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The problem of arrangement of new notions and objects in the school course of mathematics

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The purpose of this article is to invite all those who are interested in the subject, experts and the general public at large, to answer the question: "is modernization of the school* mathematics curriculum necessary, and if it is, what kind of modernization is to be implemented". This is not a simple question. In answering it, we should not display a haste while offering some arbitrary and simple-minded solutions – all of us knowing too well as to what an end such a process has led us in the past. On the other hand, we can not walk away from finding the answers as how the high school mathematics curriculum ought to look in the mid of the 21st century.

To avoid misunderstandings and misinterpretations, observe that we are talking about mathematics courses in high schools with the general education profiles, as well as about specialized schools, which are not magnet type schools with physics and mathematics emphases. Great changes happened in mathematics during the last century, and mathematics (as well as other sciences) made big advances. Mathematical methods became more general and diverse while mathematical models of natural phenomena, technical processes, and social phenomena started to reflect the essence of things more soundly, more accurately and more reliably. Mathematics is more assuredly transforming itself into a powerful instrument for analysis, investigation and prediction, while its application's significance is rising. In conjunction with computer possibilities, it allowed the creation of the principally new direction of scientific knowledge – mathematical modeling and mathematical experiment.

Almost everything has profoundly changed in the science of mathematics. On the other hand, nothing has profoundly changed in the content of courses of mathematics in Russian high school. Let us compare the curriculum from 1940 with the one from 2000: complex numbers are excluded, as well as Newton's binomial formula and also a few small subjects, while the basic notions of mathematical analysis, operations with vectors and some other little bits are included. On the whole, the old situation is preserved: mathematics leaves those who learn algebra and calculus in 18th century, while those who learn geometry in ancient Greece! The information obtained from the other subjects (though often descriptive and fragmentary) is on contemporary scientific level. What should we do now – to preserve this situation until 2050?

It seems that changes in the high school mathematics curriculum are urgent – it concerns the curriculum's content, its methodology of teaching as well as the way of presenting lessons.

Curriculum's content. The content of the high school curriculum of mathematics is the most delicate and ambiguous problem. On one hand, it is not desirable that the knowledge of the students remains at the 18th century level. On the

* Remark of the translator: in the sequel the term “school” means both elementary and secondary school.

other hand, it is obviously unreasonable to try to impose in high schools the study of abstract algebra, theory of functions of complex variables and functional analysis. But, one should not overlook the fact that, besides the specific (purely mathematical) concepts, that mathematics has produced a number of important notions and methods used in general education, all of them having a general cultural importance.

It appears that today among the general education concepts one can include the notions such as bifurcation, fractals, chaos… Physicists, as well as sociologists, biologists and philosophers are working with them. Accordingly, the high school course of mathematics is obliged to inform young people about these notions, at lest at the descriptive-visual level.

Of special interest today is the ability to perform quick calculations – to obtain at least an estimate of the value in which we are interested. Therefore, high school graduates should be able to perform the simplest calculation algorithms.

The curriculum of mathematics today should be pragmatic, and to teach people to be on the right track, to help them to solve practical problems, and to ensure their safety in the broadest sense.

Bifurcation. One of the most important notions today is the bifurcation of a process during the change of its parameter. Everybody operates with this notion now – scientists, engineers and specialists in humanities. Are schools today ready to answer this call of the time? By all means!

The most appropriate is to analyze the notion of bifurcation in the mathematical school course. For, there is no need to introduce or add here anything particularly new! There exists an abundance of examples, well known for ages, of bifurcation in algebra and in geometry – only until now the attention was not concentrated on this. Observe the change of the form of the section of the cube by a plane, orthogonal to cub’s diagonal, during the motion of the plane from one vertex of the cube to the opposite one – this is the simplest bifurcation process. The change of the number of roots of the quadratic equation

3x^2 - 5x + 2c = 0

as the parameter c “runs” over the real line – is another bifurcation process.

While presenting the subject, one has only to wish to do all necessary methodological accents and to draw practical examples. We are witnessing the growth of the epidemic of textbooks on “theory of equations and inequalities with parameters”; problems in such books are becoming more and more cumbersome and refined (sometimes only the answer takes half a page). Surprisingly, none of the large number of “creators” of this theory found the time to tell, not even casually, about the general education notion standing nearby.

Fractals. This is a remarkable mathematical notion, which turned out to be a tool for adequate description of natural phenomena (growth of crystals, passing of air bubbles through oil, formation of cracks, etc.) and for object’s description (including human organism).

To inform students about fractals is worthwhile in order to demonstrate to them the unpredictable features of science development. Understanding the process of scientific cognition of the world is one of the important characteristics of an educated and cultural person. Actually, the notion of fractals was introduced and investigated at the end of the first decade of the last century, but works of its founders did not attract anybody’s attention – the idea appeared too early, had no proper instrumental and applied founding. Only half a century later, thanks to the efforts of B. Mandelbrot and to already existing highly productive computer technique, the investigation of fractals acquired a wide scope.

In the methodological literature, they love to discuss the aesthetic influence of mathematics on students and its meaning in the upbringing of the notion of beauty. Usually, they speak about the beauty of the proof of some theorem, about the elegance of the problem’s solution, about the beauty of additional construction. However, all this is accessible and perceptible only to a student genuinely enthusiastic about mathematics. And what kind of an esthetic pleasure can be obtained, from solving a number of quadratic equations, by someone who is “deaf” for it? The paintings of M. Esher are usually cited as examples of esthetic influence, but these, by all means talented works, are variations on mathematical themes, and illustrations of different mathematical facts, whose deep understanding often remains inaccessible to students.

However, with the aid of computer, fractals generate really beautiful and original pictures, not worse than works of abstract painting.

Chaos. The ”chaos problem” now attracts a special interest of scientists; the interest of physicists, philosophers, economists, medical doctors, biologists and social scientists (even of theorists of education), is attracted by the new area of science – sinergetics.

M. Feigenbaum discovered one of the principal scenarios for passing to chaos in 1978, by performing the numerical experiment using pocket (!) calculator – the analysis of behavior of the sequence $\{ x_n \}$, generated by the mapping

$$x_n \longrightarrow x_{n+1} = \lambda x_n (1 - x_n).$$

The acquaintance with this one, and with other simplest mathematical models of the birth of chaos, being constituent element of the contemporary idea of the “nonlinear world”, will have exceptionally important methodological significance in forming the life philosophy of the young ones. It will not only enrich the mathematical course itself and make it modern, but also it will show the role of mathematics as the universal language for investigation of the nature and the society.

Computational mathematics. Teachers of older generation remember the algorithm for calculating the quadratic root of a number. This was a delicate moment in the past school curriculum – formulation of the rules took the whole page. Later on, pocket calculators appeared, which calculated the quadratic root in less than a second, and this rule disappeared from the textbooks. But who among the students from regular (not specialized) high schools understands the way this operation is performed by a calculator? And what to do in the case when the calculation is connected with the volume of a solid, when one has to find the cubic root of a number?

Having in mind the influence of practical skills (this has nothing to do with formal mathematical skills) on the development of the general intellectual potential and on the upbringing, it would be a good thing to estimate the relative importance, for those who learn, of the method to determine the mutual position of roots of two quadratic equations on one hand, and of the method of successive approximations for solution of the equation with arbitrary degree.

In general, computational mathematics is the weakling of the high school curriculum for mathematics. The widespread opinion is that such subjects are the domain for a course in informatics. Nevertheless, this is more than illogical! For instance, the school subject “Informatics” declares as its notion that of algorithm, which is in fact a fundamental mathematical notion, that has been studied by mathematicians long before the appearance of the word “informatics”. By the way, this notion has always been latently present in the school curriculum for mathematics, but the term itself was avoided for obscure reasons.

Sometime ago, a mathematics course proudly emphasized calculations with a logarithmic ruler. Now, elementary and available computational mathematical methods would allow a vivid demonstration of significance of mathematics in many important practical problems.

Laboratory work in mathematics? More and more educators and psychologists persistently are recommending the strengthening of the creative aspect in the process of learning, the adoption of research projects, and the stimulation of the original cognitive search. Of course, the widespread practice of solving nonstandard (Olympic) problems - is a form of original cognitive search. However, this is only one form among many available choices. Unfortunately, it is almost always, with “prescribed result”.

Why is it that we never thought about the place and the content of possible laboratory work in the high school mathematics? This could be done so that a student learns some phenomenon or object not only with his “head” but also with his “hands” and that he notices the laws of the real world and to try to give them adequate mathematical description. Computational theme can give a rich material for fascinating laboratory work and independent investigation – for example, Feigenbaum phenomenon and numerical consideration of various iterative processes, all having real natural scientific interpretation.

The question of laboratory work in mathematics has an additional, deep and important aspect. If we look in detail at the high school course, it is not difficult to see that it is aimed to train the computational skills and to perform transformations. In the primary school the focus is – automatism and multiplication tables. Then there follows the arithmetic calculations – either “pure” or ”text” exercises (where conceptual and formalistic approaches are still competing). After that follows the variety of algebraic and trigonometric “identical transformations”, including solving of equations and inequalities (and expressing ourselves in old, but precise language, “geometric problems with application of trigonometry”).

However, human existence needs another important practical knowledge of geometric, or space, imagination. Unfortunately, the overwhelming majority of students who passed the crucible of school course of mathematics, has meager-poor knowledge of the rules of formal transformations, and frequently lacks elementary geometric imagination. This is a serious setback to high school mathematics. From the first to the last class, it is necessary to cultivate, and to develop constantly, systematically and continuously, geometric imagination, and computational skills, logic and language.

The principal condition for developing the geometric imagination is to work with the real material. Mastering objects from the real world and activities in real world, with successive transfer of these objects and activities in the world of imagination – it is apparently the only possible way to form the spatial thinking. And its most successful realization can be made in mathematical laboratory works, in which the material would be classical objects, as well as those not included in the traditional curriculum, but nonetheless, very useful. Among such objects are knots. The work with them allows, besides the training of space imagination, and the development of creative skills, the appearance of the objects such as space curve, left and right orientation, classification principles etc. It has also a non-unimportant worldly application.

Pragmatism. It is unpleasant to say, but for catastrophes, connected with swindling "activities" of unscrupulous financiers, which affected the destiny of thousands of Russians, that not only the indifferent authorities are to blame, but also the school course of algebra. Being oriented only towards "high material", such as logarithmic and trigonometric transformations, the course did not include "trifles" such as financial pyramids, but also, it did not prepare its graduates to life conflicts. Our "natural-scientific education, the best in the world" has shown its complete incompetence in collision with creatively thinking swindlers, magicians, fortune-tellers, etc.

There is another aspect of this question. Mathematics could be a more effective mean for providing knowledge of the surrounding world and for solving practical problems, providing that the geometry course did not restrict itself on boring circles and monotonous cubes, but rather to give information on variety of geometric forms of the world, and to show and to introduce the concrete abundance of figures and solids.

Probability theory. Incidentally, let us be objective – the process of perestroika of the school curriculum of mathematics has begun. Finally, the basic notions of probability theory and of mathematical statistics are included there, and the first textbooks appeared. We had a dispute about this almost a century ago, while many countries solved this problem a long time ago. Now, our graduates will not look amazed as they hear on TV the words "confidence interval".

Methodics of teaching. The most important science, which investigates and establishes principles and methods of teaching mathematics, needs a serious conceptual perestroika. Many important changes in the curriculum of school mathematics are impossible to perform if there is no agreement that different questions, or even themes, can be investigated at the descriptive level, and without formal proofs. In this way, a student will understand the essence without learning (even without being told about!) the "strict logical foundations".

For professionals, convinced that Soviet and Russian methodology of teaching mathematics in the school and who always insisted on strict scientific and logical proving (the assertion that "our education is the best in the world" was motivated by this), it is psychologically very difficult to accept this suggestion. However, this is not a case. In the first place, it is impossible to present rigorously to students many elements of the school course – we have to seek for the "convincing surrogates" instead. Secondly, it is enough to inspect closely the classical textbooks of A. P. Kiselev (say, the places where the method of mathematical induction should work), in order to observe the "descriptive" presentation of the whole sequence of questions. Finally, our education in physics and chemistry has by no means suffered from the fact that it did not contain all exhaustive logical proofs.

The main object against the "descriptive" presentation is based on the widespread opinion (especially among mathematicians), that mathematics, mathematics only, and nothing but mathematics, can build the culture of logical thinking, and that only in the course of "rigorous" teaching of mathematics the development of the ability of correct thinking is ensured.

Surely, one should not deny that in some sense the study of mathematics "brings the mind to the order" (M. Lomonosov), but one should not exaggerate, in thinking that this is the only effective way to that goal. One can learn logic in the different way, which is not connected with formal transformations and boring reasoning, so unattractive for “non-mathematicians”. As an example, let me cite the outstanding theoretic-physicist, the Nobel price winner, L. Landau: “I don’t want to argue with the opinion, worthy of medieval scholastics, that by investigating the unnecessary things, people would learn to think logically”. In fact, is a girl, reciting mechanically like verses the learned proof of the theorem about the three perpendicular lines, mastering the logic?

“Mathematics of entrance exams”. One should not say that there are no changes in the school mathematics. But, some changes are extremely dangerous, and if we do not find healthy ways of modernization of the school curriculum in the near future – we can come to a dead end.

We can state with regret, that during the last 10-20 years, besides the “classical” elementary school and higher school mathematics, another “area” of mathematics appeared - “mathematics of entrance exams” (MEE). Examiners at universities and businesslike private tutors had made the whole “science”, containing theoretical investigation of special exam problems, having no educational value. It would be all right if it were an additional knowledge, necessary for mastering the university curriculum. However, MEE is full of themes, questions, information and problems that nobody needs (not even Mathematical Faculty of the Moscow state University) – and after the enrollment, one can (and has to!) quietly forget all about it.

As an example, let us mention a theme of the high school course – the absolute value of a real number. The absolute value of a number is far from being a conceptual mathematical invention; it is rather a comfortable, technical notion. In the school course, it has a very small area of application – for concise writing of some operations, which include square roots and logarithms, for comfortable formulation of the definition of continuous function and of the “triangle inequality”.

No, I do not suggest the excluding of this notion from the school curriculum. But I see no objective reasons for making and suggesting to students and to teachers the fundamental science-like works about the absolute value, and exercises for which the standard school course is not enough, but needs special additional training. How many pages one should write in order to explain to the student what the absolute value is, and how to deal with it? Here we have a “Manual” with the title “Solutions of problems with the absolute value” – the authors succeeded to blow it up to 340 pages! Moreover, like a scientific treatise, the book contains two chapters: “Equations with absolute values” and “Inequalities with absolute values”. The most important thing in science is – the systematization. Therefore, the first chapter contains sections “Equations with one absolute value”, “Equations with two absolute values”, “Equations with three absolute values” and “Equations with four and more absolute values”. I wonder, can you guess the titles of the sections from the second chapter?

The enthusiasm of creators of MEE did not fade after the enrichment of the absolute value theory, and the search continues. At the entrance exams, a problem appears; about its solution in the new textbook, we read the following:

“The last inequality follows from the fact that for any numbers a and b

$\max\{ |a + b|, |a - b| \} = |a| + |b|$."

It would be interesting to do an investigating experiment and to find out: would the creator of the mentioned problem, be able to find its answer in the atmosphere of the entrance exam, providing he did not know then the above formula?

Of a special interest are “new” themes, discovered and elaborated within MEE – for example “Equations and inequalities with parameters” (a huge number of publications is dedicated to this theme, and the theme itself has become an “elegant” theory with a variety of “methods”).

Constant complications of the problems on entrance exams caused the proliferation of private tutoring. Surely, there is nothing bad in additional lessons given to a student, who understands poorly the material, or who missed some lessons because of an illness (in Soviet times, this was called “not-to-leave-a-child-behind”, as we would say in today’s American parlance). However, what should one do if, in order to pass the entrance exams, it is not enough to know (not even to know well!) the material that was learned at school and the material written in the school textbook? It is understood that in the big cities, for entering the colleges, one should pay large amounts to private tutors – “specialists” in MEE. Following the obvious goals and using the obvious connections, tutors continue to untwist the wheel of complications of the exam problems and to develop successfully the MEE. What are supposed to do students from villages and small towns, those who live far away from scientific and pedagogical centers, where even now the qualified teacher is the exception?

Tutors and persons close to them are precisely those who are going to be actively against the change of the existing high school curriculum in mathematics. Incidentally, mathematicians are now constantly asking for the increase of the number of hours allotted to teaching of mathematics in schools. There is a point to think that this is not only because of curriculum’s “modernization”, but rather that is in order to learn the “important” MEE themes – the theory of the equivalence of equations and inequalities and other problems of no particular interest.

I think that the appearance of the idea of the “single state-exam” was not an accidental whim, but an objective outcome of the atmosphere of unscrupulousness of tutors. The “crisis of mathematical training” of students shows itself strikingly in problems reduced to nothing else but a sheer sophistry, as proposed sometimes at the entrance exams for colleges. We have to overcome this crisis. However, we “wanted things to be better, but the outcome was as usual” – we “jumped from a frying pan into a fire”. Instead of clever and serious problems, verifying indeed the knowledge and the understanding of mathematics, they propose tests of the type: “For what values of the argument the function, represented on the drawing, takes the minimal value? For the correct answer there is no need to understand mathematics – it is enough just to use eyesight.

The new type of teachers. Changes in the content of the mathematics high school curriculum require a radical revision of the curriculum and of the training system for teachers of mathematics, while this training should be carried on with the significant time anticipation, beginning from the student years. Simultaneously, but also with anticipation of time, it is necessary to start working on the school text-books, problem books, manuals, descriptions of laboratory works, etc.

The most serious difficulty is the re-education of existing teachers, their professional and psychological orientation on new problems. For this, it is necessary to create appropriate conditions, first of all economic ones, because the teacher with 25 hours of teaching load per week is physically unable to master essentially new ideas. To this end, our specialists in methodics of teaching mathematics should make serious efforts; many among them should drop the empty speculating, and learn to write methodics manuals in simple language, available to a working teacher.