AN INTRODUCTION TO THE CONTINUITY OF FUNCTIONS USING SCIENTIFIC WORKPLACE

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Abstract. We recommend the visual presentations shown in this paper as a presentation of the content of continuity in certain grades of secondary schools.

In this paper we construct such a programme that can be applied to almost all functions, continuous and discontinues ones, in the programme package Scientific Workplace. This procedure enables teachers to add as many examples as necessary. Thus, the students understand properly this limit process, which is supposed to make the proper accepting of the continuity of function easier.

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1. Introduction

The continuity of functions is one of the most difficult topics of calculus. In our high schools the students learn continuity in the first semester of the fourth grade approximately within only two lessons. The authors teach this content both in high school and at the university (first semester). In particular, all three authors teach in special classes of students talented for mathematics, which are known as "mathematical high schools" where the continuity is taught in the second semester of the third grade. In these classes there are more than two lessons for teaching continuity but students are still young and some of them could have difficulties in understanding this concept properly.

It is very important for teachers to provide an easier, but still correct, presentation of this difficult topic. Therefore, we suggest visual introductions and visual presentations, whenever it is possible. There is a lot of programme packages such as Scientific Workplace, Maple, Mathematica, and many others, which can be very helpful in such visual presentations.

In particular, the introduction to the continuity, should start with everyday life examples, and "the lifting pencil method" (see [5]). Such an elaboration can be presented to all students at the beginning of their learning continuity in both high school and at the university. In this paper we suggest the possible continuation of this treatment. Apart from offering a visual introduction to Heine definition of continuous functions, it includes a number of such exercises as well. Since the students are taught the limits of sequences in the third grade, the Heine definition of continuity of a function at a point can be taught before the limits of functions (or, depending on teachers' preferences, after).

We hope that the suggested visual presentations can help students to understand the continuity, and to avoid the known misconceptions, such as the break of graph. Furthermore, these visual presentations can help develop advanced mathematical thinking. The presented topic can be used not only with the above mentioned high school students, but also in the case of university students who have problems with understanding the continuity.

The authors of this paper wish to recommend the visual presentations shown in this paper (and in the paper [5]) as a presentation of the content of continuity in certain grades of secondary schools.

In this paper we construct such a programme that can be applied to almost all functions, continuous and discontinues ones, at a point x_0 in the programme package Scientific Workplace. This procedure enables teachers to add as many examples as necessary. Thus, the students understand properly this limit process, which is supposed to make the proper accepting of the continuity of function easier.

These programmes are good for teaching, but they can be used for learning, as well. Namely, the visual presentations in this paper are prepared for teachers when they are in a situation to use a computer and a video projector. The students can use these programmes, also for doing exercises. The constructions of programmes are very easy, the students can make them, with the help of teacher, in a short time, and they are very useful for learning, as well.

The authors used programme package Scientific Workplace for teaching introductory analysis to both talented high school students and university students, during 2003/04 school year, in a computer classroom. The students used computers and the same programme package Scientific Workplace. They were instructed how to construct the same visual presentations as the teacher did. The programme about introducing continuity, discussed in this paper, was one of those they were taught.

The teachers were making the interviews after the lessons, in computer classroom. All students were very satisfied and delighted with this computer method. They claim that it made the lessons more interesting and that it helped them to understand the continuity of functions. Their marks from introductory analysis were very good, better than the previous ones. The authors hope that they will apply this method in 2004/05 school year, and analyze the results afterwards.

In Section 2 we present a general procedure and in Sections 3 and 4 we apply this procedure to certain typical examples.

2. Introduction to Heine definition of continuity

In this paper we introduce the continuity of a function $f\colon A \subset \mathbf{R} \to \mathbf{R}$, at a point x_0, using the visualization leading to the Heine's definition.

It is important to point out that the point x_0, in which the continuity is examined, must belong to the domain of function f, i.e. to A.

In our presentation we take that the point x_0 is an accumulation point of the domain A. The students generally have no problems in acquiring the idea of

accumulation point. If the teacher introduces this notion for the first time, the computer can be of great help in accepting it.

Let us recall the Heine definition of a continuous function.

DEFINITION 1. *A function $f \colon A \subset \mathbf{R} \to \mathbf{R}$ is continuous at the point $x_0 \in A$, where x_0 is an accumulation point of the domain A, if for every sequence $(x_n)_{n \in \mathbf{N}}$ of elements from A it holds that $\lim_{n \to \infty} f(x_n) = f(x_0)$.*

In this paper the authors consider some examples of continuous and discontinuous functions at interesting points x_0 from the domain of f. Various sequences $(x_n)_{n \in \mathbf{N}}$ tending to x_0, and the corresponding sequences of the values of function f, i.e., $(f(x_n))_{n \in \mathbf{N}}$, are considered.

The programme package Scientific Workplace is used for drawing the points $(x_n, 0)$, $(0, f(x_n))$, $(x_n, f(x_n))$, for $n = 1, 2, \ldots, 10$, or $n = 1, 2, \ldots, 15$, in (x, y)-plane, for chosen sequences $(x_n)_{n \in \mathbf{N}}$, including the examples of both, continuous and discontinuous, functions at the point x_0.

Graphically, it is shown that:

- when the sequence $(x_n)_{n \in \mathbf{N}}$ tends to x_0, then the points $(0, x_n)$, for $n = 1, 2, \ldots, 10$, or $n = 1, 2, \ldots, 15$, are getting closer to the point $(0, x_0)$, as n increases.

If f is a continuous function at the point x_0, then, it is shown graphically that:

- if the points $(0, x_n)$, are getting closer to the point $(0, x_0)$, with n increasing, then the points $(0, f(x_n))$ are getting closer to the point $(0, f(x_0))$ with n increasing;

- if the points $(0, x_n)$, are getting closer to the point $(0, x_0)$, with n increasing, then the points $(x_n, f(x_n))$ are getting closer to the point $(x_0, f(x_0))$ with n increasing.

If the function f has a discontinuity at the point x_0, then, graphically, the following is shown:

- if the points $(0, x_n)$, are getting closer to the point $(0, x_0)$, with n increasing, then the points $(0, f(x_n))$ are not getting closer to the point $(0, f(x_0)$ with n increasing.

- If the points $(0, x_n)$ are getting closer to the point $(0, x_0)$ with the increasing n, then the points $(x_n, f(x_n))$ are not getting closer to the point $(x_0, f(x_0)$ with n increasing.

The different visual presentations for the same examples of function, but for the same sequences are given. Namely, using the programme package Scientific Workplace we can present:

- all interesting points on x-axis together,
- all interesting points on y-axis together,
- all interesting points in (x, y)-plane together,
- the graph of a function or without it,

using as many colors as useful.

Moreover, we can present the animations with an increasing or decreasing number of interesting points, in each of the cases mentioned above.

3. Case of continuous functions

First, we can consider the linear function, for example $f(x) = 2x - 1$, whose domain and co-domain is **R**. Almost all students treat this function as being continuous over its domain. Let us choose the point $x_0 = 2$, (this point belongs to the considered domain of function) and $f(2) = 3$ holds.
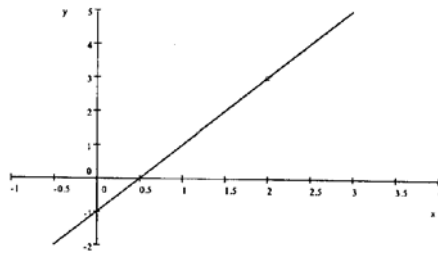

Fig. 1

The graph of the function f is drawn using a computer and Scientific Workplace.

Let us denote the point $(2, 3)$ and remember that it belongs to the graph of the function f. In order to connect the continuity at the point 2, and the sequences $(x_n)_{n \in \mathbf{N}}$, tending to 2, the teacher can ask the following question:

What happens with the values of the function $f(x) = 2x - 1$, when "x is close to 2, but $x \neq 2$", or when "x approaches 2, but x is not equal to 2"?

In order to answer this question one can take different sequences $(x_n)_{n \in \mathbf{N}}$, which satisfy that $\lim_{n \to \infty} x_n = 2$, and show graphically that the sequences of the corresponding values of functions f, $f(x_n)$, have the limit 3 (namely, $\lim_{n \to \infty} f(x_n) = 3$). Actually, we will show that when x approaches 2, then $f(x)$ approaches 3, by taking several sequences. We think that it is not necessary to use phrase "for every sequence" at this place. However, the role of accumulation point has to be explained.
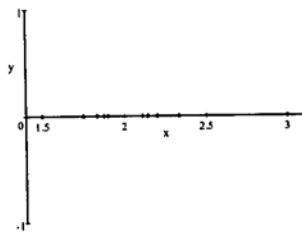

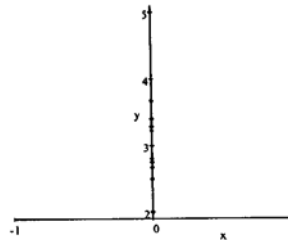Fig. 2   Fig. 3

First, we can consider the sequence given by $x_n = 2 + \frac{1}{n}$, tending to 2 from the right-hand side, and the points

- $(x_n, 0)$, $n = 1, \ldots, 10$, on x-axis;
- $(0, f(x_n))$, $n = 1, \ldots, 10$, on y-axis;
- $(x_n, f(x_n))$, $n = 1, \ldots, 10$, in (x, y)-plain,

and then, show the visual presentation. After that, we can change the sequence, for example $x_n = 2 - \frac{1}{n}$, tending to 2 from the left-hand side, and show the corresponding points. In this paper we shall show only the visual presentation for the sequence given by $x_n = 2 - \frac{(-1)^{n+1}}{n}$, and the corresponding points.

- $(x_n, 0)$, $n = 1, \ldots, 10$, on x-axis (Fig. 2);
- $(0, f(x_n))$, $n = 1, \ldots, 10$, on y-axis (Fig. 3);
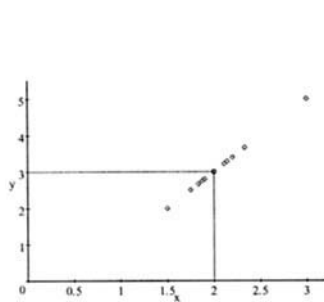- $(x_n, f(x_n))$, $n = 1, \ldots, 10$, in (x, y)-plain (Figs. 4 and 5).
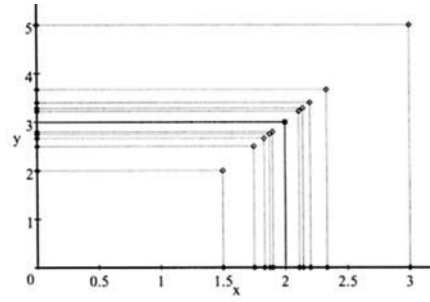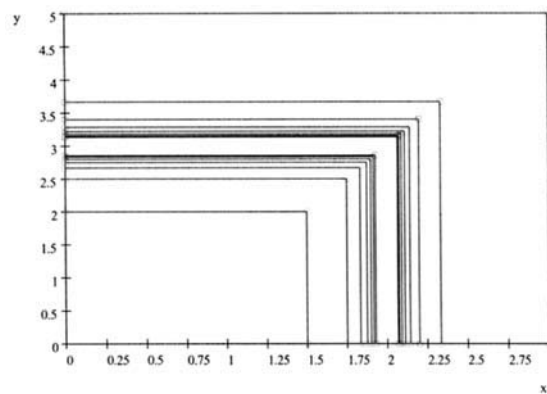

Fig. 4

Fig. 5

Fig. 6

In Figures 5 and 6, the horizontal and the vertical lines corresponding to the points $(x_n, 0)$, $(x_n, f(x_n))$, $(0, f(x_n))$, $n = 1, \ldots, 10$, or $n = 1, \ldots, 15$, are drawn in order to enable better understanding of the connection between the sequences

$(x_n)_{n \in N}$, and $f(x_n)_{n \in N}$, when n increases. (In Fig. 6, all points and lines are drawn for $n = 1, \ldots, 15$).

In Figures 5 and 6 it can be seen that the values x_n are tending to 2, when n is increasing, and the corresponding values of the function f, $f(x_n)$, are tending to 3, when n is increasing. Now, we can say that:

- the function f is continuous at $x = 2$ and therefore it has this property.
- If x is very close to 2, then the corresponding value $f(x)$ is very close to 3.

It can be seen from the two figures, that there exist two empty spaces: surrounding number 2, on x-axis, and number 3 on y-axis, as the horizontal and vertical strips surrounding lines $x = 2$ and $y = 3$. The reason for this phenomenon is, of course, the limit process.

In previous presentation, all employed points (and lines) appear together. However, we have also prepared the animations, for each case, in which points appear one after another. It can help students to understand the two considered sequences, $(x_n)_{n \in \mathbf{N}}$, $(f(x_n))_{n \in \mathbf{N}}$, their limits, and their connections.

Further on, in the visual presentation, we can easily change and consider more sequences $(x_n)_{n \in \mathbf{N}}$ tending to 2, and the corresponding $(f(x_n))_{n \in \mathbf{N}}$ tending to 3.

Moreover, we can change the point x_0 in which the continuity is examined and for each point we can construct different sequences $(x_n)_{n \in \mathbf{N}}$, tending to x_0, and exhibit the corresponding visualization.

Then we can take as many continuous functions as we find necessary and apply the mentioned procedure.

All these changes can be done very quickly, even immediately, with few alternations. This means that the teacher does not need to spend too much time in order to show the mentioned and some other visual presentations.

This programme can be applied by students also, if they have computers. At the beginning, they can use the same examples as the teachers, but later they can consider different cases, changing colours, thickness, adding and omitting some points and lines.

For further consideration we suggest the following functions: $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$, $f(x) = \sin x$, $f(x) = \cos x$, and the points $2, 0, -2$, and so on.
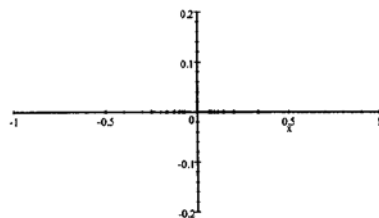

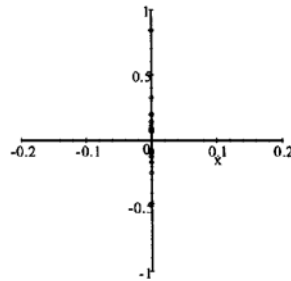Fig. 7   Fig. 8

For example, by taking $f(x) = \sin x$, and the point $x_0 = 0$, we can first consider the sequence $x_n = (-1)^{n+1}\frac{1}{n+1}$, tending 0, when n tends to ∞. The points

- $(x_n, 0)$, $n = 1, \ldots, 15$, are shown on x-axis (Fig. 7);
- $(0, f(x_n))$, $n = 1, \ldots, 10$, are shown on y-axis (Fig. 8);
- $(x_n, f(x_n))$, $n = 1, \ldots, 10$, are shown in (x, y)-plain (Figs. 9,10).
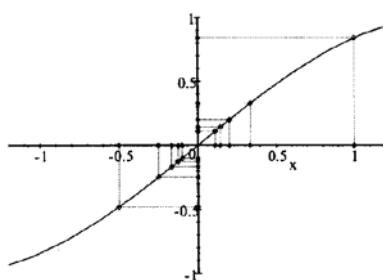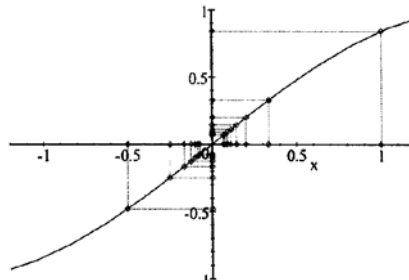

Fig. 9   Fig. 10

The only change in the programme is a change of the function and the intervals.

Let us remark that in this paper we did not pay attention to the quantifiers "for every" and "there exist". The reason for this is that this programme, or in general the computer, can show only the finite number of elements of each sequence. We hope that the use of these quantifiers will be elaborated by us later on.

In this paper the functions, as $f(x) = \frac{1}{x}$, having the break in graphs are not considered, because it would only be the repetition of the presentation in [5], and in our opinion, the elaborated programme does not help much in overcoming the existing conflicts.

4. Case of discontinuous functions

The procedure presented in the previous section can be applied to a function f which is not continuous at some points x_0. We choose sequences $(x_n)_{n \in \mathbf{N}}$ tending to x_0, but the corresponding sequence of function values f, i.e., $(f(x_n))_{n \in \mathbf{N}}$, does not tend to $f(x_0)$. For example, we can consider the function

$$f(x) = \begin{cases} 2x - 1, & \text{if } x < 0, \\ -1/2, & \text{if } x = 0, \\ \sin x, & \text{if } x > 0, \end{cases}$$

and the sequence given by $x_n = (-1)^{n+1}\frac{1}{n}$.

The corresponding points and lines are shown in Figures 11 and 12. The point $x_0 = 0$ belongs to the domain of f and $f(0) = -0.5$. We can remark that although $x_n \to 0$, the values of the function f, $f(x_n)$, do not tend to -0.5. The last two figures can be used for the following explanation:
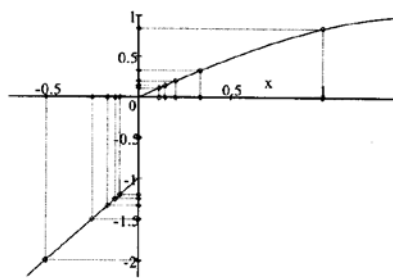
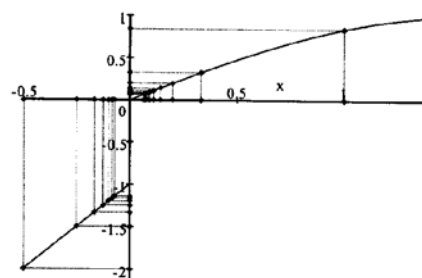Fig. 11   Fig. 12

- the subsequence given by $x_{2n-1} = (-1)^{2n}\frac{1}{2n-1} = \frac{1}{2n-1}$ of the sequence $(x_n)_{n\in\mathbf{N}}$ tends to 0 from the right-hand side and the corresponding values of function f tend to 0.

- the subsequence given by $x_{2n} = (-1)^{2n+1}\frac{1}{2n} = -\frac{1}{2n}$ of the sequence $(x_n)_{n\in\mathbf{N}}$ tends to 0 from the left-hand side and the corresponding values of function f tend to -1.

Next, we can choose various functions with the discontinuity at some points. The function $f(x) = \operatorname{sgn} x$ is interesting for consideration, because there are sequences tending to 0, such that the corresponding sequences of values of function f take two distinct values 1 and -1 and do not tend to 0.

Besides, this programme is very good for exhibiting functions which are continuous from the left-hand or from right-hand side. For example, we can consider the functions

$$f(x) = \begin{cases} \sin x, & \text{if } x \leqslant 0, \\ \cos x, & \text{if } x > 0, \end{cases} \qquad g(x) = \begin{cases} 2x - 1, & \text{if } x < 0, \\ x^2, & \text{if } x \geqslant 0, \end{cases}$$

and the similar examples.

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