider one more example.

b) $14848:64 = 111$

$$\begin{array}{r}
 \underline{64} \quad 111 \\
 84 \quad \underline{1} \\
 \underline{64} \quad 232 \\
 204 \\
 \underline{64} \\
 140 \\
 \underline{64} \\
 76 \\
 \underline{64} \\
 128 \\
 \underline{64} \\
 64 \\
 \underline{64} \\
 0
 \end{array}$$

We divide 148 hundreds by 64, subtracting twice. We get 2 hundreds and 20 hundreds remain.

We bring down 4 tens, continuing to divide 204 tens by 64. We get 3 tens and 12 tens remain.

Bringing down 8 units, we continue to divide 128 units by 64. We get 2 units.

Altogether, it is 232 units.

3. Long division splits into the short ones

In the example a) of the previous section, we divided by 84 the following numbers:

386, obtaining the quotient 4 and the remainder 50,
504, obtaining the quotient 6 and the remainder 0.

In the example b) of the same section, we divided by 64 the numbers:

148, obtaining the quotient 2 and the remainder 20,
204, obtaining the quotient 3 and the remainder 12,
128, obtaining the quotient 2 and the remainder 0.

The term *long division* will be taken here to mean the cases of division when the corresponding quotients have two or more digits. Thus, we see that the procedure of long division splits into the cases of division when quotients are one-digit numbers. Such a case of division we will call *short division*. Also, let us call the numbers such as 386 and 504 are in the example a), and 148, 204 and 128 are in the example b), *main parts of the dividend*. Using these terms, we see that each long division reduces to the cases of short division of the main parts of dividend.

Let us also observe that the number of main parts of dividend is equal to the number of places of quotient. By selecting the first main part, the remaining digits of the dividend are in the one-to-one correspondence with the main parts that follow. Thus, we see that the number of those remaining digits plus one is equal to the number of places of quotient. For example, when the numbers:

805, 8054, 80542, 3846, 38462, 384624

Are divided by 789, we know in advance that the number of places of the corresponding quotients will be: 1, 2, 3, 1, 2, 3 respectively.

To the cases of short division, the term *guide number of dividend* will also be related. Namely,

(i) The number represented by the first digit of dividend is its guide number if both dividend and divisor have the same number of digits,

(ii) The number represented by the first two digits of dividend is its guide number if dividend has a digit more than divisor.

Before we proceed further describing the ways of successive determination of true digits, let us state two simple cases of short division, when the true quotient digit is easily found:

(i) Dividend is less than divisor and then, the true digit is 0,

(ii) Dividend is not less than divisor but its guide number is less than the doubled value of the number represented by the first digit of divisor and then the true digit is 1. These two cases will be called *trivial*.

For example, when the divisor is 704, and when dividend belongs to the range of numbers from 0 to 703, the true digit is 0 and when it belongs to the range from 704 to 1399, the true digit is 1.

4. First variant of “increase-by-one” method

Division by one-digit is just a continual repetition of division with remainder which is usually covered first within the block of numbers 1 – 100. (See, for example, this author’s paper: A broader way through the themes of elementary school mathematics VI, This Teaching, 2004, vol. VII, 1, pp. 35 – 52, on the Web: <http://elib.mi.sanu.ac.yu/journals/tm>).

Here is a couple of examples:

$$\begin{array}{r} 7488:8 = 936, \\ \underline{28} \end{array} \quad \begin{array}{r} 5313:7 = 759 \\ \underline{41} \end{array}$$

$$\begin{array}{r} 48 \\ 0 \end{array}$$

$$\begin{array}{r} 63 \\ 0 \end{array}$$

where all short divisions are performed mentally, producing in that way true digits and corresponding remainders. Division by 10 is also added to these simple cases of division that are performed mentally.

Now we turn our attention to the cases of division by divisors having two or more digits. Since in the more complex cases of such division the true digits cannot be found at once, we apply the “*increase-by-one*” method for their successive determination. This procedure consists of the performance dependent of one of the following steps:

(i) Dividend is less than divisor, when the true digit is 0,

(ii) Dividend is not less than divisor but both have the same number of digits and their first digits are equal, when the true digit is 1,

(iii) In all remaining cases, the first digit of divisor is increased by one and the guide number of dividend is divided by that number, obtaining so an apparent quotient. Then, this quotient is used to multiply the divisor and to subtract that product from the dividend. If the remaining number is not less than the divisor, the same procedure is repeated again.

Now we use a number of concrete examples to demonstrate how this method works. To shorten the way of expression, when a number is mentally divided by another, we will not write remainders in the part of the text describing the procedure in words. Instead of it we will write the ellipsis.

c) $386:84 = 4$

$$\begin{array}{r} 336 \\ \underline{50} \end{array}$$

8 is increased by 1: 38 divided by 9 is 4 ...

By multiplying: $84 \cdot 4 = 336$ and subtracting, 50 remain.

d) $504:84 = 5$

$$\begin{array}{r} 420 \\ \underline{84} \quad \underline{1} \\ 84 \\ \underline{84} \\ 0 \end{array}$$

50 divided by 9 is 5 ...

After multiplying and subtracting, 84 remain.
Division continues: 84 divided by 84 is 1.
The remainder is 0.

e) $1484:451 = 2$

$$\begin{array}{r} 902 \\ \underline{582} \quad \underline{1} \\ 451 \\ \underline{131} \end{array}$$

14 divided by 5 is 2 ...

$451 \cdot 2 = 902$ and after subtracting, 582 is obtained.

5 divided by 5 is 1. $451 \cdot 1 = 451$ and after subtracting, 131 is obtained as the remainder.

Now we demonstrate this method on a couple of examples of long division.

f) $5561:83 = 66$

$$\begin{array}{r} 498 \\ \underline{581} \quad \underline{1} \\ 498 \\ \underline{83} \\ 83 \\ \underline{83} \\ 0 \end{array}$$

55 divided by 9 is 6 ...

58 remain.

1 is taken down. 58 divided by 9 is 6 ...

6 is written in the position of the second place of the quotient.

83 divided by 83 is 1.

The remainder is 0.

g) $1060164:601 = 1653$

$$\begin{array}{r}
 \underline{601} \\
 4591 \\
 \underline{3606} \\
 985 \\
 \underline{601} \\
 3846 \\
 \underline{3005} \\
 841 \\
 \underline{601} \\
 2404 \\
 \underline{1803} \\
 601 \\
 \underline{601} \\
 0
 \end{array}$$

h) $1060164:1764 = 501$

$$\begin{array}{r}
 \underline{8820} \\
 1781 \\
 \underline{1764} \\
 1764 \\
 \underline{1764} \\
 0
 \end{array}$$

10 divided by 2 is 5.

$176 < 1764$ and the true quotient digit is 0.

The following two examples may be considered as “unfavorable” for the application of this form of “increase-by-one” method, since even five steps are needed to arrive at the true digit.

i) $99:11 = 4$

$$\begin{array}{r}
 \underline{44} \quad 2 \\
 55 \quad 1 \\
 \underline{22} \quad 1 \\
 33 \quad \underline{1} \\
 \underline{11} \quad 9 \\
 22 \\
 \underline{11} \\
 11 \\
 \underline{11} \\
 0
 \end{array}$$

j) $987:101 = 4$

$$\begin{array}{r}
 \underline{404} \quad 2 \\
 583 \quad 1 \\
 \underline{202} \quad 1 \\
 381 \quad \underline{1} \\
 \underline{101} \quad 9 \\
 280 \\
 \underline{101} \\
 179 \\
 \underline{101} \\
 78
 \end{array}$$

In the section that follows, this method will be refined further. At the end of this section, some relevant comments have to be made.

Let us observe that the guide number of dividend together with 0’s put in all its other places, makes dividend to be rounded down. Increasing by one the first digit of divisor and putting 0’s in all its other places, the divisor is rounded up. The ratio of these two rounded numbers is equal to the ratio of the guide number to the number obtained by increasing the first digit of divisor. On the other hand, that ratio is equal or less than the ratio of dividend to divisor. It is just this fact that makes the “increase-by-one” method to lead uniformly towards the true digits.

In an “increase-by-one” method, suggested by T. E. Grossnickle (The Elementary School Journal, 32, 1931, pp. 299 – 306), when the first digit of divisor is 5 or less, it stays

unchanged and when 6 or more, it is increased by one. Applying his method, estimated quotients come closer to the true ones than in the case of our method, but the changes may be increases and decreases. Nevertheless, the mentioned uniformity of our method is a feature which makes it to be a convenient ground for further refinements.

To point out an advantage and a disadvantage of the “increase-by-one” method, let us observe that procedure of division is reduced, in each step, to an easier case of division (with a smaller dividend) but the number of these steps may be even five.

As for children, they have a direct method to learn to use without being under strain to know how to estimate. And only after they have mastered this method, they should proceed with further, more efficient procedures.

Let us also add that, in applying this method, short division within the block of numbers up to 100 is the only division done mentally.

5. Second variant of “increase-by-one” method

Our considerations in this section are also confined to the cases of short division. Exempting again the division by one-digit divisors as well as the cases when dividend is smaller than divisor or if not smaller, they both, dividend and divisor, have the same number of digits and their first digits are equal, we subject all other cases to a stronger variant of “increase-by-one” method.

In this context, if both dividend and divisor have the same number of digits, the *second guide number of dividend* is the number represented by its first two digits, and if not, such number will be that represented by the first three digits of dividend.

In the examples that follow, the digits of the second guide numbers are colored blue:

$$8542:2531, \quad 1327:459, \quad 564:81.$$

In describing this new method, two cases are distinguished:

(i) When divisor is two-digit, the trial digit is equal to the number of times the divisor is contained in the second guide number.

(ii) When divisor has three or more digits, the number represented by the first two digits of divisor is increased by one. Then the trial digit is equal to the number of times this increased number is contained in the second leading number.

Multiplying the divisor by the trial number and subtracting that product from the dividend, a number remains. If the remaining number is not less than the divisor, the same procedure continues.

To interpret this second variant, first we will redo examples under i) and j).

$$\text{k) } 99:11 = 9$$

$$\begin{array}{r} 99 \\ 0 \end{array}$$

99 divided by 11 (no increasing) is 9.

$$\text{l) } 987:101 = 8$$

$$\begin{array}{r} 808 \\ 179 \\ \hline 101 \\ 78 \end{array}$$

98 divided by 11 is 8 ...

179 divided by 101 is 1 ...

78 remain.

$$\text{m) } 2838:473 = 5$$

$$\begin{array}{r} 2365 \\ \hline 1 \end{array}$$

283 divided by 48 is 5 ...

$$\begin{array}{r} 473 \quad 6 \\ \underline{437} \\ 0 \end{array}$$

437 divided by 437 is 1.

n) $51732:6458 = 7$

$$\begin{array}{r} \underline{45206} \quad \frac{1}{8} \\ 6526 \\ \underline{6458} \\ 68 \end{array}$$

517 divided by 65 is 7 ...

6526 divided by 6458 is 1 ...

68 remain.

o) $91107:10123 = 8$

$$\begin{array}{r} \underline{80984} \quad \frac{1}{9} \\ 10123 \\ \underline{10123} \\ 0 \end{array}$$

91 divided by 11 is 8 ...

10123 divided by 10123 is 1.

This second variant of “increase-by-one” method is evidently much more efficient than the first one. Its noticeable disadvantage is the hard mental performance of short division, when divisor is a two-digit number and dividend is any number less than tenfold value of that divisor. In particular, such calculations are difficult for children to perform them mentally.

For a rapid but mentally less demanding performance of division procedures, we suggest a combined method that will be exposed in the next section.

6. Combining two variants

To facilitate this exposition, we will use here the shortened forms of some frequently involved expressions. Thus, instead of using the term “first (second) variant of “increase-by-one” method, we will speak of “first (second) method.” Also the guide number of dividend related to the first variant will be called “first guide number” and that related to the second variant “second guide number”. For instance, in the case of examples:

$$833:249, \quad 2533:349$$

The respective first guide numbers are 8 and 25, while the second ones are 83 and 253.

If the full extent of mental division by two-digit divisors has to be considered as difficult for children, the inclusion of those cases when divisors are less than 200 requires only some extra practice in mental performing of such divisions. Not including two trivial cases, the first step in performing short division in this combined way will be the application of the second method if the guide number is less than 200 and, if not, the first method will be applied. Thus, this procedure of division starts with the comparison of the second guide number and the number 200. How it proceeds further will be seen in a number of examples that follow.

p) $6063:1294 = 4$

$$\begin{array}{r} \underline{5176} \\ 887 \end{array}$$

$60 < 200$ and the second method applies.

60 divided by 13 is 4 ...

q) $387:129 = 2$

$$\begin{array}{r} \underline{258} \quad \frac{1}{13} \end{array}$$

$38 < 200$ and the second method applies.

38 divided by 13 is 2 ...

$$\begin{array}{r} 129 \quad 3 \\ \underline{129} \\ 0 \end{array}$$

129 divided by 129 is 1.

r) $3258:638 = 4$

$$\begin{array}{r} \underline{2552} \quad \underline{1} \\ 706 \quad 5 \\ \underline{638} \\ 68 \end{array}$$

$325 > 200$ and the first method applies.
 32 divided by 7 is $4 \dots$
 $70 < 200$ and the second method applies.
 70 divided by 64 is $1 \dots$

s) $2992:374 = 7$

$$\begin{array}{r} \underline{2618} \quad \underline{1} \\ 374 \quad 8 \\ \underline{374} \\ 0 \end{array}$$

$299 > 200$ and the first method applies.
 29 divided by 4 is $7 \dots$

t) $63824:7856 = 7$

$$\begin{array}{r} \underline{54992} \quad \underline{1} \\ 8832 \quad 8 \\ \underline{7856} \\ 976 \end{array}$$

$638 > 200$ and the first method applies.
 63 divided by 8 is $7 \dots$
 $88 < 200$ and the second method applies.
 88 divided by 79 is $1 \dots$

u) $1435:179 = 7$

$$\begin{array}{r} \underline{1253} \quad \underline{1} \\ 182 \quad 8 \\ \underline{179} \\ 3 \end{array}$$

$143 < 200$ and the second method applies.
 143 divided by 18 is $7 \dots$
 182 divided by 179 is $1 \dots$

Let us also consider this combined method being applied to a couple of examples of long division.

v) $241613:41 = 4773$

$$\begin{array}{r} \underline{164} \quad \underline{112} \\ 77 \quad 5893 \\ \underline{41} \\ 366 \\ \underline{287} \\ 79 \\ \underline{41} \\ 381 \\ \underline{287} \\ 94 \\ \underline{82} \\ 123 \\ \underline{123} \\ 0 \end{array}$$

24 divided by 5 is $4 \dots$ (first method)
 77 divided by 41 is $1 \dots$ (second method)
 36 divided by 5 is $7 \dots$ (first method)
 79 divided by 41 is $1 \dots$ (second method)
 38 divided by 5 is $7 \dots$ (first method)
 94 divided by 41 is $2 \dots$ (second method)
 123 divided by 41 is 3 , (second method)

w) $1864825:3845 = 484$

$$\begin{array}{r} \underline{15380} \quad \underline{1} \\ 32682 \quad 485 \end{array}$$

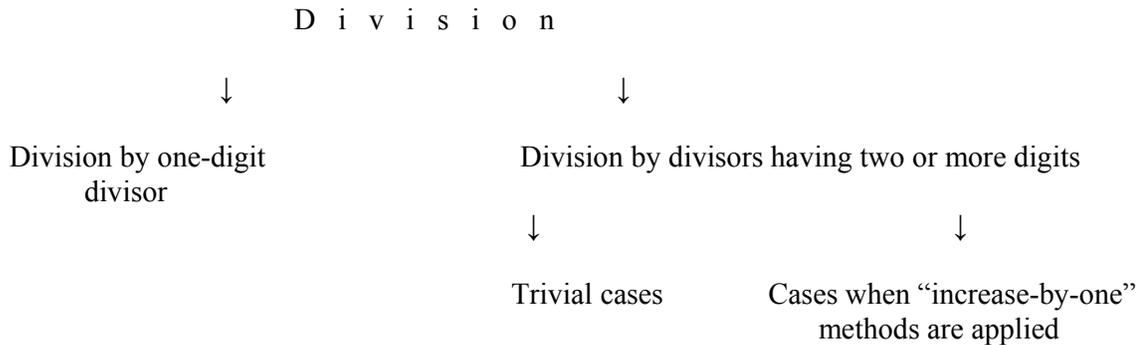
186 divided by 39 is $4 \dots$ (second method)
 32 divided by 4 is 8 (first method)

$$\begin{array}{r}
 30760 \\
 \underline{19225} \\
 15380 \\
 \underline{3845} \\
 3845 \\
 \underline{3845} \\
 0
 \end{array}$$

192 divided by 39 is 4 ... (second method)

3845 divided by 3845 is 1

At the end, we use the following scheme to show how the cases of division are classified according to the way of their performance.



7. Addendum

Inspecting the following example

$$\begin{array}{r}
 \text{x) } 3499:401 = 6 \\
 \underline{2406} \quad \underline{2} \\
 1093 \quad \quad 8 \\
 \underline{802} \\
 291
 \end{array}$$

34 divided by 5 is 6 ... (first method)

109 divided by 41 is 2 ... (second method)

we see that 348 as the second guide number of dividend is larger than 200 and, in the first step of this procedure, the first method is applied. The remaining number 1093 is not less than the divisor and its guide number is 109. Being that number less than 200, in the second step, the second method is applied. Now we will prove that this same pattern occurs in all such cases.

Let $abc\dots$ be a dividend and abc its second guide number, where the letters a, b, c denote its digits. Let $\alpha\beta\dots$ be a divisor, where the letters α, β denote its digits. Let us suppose that $abc > 200$ and that the first method applies. If ab divided by $\alpha + 1$ is k , then k is the unique natural number such that

$$(k + 1)(\alpha + 1) > 10a + b \geq k(\alpha + 1).$$

If the remaining number is not less than divisor, let δ be its second guide number. Then,

$$\delta \leq 100a + 10b + c - k(10\alpha + \beta) = 10(10a + b) + c - k(10\alpha + \beta)$$

$$\begin{aligned}
&\leq 10(k+1)(\alpha+1) + c - k(10\alpha + \beta) \\
&= 10\alpha k + 10\alpha + 10k + 10 + c - 10\alpha k - \beta k \leq 10(\alpha + k + 1) + c \\
&\leq 10 \cdot 19 + 9 = 199.
\end{aligned}$$

Thus, as a second step in this procedure of division, the second method can be applied if, of course, that case is not trivial.