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To the sacred memory of my father
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ON THE PSYCHO-LINGUISTIC BASIS OF TEACHING MATHEMATICS IN ENGLISH AS A FOREIGN LANGUAGE

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Abstract. This paper is an attempt of the author to answer the question "What is the psycho-linguistic basis of teaching Mathematics in English to non-native speakers?" The idea presented is based on Piaget's Theory of Cognitive Development and Chomsky's Extended Standard Theory (Transformational Syntax). The core of the author's idea is the proposal that there exists a connection between Piagetian operations and operational schemes, on one hand, and Chomsky's Deep Structures from which the sentences describing these operations have been derived, on the other. Two SAT coaching courses have been explored and much literary evidence found in support of this idea. As many as possible examples taken from these books have been included to help teacher's practice. At the end, the author expresses his conviction that this approach can be most generally applied to the formation of a psycho-linguistic basis of using English for specific purposes.

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1. Introduction
(Is it easy to teach Mathematics in English or is it a problem?)

When speaking on teaching Mathematics in English we usually have in mind students at a certain level of knowledge both in the language and Mathematics. As regards to English these students are at or up the intermediate level. The same could be said about their abilities to deal (adequately to their age) with mathematical matter. In the presentation below our attention will be focused on the intermediate language level students who have just begun studying Mathematics in English. In most cases they are students who have just passed the preparatory class in schools providing intensified teaching in English. A psychological characteristic of these intermediate level students will be given below. What is typical for such students is that their knowledge of English does not appear to be mathematically oriented; for example they have perceived the term triangle but usually cannot describe in English a classification such as scalene, isosceles, equilateral or acute, right, obtuse triangles. Moreover, and it is the most important thing, they have not taken in any ability to think on mathematical objects and relations using English terminology and expressions, which are sometimes very specific ones. So, the teacher who intends to teach this subject in English faces a variety of problems concerned with building and developing of the student's:

a) mathematical vocabulary in English;

b) system of vehicles of expressions which linguistically provide the specific process of mathematical thinking.

Our presentation elaborates the second group of problems. The basic of them are as follows:

a) finding an appropriate theory of the forms of human thinking;

b) carrying out a specific study to reveal the typical activities in the process of learning and using Mathematics;

c) carrying out a study on the specific linguistic structures describing the forms and activities mentioned in a) and b);

d) finding a theory of specific to Mathematics transformations of the structures mentioned in c).

The next part is going to be devoted to solving the problems just brought.

2. The three *corner stones* of the study

The thesis presented is methodologically based on the nature of mathematical knowledge, Piaget's concept of mental activities, and Chomsky's Transformational Syntax. Each one of these corner stones is going to be presented separately.

2.1. Briefly on Mathematics Itself

After Lawrence [6], Mathematics is an example of a postulational system in which a beginning set of postulates and undefined terms are used as a starting point in developing new relationships that are expressed as statements called theorems. These theorems, together with the postulates and defined and undefined terms, can be used to prove other theorems. To "prove" means to demonstrate a logical chain of reasoning which uses undefined terms, definitions, postulates, and previously established theorems to arrive at a new generalization. In this way an expanding set of related mathematical properties and relationships can be discovered, as well as an appreciation for the nature of a mathematical proof. Such a presentation of the entity of Mathematics as a science, albeit very brief, keeps its plausibility when looking at it as a subject to be taught and learnt. Hence, a list of basic activities typical of teaching and studying Mathematics must include:

a) defining a notion;

b) formulation of theorems, assertions and problems;

c) creating and describing exact logical chains of reasoning which lead from given presuppositions to certain conclusions.

At a lower level of action other activities appear to be often used:

a) notation of a mathematical object;

b) relative comparison between two variables.

It can be looked at all these activities as a result of some mental operations. Which are they and how do they function? From the author's standpoint the most suitable psychological theory to answer these questions appears to be Piaget's Theory of Cognitive Development.

2.2. Some Basic Concepts of Piagetian Theory of Cognitive Development

One of the basic ideas of Piagetian theory is that the human behaviour and thinking (as a mental behaviour) follow certain models which Piaget calls schemes. He points at two classifications of these schemes:

a) depending on their complexity:

– simple, concerned with some elementary action; e.g. addition of integers;

– complex, concerned with human behaviour under more complicated conditions; e.g. finding a contradiction as a result of some impossible presupposition;

b) depending on the object which a person deals with:

– behavioural, concerned mainly with physical objects;

– cognitive, concerned with objects from the mental reality (problems to be solved, concepts to be classified).

The paper presented puts the cognitive schemes in the focus of the study. Three processes, interrelated with each other, can be distinguished and observed as three aspects of the cognitive act.

2.2.1. Assimilation. It is a process of incorporating a new object or event into an existing scheme. For example, having in mind that $\frac{A}{A} = 1$ the student can guess that $\frac{a-b}{a-b} = 1$. In this way, he has already realized the incorporation of a new object $\frac{a-b}{a-b}$ into the existing scheme $\frac{A}{A} = 1$.

2.2.2. Accommodation. Accommodation as a mental process arises when the old scheme refuses to assimilate a new object or event. In such a moment consciousness might change the existing scheme to fit new reality. For example, the scheme $\frac{A}{A} = 1$ will not accept the new object $-\frac{b-a}{a-b}$. But after doing the following transformation $-\frac{b-a}{a-b} = -\frac{-(a-b)}{a-b} = \frac{a-b}{a-b} = 1$ a new (expanded) scheme $-\frac{-A}{A} = 1$ arises and becomes a fact of consciousness.

2.2.3. Equilibration. Let us go back to the situation that could not be fully handled by the existing scheme. This, in Piaget's theory, creates a state of disequilibrium between what is understood and what is encountered. People naturally try to reduce such imbalances by focusing on the stimuli that cause the disequilibrium and developing new schemes or adapting old ones until equilibrium is restored. This process of restoring balance is called equilibration.

Let us look at how these three processes work together in an example that has place in teaching Mathematics in English as a foreign language.

Let S be the system of mathematical knowledge of a student. Let S_E be that part of S which he is able to express in English. Let T_E be that part of the system of English language which the student is able to use. In this case it can be said that there exists a balance between the mathematical knowledge S_E and the ability to express it in English. In other words, the scheme $S_E \sim T_E$ is a balanced one. Let

P be the student's ability to perform $\frac{a^2-ab}{a-b} = \frac{a(a-b)}{a-b} = a$. Giving the example the teacher says:

(1) *Let us factor out.*

Here a familiar phrase is used instead of the full phrase:

(2) *Let us take the common factor a out of brackets.*

What would happen after that if $P \in S$ but $P \notin S_E$? Obviously in this case the connection between the equality $a^2 - ab = a(a-b)$ and the term *factor out* does not exist. So, the scheme is out of balance. There exist two possibilities to restore it:

a) The student would detect himself the connection, and in this way he would reach an expanded scheme $(S_E \cup P) \sim (T_E \cup (1))$.

b) Using (2) the teacher would help the student to create such an extension.

The first way is obviously more fruitful one.

In order to solve more complicated problems some schemes must be used in strict sequence. In such a case it is said that a routine is applied. The difference between schemes and routines is a very subtle one sometimes. They both build the base of human thinking.

At this stage of presentation of Piaget's theory only one term is left to be introduced and it will be possible to define the most important notion of operation.

Let us take the set of all triangles. It is possible to separate from it the subset of all isosceles triangles. Then the second set can again be thought together with the rest part of the first set. This mental ability is very important aspect of thinking, according to Piaget, and simply means the capacity for changing direction in one's thinking so that one can return to a starting point. This ability of human thinking is called by Piaget reversibility. The reversible mental routines are known as operations. Piaget has described certain groups of operations in [8, pp. 101–106]. The group of logical operations (classification, unifying asymmetrical relations, substitution, etc.) appears to be very important for the study carried out.

Piaget divides the cognitive development of children and adolescents into four stages: sensorimotor (birth to 2 years), preoperational (2 to 7 years), concrete operational (7 to 11 years) and formal operational (11 years to adulthood). The students who have just passed the preparatory class belong to the fourth group. At this stage abstract and symbolic thinking is possible. They can solve problems through the use of systematic experimentation and hypothesis. These students can think on propositions as objects. In this way, they are able to use the whole group of the propositional operations (implication, disjunction, conjunction, etc.). At this stage several higher level operations (so called operational schemes) become able to be used. The most important of them is proportion [8, p. 590]

At the end of this part it is necessary to consider the next situation, presented by the implication: *If* $0 \leq a < b$, *then* $\sqrt{a} < \sqrt{b}$.

Then we are going to interest in all positive integers x satisfying the condition $\sqrt{x} < \sqrt{b}$. How could we express such a fact? Obviously another sentence is needed.

If $0 \leq b$*, which are all positive integers* x *satisfying* $\sqrt{x} < \sqrt{b}$*?*
How can it be formed? Which are the rules of its formation? The answer of these questions is given by Chomsky's Transformational Syntax.

2.3. Just a Few Words on Chomsky's Deep Structures

Let us imagine a teacher who is explaining how the equation

$$a\cos^2 x + b\sin^2 x + c = 0$$

could be solved. He asks, "How can I do it?" and repeats the question, saying, "I can do it how? By substituting the expression $(1 - \sin^2 x)$ for the expression $\cos^2 x$." That is why, we can say: "The teacher will put an expression in the place of the expression $\cos^2 x$." So, the question *What expression?* is relevant to the explanation. There are two ways of correct asking such a question (so called echo and non-echo questions):

(1) *The teacher will put what expression in the place of the expression* $\cos^2 x$*?;*

(2) *What expression will the teacher put in the place of the expression* $\cos^2 x$*?*

As is known, questions which involve the use of an interrogative word beginning with *wh- (who, why, what, when, which* – but also *how*) are called *wh*-questions. So, (1) is an echo question. This sentence is grammatical. Following word for word [9, pp. 154–156], we can say that it seems as if the sentence-initial *wh*-phrase *What expression* in (2) behaves as though it had actually occurred after the verb, rather than at the beginning of the sentence. The author captures this intuition postulating that the *wh*-phrase in each case does actually originate (in a metaphorical sense) after the verb, and only subsequently gets moved into initial position by a rule which Chomsky calls WH-MOVEMENT. In particular, we might propose an analysis along the following lines:

The teacher will put what expression in the place of the expression $\cos^2 x$?

$$\Vert$$

($\star$) WH-MOVEMENT

$$\Downarrow$$

What expression will the teacher put__in the place of the expression $\cos^2 x$?

(__) in the second sentence serves to indicate the position that *what expression* occupied before it was moved. An analysis like ($\star$) presupposes that we recognize two levels of structure in syntax:

a) The familiar SURFACE STRUCTURE (2);

b) An additional level of abstract structure (1) which underlies it and which Chomsky refers to as DEEP STRUCTURE.

The Deep Structure in this case is related to the Surface Structure by a transformation (i.e. movement rule) called WH-MOVEMENT. The constituent analysis of (1) leads to the following set of sentence-formation rules:

S → NP – AUX – VP;
NP → DET – N;

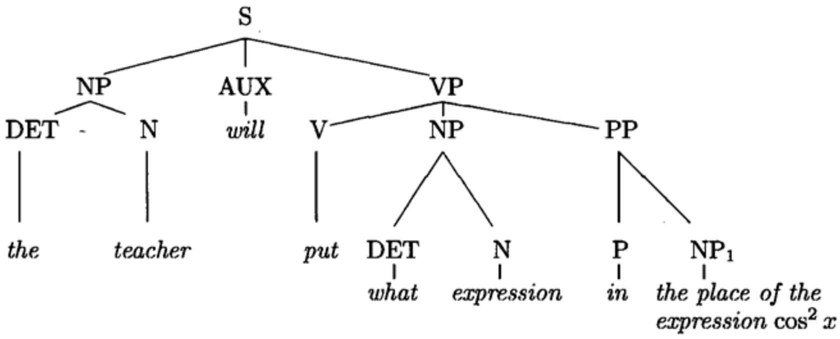
VP → V – (NP) – PP;
PP → P – NP_1;
NP_1 → [the place of the expression $\cos^2 x$].

The abbreviations used denote the following lexical categories:

NP – Noun Phrase;
AUX – Auxiliary;
VP – Verb Phrase;
PP – Prepositional Phrase;
P – Preposition.

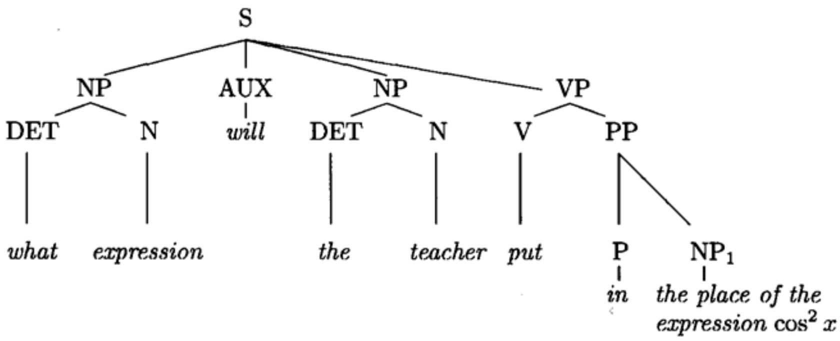
By applying these rules and inserting appropriate lexical items under the appropriate lexical categories we generate the Deep Structure associated with the sentence (1).

(1)(a)



Following the same way we obtain the Surface Structure of the sentence (2).

(2)(a)



Now we can specify the scheme ($\star$). To derive the structure (2)(a) from (1)(a) we need two movement operations, one involving the preposing of the *wh*-phrase *what expression,* and another inverting the subject NP *the teacher* and the auxiliary *will.* The author of [9] argues that two distinct movement rules are involved respectively: WH-MOVEMENT, and NP – AUX INVERSION. His reasons in this way are:

a) Inversion may take place in sentences not containing a *wh*-phrase.

Will the teacher put an expression in the place of the expression $\cos^2 x$*?;*

b) Inversion may take place in sentences containing a non-preposed *wh*-phrase – e.g. in echo questions like that uttered by speaker B:

Speaker A: Will the teacher put an expression in the place of the expression $\cos^2 x$*?*

Speaker B. Will the teacher put what expression in the place of the expression $\cos^2 x$*?*

c) Embedded *wh*-questions show preposed *wh*-phrases without inversion.

I don't know what expression the teacher will put__in the place of the expression $\cos^2 x$*?.*

The movements mentioned above are not the only ones which the author of [9] has presented. But they are the most important for our presentation because of the frequency of their appearance in the literature explored.

At this stage we are ready to present the specification of the scheme ($\star$). The new scheme is pointed in [9] as a model of grammar (at a certain level).

	1. Sentence – formation Rules
BASE:	2. Lexicon
	3. Lexical Insertion Rules
	$\Vert$
	DEEP STRUCTURES
	$\Vert$
TRANSFORMATIONAL	4. Movement Rules
COMPONENT:	$\Downarrow$
	SURFACE STRUCTURES

This model appears to be the corner stone of our presentation as it will be seen further.

3. The main idea. The main purpose

Hamlet. " . . . suit the action to the word, the word to the action, . . . "
Hamlet [3.2,(10–20)]

As well as operations and operational schemes serve as a base of human behaviour (mental, verbal and physical one (except the first signalling reflexes)), the Deep Structures serve as a base of generating correct sentences, and, consequently creating meaningful texts. The basic hypothesis on this matter is that there exists a connection between operations and operational schemes on one hand, and certain Deep Structures (or sets of Deep Structures) on the other. Which are these Deep Structures and how can they be found? Let G be an operation or operational scheme. It can be verbally described by a well-formed sentence (sometimes not so simple) or by a set of sentences. Such a sentence (a set of sentences) could be one for a certain operation but often more. In this case let SG be one of them. Then the question "Which of them?" sounds quite natural. We will try to explain it below. For the time being let DSG be the Deep Structure (the set of Deep Structures) from which SG has been derived. If we assume that DSG is relevant to G, an immediate question arises: "Is there any criterion to choose the sentence (the set of sentences) SG from the totality of all possible ones?". Let us make it clearer using some particular operational scheme, say deduction. Here you are an ancient example of deduction:

	Men are mortal.	– General precondition
	Socrates is a man.	– Particular precondition
($\star$)	*Therefore, Socrates is mortal.*	– Conclusion (drawn from the preconditions) about the particular subject.

This set of sentences presents a deduction. Let us write its sentences in a formal way using the symbols of the theory of sets.

Let B be the set of all mortal beings, A be the set of all men, and x be Socrates. Now the deduction ($\star$) can be written in a number of ways:

(1) $(A \subset B \wedge x \in A) \rightarrow x \in B$ or verbally

(1a) *Men are mortal and Socrates is a man.*
Therefore, Socrates is mortal;

(2) $(x \in A \subset B) \rightarrow x \in B$

(2a) *Socrates is a man and every man is mortal.*
Therefore, Socrates is mortal.

(3) $(A \cap B = A \wedge x \in A) \rightarrow x \in B$

(3a) *Men are a part of mortals and Socrates is a man.*
Therefore, Socrates is mortal.

Structurally (1a) is closer to ($\star$) than (2a) and (3a). In a fairly obvious sense we can say (1a) is as close as possible to ($\star$). Then the set of the Deep Structures from which the sentences (1) have been derived is relevant to the deduction ($\star$).

The author could be asked a question. We agree that structurally ($\star$) and (1a) are very similar. Nevertheless, why do you prefer (1a) rather than ($\star$)? They mean the same. Of course, it is true. But the form (1a) combines in a whole the set of preconditions and separates clearer the precondition part from the conclusion.

At this stage we are able to answer the question about the criterion of choice, asked above. Simply choose the sentence (the set of sentences) which is as close as possible to the formal description of the operation (operational scheme). The Deep Structure (the set of Deep Structures) from which this sentence (set of sentences) has been derived is relevant to the operation (operational scheme) explored.

The reader could surely insist on specification of the phrase "to be as close as possible" and he will be right. Such a specification however, undoubtedly will make the presentation quite complicated and thus put it out of the teacher's need.

At this stage the author is able to present how operations (operational schemes) and Deep Structures function together in the process of teaching Mathematics in the high school. It appears to be an important purpose of the paper presented. In addition to it the author presents a lot of examples in this sense, chosen from the well-known books [3] and [10], to help the teacher's practice. Thus, a partial answer of the question "How can Mathematics be taught in English?" will be given.

4. Operations, linguistic structures, and how they function together

4.1. Introduction and Frequency Characteristics

This part presents the connection between the typical activities in Mathematics (and Piagetian operations with which they are related to) on one hand, and their relevant syntactic structures, on the other hand. This presentation is based on the author's investigation of two well known SAT coaching courses [3 – the mathematical units only], and [10]. The subject matter of this study appears to be the conditional sentences because of the great variety of their functions. Before exposing the essential results some preliminary notes must be presented and some abbreviations introduced. The basic classification used here is given in [2]. The following abbreviations concerned with the different types of conditionals will be often used.

Basic form of Type 1 (or 2, 3) conditionals	–	TIB, TIIB, TIIIB
Type 1, Variation 1 (or 2, 3) conditionals	–	TIV1 (or 2, 3)
Type 2, Variation 1 (or 2, 3) conditionals	–	TIIV1 (or 2, 3)
Type 3, Variation 1 conditionals	–	TIIIV1
Mixed types	–	Mixed

In addition to them, some special kinds of conditionals have to be explicitly noted.

Type 1, Variation 0 (TIV0). Its structure is: If I (do) ... , I (do) ... .
Type 2, Variation 0 (TIIV0). Its structure is: If I (did) ... , I (did) ... .
If command (IFC). Its structure is: (Do) ... if + present.
For ... , Let ... conditionals. (For ... , let ...)

Example
"For $z \neq 0$, let $\textcircled{z}$ be defined by $\textcircled{z} = \dfrac{z}{1+\frac{1}{z}}$" [3, p. 457]

Let (sth) be ... appears to be a Present Subjunctive. The sentence-construction
For (conditions to be satisfied), let (sth) be ... .
expresses the fact that something exists or is true upon certain conditions. That is why the structure above is similar in meaning to the structure *If ... , then ...* . But this is a conditional sentence. So the structure *For ... , let ...* appears to be a structure presenting conditionals. No *If*, no *then*. Just a feeling that the meaning is the same.

The study shows a density of appearance of conditionals as follows:

SAT-MATH WORKBOOK	– 1.7 conditionals per page;
SAT	– 3.1 conditionals per page.

Such a difference could be explained by the fact that SAT-MATH WORKBOOK is intended for high school applicants but SAT – for college ones. The higher the grade of education, the more complicated the language in use.

The abbreviations introduced provide a possibility to tabulate the results of the study carried out.

Table 1. Distribution of the conditionals
in accordance with their types and variations

	TIV0	TIIV0	TIB	TIIB	TIIIB	TIV1	TIIV1	TIIIV1	IFC	Mixed	For ... Let ...	Total
SAT-MATH WORKBOOK	240	24	35	2	2	61	8	1	78	11		462
%	51.9	5.1	7.5	0.4	0.4	13.1	1.6	0.2	17.6	2.2		100
SAT	356	18	66	11	3	45	1		38	18	1	558
%	63.8	3.2	11.8	2	0.5	8.2	0.2		6.9	3.2	0.2	100
TOTAL	596	42	101	13	5	106	9	1	116	29	1	1020
%	58.4	4.1	9.9	1.3	0.5	10.4	0.9	0.1	11.5	2.8	0.1	100

Table 1. shows that the most often used types of conditionals both in [3] and [10] are TIV0, IFC, TIV1 and TIB ones.

Sometimes the conditional word is different from *if*. The list of such words (they are included in Table 1.) and the frequency of their occurance follows:

SAT-MATH WORKBOOK

Given – 1
Assuming – 1
Whether or not – 1
Unless – 1
Whenever – 2

SAT

If and only if – 1
Only if – 1
Only when – 1
Only if + when – 1
When + if and only if – 1
If + assuming – 1
When + if – 1

Table 2. shows that the conditionals are the most often used in describing problems, giving explanations, hints, instructions, solutions, and presenting theorems and assertions which is typical for Mathematics.

The most interesting results are shown in Table 3. and especially in its last column.

In addition to it the study reveals that the WH-Transformed conditionals appear to be the most often used device for introducing the purpose of a problem. A usage like this has been observed 111 times in [10] (for this purpose only) and 205 times in total of 207 occurrences in [3]. Such a percentage of appearance, so high, seems to be directly concerned with the answer of the question how should Mathematics be taught in English. If the necessity of understanding questions could be left apart as a *præsumptio sine qua non*, the ability to make questions on the

Table 2. Distribution of the conditionals
in accordance with their functions in the text

	Theorems and Assertions	Definitions	Rules	Problems	Explanations Hints. Instructions Solutions. Answers	Common instructions to the user	Total
SAT-MATH WORKBOOK	56	4	31	269	85	17	462
%	12.1	0.9	6.7	58.2	18.4	3.7	100
SAT	54	21	15	306	129	33	558
%	9.8	3.7	2.7	54.8	23	6	100
TOTAL	110	25	46	575	214	50	1020
%	10.8	2.5	4.5	56.3	21	4.9	100

Table 3. Distribution of WH-transformed conditionals
in accordance with their types and variations

	TIV0	TIIV0	TIB	TIIB	TIV1	TIIV1	Mixed	When	Total	% of the total amount
SAT-MATH WORKBOOK	46	17	21		22		5		111	24
SAT	108	15	38	6	18	1	9	12	207	37.1
TOTAL	154	32	59	6	40	1	14	12	318	31.2

base of some initial conditions appears to be a very important thing in the way of comprehension of the matter taught.

The following units of this part are going to be devoted to the question how conditionals have been used for presentation of the main mathematical activities.

4.2. Notation of a Mathematical Object by a Symbol

"Take care of the sense and the sounds will take care of themselves" [4]

The syntactic structure which is the most often used to note an object is

LET *symbol* BE *noun phrase.*

and it presents an analytical form of the Present Subjunctive. The structure is relevant to the operation which appears to be the reverse one to the basic operation – unifying elements into a class [8, p. 102]

Examples:

1. *"Let x be the list price ... "* [3, p. 345];
2. *"Let x% be the percent of increase ... "* [3, p. 346].

In case of using some mathematical expressions this structure might undergo some changes.

Examples:

1. *"Let $x =$ side of original square."* [3, p. 684)]

which means

1(a) *Let x be the side of the original square.*

Here the usage of the symbol = instead of *be* gives rise to just a slight change in the initial sentence (if it is a change). Simply 1(a) is another way of presenting of the original syntactic structure;

2. *"Let $OA = OB = x$."* [10, p. 252]

which means

2(a) *Let x be the segment OA which is equal to the segment OB.*

In this case a relative clause appears to be "hidden" in the mathematical equality.

4.3. Defining a Notion

"It says 'Bough-wough!' " cried a Daisy: "that's why its branches are called boughs!" [5]

The syntactic structure which is the most often used to define a notion is

IF *object(s) and condition(s) to be satisfied,*

(THEN) *object(s)* BE + present (CALLED) *noun phrase*

and it presents a conditional.

The structure is relevant to the first basic operation – unifying elements into a class. [8, p. 102].

Examples:

"If $x^2 = y$, then x is called a square root of y." [3, p. 323] $(x, y > 0)$;

Sometimes the initial syntactic structure undergoes changes:

1. *"The principle square root of a number is its positive square root if the number has two roots."* [3, p. 323];

2. *"Two quantities are said to vary directly if they change in the same direction."* [10, p. 46].

Sometimes *when* might be used instead of *if.*

"S(x) is defined as follows:

$$\begin{aligned} S(x) &= 1 && \text{when} \quad x > 1, \\ S(x) &= x && \text{when} \quad -1 \le x \le 1, \text{ and} \\ S(x) &= -1 && \text{when} \quad x < -1.\text{" [3, p. 296].} \end{aligned}$$

The structure (FOR ... LET ...) used in giving definitions has been already presented.

4.4 Formulation of Theorems, Problems, Rules, and Giving Explanations

"If you knew Time as well as I do, said the Hatter, you wouldn't talk about wasting it . . . " [4]

4.4.1. Theorems. The syntactic structure which is the most often used to formulate a theorem is

IF *object(s) and condition(s) to be satisfied*, (THEN) *sth. is true*
and it presents a conditional.

The structure is relevant to the propositional operation implication.

Examples:

1. *"If $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$"* [10, p. 24];

2. *"If two sides of a triangle are congruent, the angles opposite these sides are congruent."* [3, p. 292].

Sometimes the if- or the main-part of a conditional describing a theorem appears to be a compound or complex sentence.

Examples:

1. *"If two lines are cut by a transversal and a pair of alternate interior angles are congruent, the two lines are parallel."* [3, p. 292].
Here the if-part is a compound sentence;

2. *"If two sides of a triangle are not congruent, the angles opposite these sides are not congruent and the greater angle lies opposite the greater side."* [3, p. 293].
Here the main-part is a compound sentence;

3. *"If point $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, the coordinates of the midpoint of the segment whose endpoints are P_1 and P_2 are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$"* [3, p. 372].
Here the if-part is a compound sentence and the main-part is a complex sentence;

4. *"In direct proportion the two variables are so related that, if both quantities are multiplied or divided by the same number, the ratio is unchanged."* [3, p. 354].
In this example the whole conditional sentence appears to be the relative clause of a complex sentence.

A condition may be expressed by conditional word different from *if*, say *when.*

1. *"When two negative numbers are added, their sum will be negative."* [10, p. 208];

2. *"When the numbers to be averaged form an evenly spaced series, the average is simply the middle number."* [10, p. 86].

To note the fact that the two implications presented by the following syntactic structure:

(1) if A, then B, and if B, then A are both true, the expression *if and only if* is traditionally used. So the group (1) of syntactic structures could be written as only syntactic structure.

(2) A *if and only if* B or shortly A *iff* B.
A conditional structure equivalent in meaning to (2) is

The necessary and sufficient condition A to be true is B to be true.

Example:

"When both b and d are positive, $\frac{a}{b} > \frac{c}{d}$ *if and only if* $ad > bc$*."* [3, p. 327].

The main-part of this *when*-conditional appears to be an *iff-conditional*.

4.4.2. Problems. The aim of this unit is to present linguistically some ways of introducing a problem. It will be useful to recall that a problem consists of two parts – information given (condition of the problem) and purpose of the problem. Sometimes a piece of the initial information might be included not in the conditional part but in that which introduces the problem's purpose. The examples considered are going mainly to treat such degenerate cases.

A. A problem may be introduced by the so called IF command sentence. Its structure is

IF *condition(s) to be satisfied*, VERB + IMPERATIVE *something to be done.*

Example:

"If the wife receives $\frac{1}{3}$ *of the estate and each son receives* $\frac{1}{2}$ *of the remainder, find the value of the entire estate if each son receives* $4,000 *as his share."* [3, p. 339].

Obviously an if- clause occurs in the command part; the whole command part is an IF command sentence.

B. A problem can be introduced by a question which has not undergone WH-Movement.

Example:

"If each is increased by A units, the perimeter is increased by how many units?" [3, p. 371].

The main part of this conditional is presented by the Deep Structure

$[_{\text{S}}[_{\text{NP}}[_{\text{DET}}\ \textit{the}][_{\text{N}}\ \textit{perimeter}]][_{\text{AUX}}\ \textit{is}][_{\text{VP}}[_{\text{V}}\ \textit{increased}][_{\text{PP}}[_{\text{P}}\ \textit{by}]$
$[_{\text{NP}}\ \textit{how many units}]]]]$

because of the position of the question word *how*.

The result of applying WH-MOVEMENT (to the WH-NP *how many units*) and NP-AUX INVERSION to this DEEP STRUCTURE is the SURFACE STRUCTURE

How many units is the perimeter increased by?

C. More often a problem occurs introduced by a WH-Transformed conditional. The respective structure in this case is

IF *condition(s) to be satisfied*, WH- *question?*

Examples:

1. *"If m men do a job in 10 days, how long will it take 10 men to complete this task, assuming that they work at the same rate?"* [3, p. 356].

In this example a conditional clause introduced by *assuming* is embedded in the main-part;

2. *"If* $xy = k$*, k a constant, and if* $y = 5$ *when* $x = 7$ *what is the value of x when* $y = 32$*?"* [3, p. 619].

The syntactic structure is

IF-part: If + assertion and If + embedded conditional

Main-part: a WH-Transformed conditional (the conditional word is *when*).

In its turn, the main-part of this WH-transformed conditional obviously derives from the Deep Structure

$$[_{\mathrm{S}}[_{\mathrm{NP}}[_{\mathrm{DET}}\ the][_{\mathrm{N}}\ value][_{\mathrm{PP}}[_{\mathrm{P}}\ of][_{\mathrm{NP}}[_{\mathrm{DET}}\ e][_{\mathrm{N}}\ x]]]][_{\mathrm{VP}}[_{\mathrm{V}}\ is][_{\mathrm{WH}}\ what]]]$$

where S-sentence, e-empty (no DET).

D. A problem could be introduced by a conditional and a set of possible answers.

Example:

"Given the number 83.21p*, in order for this number to be divisible by* 3, 6*, and* 9*, p must be*

(A) 4 *(B)* 5 *(C)* 6 *(D)* 0 *(E)* 9 [10, p. 21].

Obviously the conditional word is *given.* [2, p. 282].

4.4.3. Rules. A rule is an explanation what must be done with certain object upon certain conditions. So, conditionals may take part in presenting rules. Some different ways of doing it follow:

A. Introducing a rule by an IF command conditional.

Examples:

1. *"If there is an odd number of digits before the decimal point, insert a zero at the beginning of the number in order to pair digits."* [10, p. 122];

2. *"Division follows the same rule as multiplication: divide absolute values and use a positive sign for your answer if the original signs were the same, or a negative sign if they were opposite."* [3, p. 319].

This sentence appears to be a mixed type IF command conditional. Its structure is

... COMMAND + (if + ... + Be + simple past + ...) (COMMAND+ VERB DELETION) + (if + ... + Be + simple past + ...);

B. Conditionals different from IF command ones have been also used in [3] and [10] to present rules.

Examples:

1. *"This law states that, if a sum is to be multiplied by a number, we may multiply each addend by the given number and add the results."* [3, p. 320].

The main-part of this TIV1 conditional is a compound sentence;

2. *"If the rate is given in miles per hour*
the time must be in hours ← TIV1
and the distance will be in miles." [10, p. 153] ← TIB

Obviously, the main-part is a compound sentence and its two clauses present variations of conditionals;

3. *"Two inequalities may be added if the inequality signs have the same direction by simply adding the left sides, adding the right sides, and keeping the same inequality sign."* [3, p. 326].

Here the if-part is embedded in the main-part;

4. *"If an expression consists of two terms which are separated by a minus sign, the expression can always be factored into two binomials, with one containing the sum of the square roots and the other their difference."* [10, p. 136].

The if-part of this sentence appears to be a complex one;

C. In some cases other kinds of conditional words have been used instead of *if*.

Examples:

1. *"In other words, the fractions may be inverted only if the inequality symbol is reversed when both fractions have the same sign."* [2, p. 327];

2. *"When numbers are reversed around a minus sign, they may be turned around by factoring out a* (-1)*."* [10, p. 130];

3. *"Whether an equation involves numbers, or only letters, the basic steps are the same."* [10, p. 97].

Here an interesting transformed form of *whether or not* [2, p. 282] can be viewed. One might say that *only letters* does not mean *not* but it is clear from the context that it is not true.

4.4.4. Giving Explanations. An explanation often explores different ways of solving a problem to point the right or easier one. So, conditionals may take part in giving explanations or hints to the students. But some peculiarity has to be noted in this case. When solving a problem it will be useful to speculate on the possible result of doing something or, in other words, to imagine a possible implication. To describe such kinds of explorations the student often needs TII and TIII conditionals. The examples below have been selected with a view to these needs.

Examples:

1. *"We may not assume that* ABC *is a right triangle. If it were a right triangle, and if we assumed that side* AC *is the hypotenuse, then the area of* ABC *would be* $\frac{1}{2}(5)(8)$ *or* 20*."* [3, p. 582];

2. *"If we were asked to solve for both* x *and* y*, we would now substitute* $2\frac{1}{2}$ *for* x *in either equation and solve the resulting equation for* y*."* [10, p. 100].

This TIIV1 conditional presents a possible way of solving simultaneous equations;

3. *"If we had solved without factoring, we would have found* $16x^2 = 25$*."* [10, p. 102].

This TIIIB conditional presents a true result of something which has not happened (now and in the past);

4. *"Remember that if the terms had been reversed around a plus sign, the factors could have been cancelled without factoring further, as* $a + b = b + a$*, by the commutative law of addition."* [10, p. 130].

4.5. Comparison of Two Variables

"The more there is of mine, the less there is of yours." [4]

This unit describes and explores syntactic forms expressing the relation between two variables which vary as follows:

As the first decreases/increases, the second does also;

As the first decreases/increases, the second increases/decreases. [10, p. 46]

The typical and the most well-known mathematical examples in this sense are:

(1) the direct proportion $y = kx$ (linear monotony);

(2) the inverse proportion $y = \frac{k}{x}, x, y \neq 0$ (hyperbolic monotony).

For the sake of simplicity, it is supposed here and below that $k > 0$.

As it was mentioned above, the ability to deal with them depends on the level of development of an important operational scheme which has been initially introduced as a psychological object by Piaget and called a PROPORTION [8, p. 590]. As before, the aim of the research will be finding of the Deep Structure, relevant to this operational scheme.

Let us introduce the verbal descriptions of (1) and (2) [11, pp. 75,76].

(1)(a) A direct proportion is indicated when two quantities are so related that an increase in one **causes** a corresponding increase in the other or when a decrease in one **causes** a corresponding decrease in the other. (The bold face is mine.)

The following [11, p. 75] presents a list of typical quantitative expressions in which the variables are directly related:

(3)(a) *The faster the speed, the greater the distance covered;*
(b) *The more men working, the greater the amount of work done;*
(c) *The faster the speed, the greater the number of revolutions;*
(d) *The higher the temperature of gas, the greater the volume;*
(e) *The taller the object, the longer the shadow;*
(f) *The larger the quantity, the greater the cost;*
(g) *The smaller the quantity, the lower the cost;*
(h) *The greater the length, the greater the area;*
(j) *The greater the base, the larger the discount, commission, interest and profit.*

(2)(a) An inverse proportion is indicated when the quantities are so related that an increase in one **causes** a corresponding decrease in the other, or vice versa.

The following [11, p. 76] are quantitative expressions in which the variables are inversely related:

(4)(a) *The greater the speed, the less the time;*
(b) *The lower the speed, the longer the time;*
(c) *The greater the volume, the less the density;*
(d) *The more men working, the shorter the time;*

(e) *The fewer men working, the longer the time.*

Obviously, both /3/ and /4/ sound as discourses.

The two SAT courses [3] and [10] also provide a variety of syntactic structures which are similar or very close to the sentences here quoted. They occur 21 times in [10] and 5 times in [3]. The following set of such sentences is extracted from them.

(5)(a) The more kilometers, the more miles. [10, p. 54]

(b) The larger the diameter, the smaller the number of revolutions per minute.

(c) The smaller the diameter, the greater the number of revolutions per minute. [10, p. 48]

(6)(a) In conversing the kilometers to miles, the greater the number of miles, the greater *will be* the number of kilometers. [3, p. 354]

(b) The more people, the less time it *will last.* [10, p. 48]

(7)(a) The more teeth the gear *has*, the fewer revolutions *it will make* per minute.

(b) The less teeth it *has*, the more revolutions *it will make* per minute. [10, p. 48]

(c) In conversing U.S. dollars to British pounds, the more dollars you *exchange*, the greater *will be* the number of pounds you *will get.* [3, p. 354]

What may be seen at a first glance at (5)(a), (b), (c) is that there are no verbs at all, so that the question "Are these syntagma sentences?" could sound quite natural. If so, what kind of sentences do they belong to? It is easy to answer the question because (3) and (4) provide a lot of literary evidence in this sense. Simply the sentences (3), (4), (5) reflect the common syntactic fact of verbal ellipsis [7, pp. 26, 118]. In order to answer the second question it is necessary to recover such a sentence, say (5)(b). This sentence can be thought as a small discourse extracted from the text:

> ($\star$) "c) *When two pulleys are connected by a belt, the diameter of a pulley varies inversely as the number of revolutions per minute. The larger the diameter, the smaller the number of revolutions per minute. The smaller the diameter, the greater the number of revolutions per minute. The product of the diameter of a pulley and the number of revolutions per minute remains constant.*" [10, p. 48]

The first sentence of this text contains certain anaphoric information that can be useful in finding the full (recovered) form of (5)(b) in accordance with the sentence formation rules. The way to this recovery can be partially elicited from the second part of each sentence 6 and from the sentences 7 (see the italicised words).

More concretely, it is obvious that:

– there is no verbal ellipses in the second part of all the sentences (6) and (7); all these verbs are in the future simple;

– there is no verbal ellipses in the first part of all the sentences (7); all these verbs are in the present simple;

– there is no nominal (or clausal) ellipses in the second part of the sentences (7);

– the personal pronouns *it* and *you* in the second parts of (7)(a, b) and (7)(c) respectively involve an anaphoric connection between the two parts of every sentence (7).

These remarks trace out the way of recovering. Using the appropriate symbol (0) introduced in [7, p. 26] sentence (5)(b) can be rewritten as

(5)(b1) *The larger the diameter* $(0_1)(0)$*, the smaller the number of* (0_2) *revolutions per* (0)*.*

From a pragmatical point of view it is clear that (is), (of a pulley) and (its) must be substituted in (5)(b1) for $(0), (0_1), (0_2)$ respectively. So (5)(b1) becomes

(5)(b2) *The larger the diameter of a pulley is, the smaller the number of its revolutions per minute is.*

Let us consider the meaning of this sentence. To catch it we have to remember that:

– the definition of proportion, the most generally saying, expresses the fact that something will happen upon certain condition(s) (see (1)(a), (2)(a) and the bold-faced word **causes**);

– the structure of the sentence (5)(b2) is

"The + comparison + the " [2, p. 119];

– the construction just mentioned can be used with adjectives and adverbs to show cause and effect [2, p. 119].

These three things together immediately lead to the conclusion that (5)(b2) has the meaning of a typical conditional sentence and it can be consequently written as

(5)(b3) *If the diameter of a pulley is larger, then the number of its revolutions per minute is smaller.*

The comparison operator acting here by the exponent *-er/than* (*than* is omitted in (5)(b3)) of the paradigm of grammatical means of expressing inequalities requires two items as a general condition to function. The second item (namely the one omitted along with *than*) is semantically leading in relation to the first item and plays the role of a norm. Where is this second item in the sentence (5)(b3)? Obviously, it is not apparently mentioned. Let's remember however, that (5)(b3) is a partial recovering of (5)(b) which has been extracted in its turn from the text ($\star$). This text gives the missing piece of information as an anaphoric one – "When **two** pulleys are ... " (The lifting is mine.) So, the second item necessary for the function of the comparison operator exists and the final recovering of the sentence (5)(b) is

(5)(b4) *If the diameter of a pulley is larger than the diameter of another, then the number of its revolutions per minute is smaller than the ones of the second pulley.*

The sentences (5)(b) and (5)(b4) mean the same. We can say (5)(b4) is a Deep Stricture relevant to the inverse proportion described by the Surface Structure (5)(b).

Moreover, the Deep Structure above fully reflects the connection between the sense of the inverse proportion (that is an operational scheme) and the syntactic structure *If ... , then ...* expressing the implication CAUSE → EFFECT. The difference between (5)(b) and (5)(b4) is that the second is closer to the natural human imagination and way of thinking than the familiar sentence (5)(b).

Let us revert to the operational scheme PROPORTION in general. Having in mind that the value of kx (and $\frac{k}{x}$) is received from the given value of x, the verbal presentation of the two proportions may be generalized as it is made below.

DIRECT PROPORTION ($y = kx$). The syntactic structure is

(8) *If x is the larger value of two given ones, then kx will be the larger value of the two received ones.*

and it originates from the Deep Structure

$[[_{\text{COND}}\ if][[_{\text{NP}}\ x][_{\text{V}}\ is][_{\text{CCPLP}}$ *the larger value of the two given ones*$]][_{\text{IMPL}}$ *then*$]$
$[[_{\text{NP}}\ kx][_{\text{AUX}}$ *will*$][_{\text{V}}$ *be*$][_{\text{CCPLP}}$ *the larger value of the two received ones*$]]]$.

INDIRECT PROPORTION ($y = \frac{k}{x}$). The syntactic structure is

(9) *If x is the larger value of two given ones, then $\frac{k}{x}$ will be the smaller value of two received ones.*

and it originates from the Deep Structure

$[[_{\text{COND}}\ if][[_{\text{NP}}\ x][_{\text{V}}\ is][_{\text{CCPLP}}$ *the larger value of the two given ones*$]][_{\text{IMPL}}$ *then*$]$
$[[_{\text{NP}}\ \frac{k}{x}][_{\text{AUX}}$ *will*$][_{\text{V}}$ *be*$][_{\text{CCPLP}}$ *the smaller value of the two received ones*$]]]$.

The abbreviations used above mean as follows: COND ≡ Conditional word; its paradigm includes the conditional words *if, when, given, assuming*, etc. [see Part 4]; IMPL ≡ Imply; its paradigm includes *then, it implies*, etc; CCPLP ≡ Comparative Complement Phrase.

The corresponding Surface Structures derived from them are:

(8)(a) *The larger x, the larger kx;*

(9)(a) *The larger x, the smaller $\frac{k}{x}$.*

Having read the whole text above, one could say: "The more words, the less use." Maybe he will be right. But do not forget that the sentences (8)(a) and (9)(a) are very familiar for the native speaker. As far as it is concerned with the non-native beginner, he ought to understand these structures before practising them. The better understanding, the more correct usage. But in our opinion, the best comprehension is rooted in the origin and the origin is the Deep Structure. Besides, just for the sake of meaningful practice it is very important to convey as more as possible different real situations which can be accounted for by such a syntactic structure.

So far, we have looked at the sentences (5)(b), (8)(a), (9)(a) as discourses and transformed them into meaningful texts in accordance with the sentence formation rules. It will be tempting to trace the back way (if it exists) which, however, goes out of the teacher's needs.

Here we can formulate the main question on the matter just discussed.

Is the Surface Structure mentioned above a result of making discourse from a text when the intuition is the basic thing we can rely on, or does it descend from a Deep Structure which has undergone a sequence of transformations, or both?

At this stage we have not any explicit answer of this question.

5. Conclusions

Being at the end of the paper, the author is intelligibly tempted to ask the reader whether now he is closer to the answer of the question "How can Mathematics be taught in English?" than in the beginning, or he is as far from it as before or even more. The answer to this question requires to begin *ab ovo* and track out again the main points of the way covered.

In the process of study some typical situations, inherent in teaching Mathematics have been revealed. These typical situations require specific devices for their elaboration in a pedagogical aspect. A suitable environment in this respect has been found out to be the system of Piagetian operations and operational schemes which reflect basic forms of human thinking. The most important vehicles of expressions describing these operations and the way they function have been presented.

Several points of the thesis have to be especially noted:

1. The highlight has been focused on the implication and on its expressing. The situations involving usage of implication have been presented as phrasing in a positive (*If A, then B*) and interrogative (*If A, (then) question about element(s) of B*) mode.

In addition to the respective syntactic structures a lot of literary examples of structure deflection have been offered;

2. WH-Transformation as a means of description of the problem's purpose has received a special attention. We should have in mind that the precision of presentation of the problem's purpose depends to a great extent on the accuracy of formulation of the respective WH-Transformed sentence.

All mentioned above points out at covering a great part of the main purpose of teaching – a permanent development of the student's ability to think in English. Such an approach could be successfully used to teach some other subjects in English, say Physics or Chemistry. In this sense the scheme *Piagetian operations and Deep Structures* (concerned with the description of these operations) could be thought as a psycho-linguistic base of teaching and use of English for specific purposes.

In the end, it will be useful to mention another aim that could be reached when teaching Mathematics in English – to revise the usage of some important grammatical forms. The appendices included give an illustration to this opportunity.

The paper, however, does not present a systematic methodology of teaching Mathematics in English. Some important subjects have been left out of the study. For instance, the problems concerned with:building up a special mathematical vocabulary; larger presentation of the problem's purposes without using WH-transformed conditionals; giving definitions in a direct way (without conditionals), etc. In spite of all, the paper presents important ideas and notions which a teacher has to keep in mind. Thus, he will be closer to a greater effectiveness in classroom.

Appendix 1.

> *"Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!" [5]*

"A boat sails M miles upstream at the rate of R miles per hour. If the rate of the stream is S miles per hour, how long will it take the boat to return to its starting point?" [3, p.351]

Solution:

First it is clear that S must be positive. Imagine in the past S was negative. Now it would be right to say:

If S had been negative, the river would have flowed back to its spring;

If S had been 0, the river wouldn't have flowed.

Let X m/h be the speed of the boat sailing on still water. Obviously, X is greater than S.

If X were less than S, or equal to S, the boat wouldn't move upstream.

Having in mind the meaning of the values X, S, R, it is clear that $X - S = R$. Hence, $X = R + S$.

On the other hand, the speed of the boat downstream is $X + S = R + S + S = (R + 2S)$ m/h. So, the unknown value of time is $\frac{M}{R+2S}$ h.

Appendix 2.

When two pulleys are belted together the revolutions per minute (rev/min)vary inversely as the size of the pulleys. A 20 cm pulley running at 180 rev/min drives an 8 cm pulley. Find the revolutions per minute of the 8 cm pulley.

Solution:	20 *cm makes* 180 *rev/min*	$8x = 3600$
	8 *cm makes ? rev/min*	$x = 450$ rev/min
	$\frac{8}{20} = \frac{180}{x}$	

Explanation: First make a table of corresponding values. Put like numbers together. *The smaller the pulley, the greater the number of revolutions;* the quantities are in inverse ratio. Inverting the first ratio, write the proportion. Solve for x. [11, p. 76]

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