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EXPLORATIONS AND DISCOVERIES IN THE CLASSROOM

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Abstract. One way of introducing youngsters to independent study is to get them involved in work on projects. The aim of this paper is to describe problems selected for five projects on topics from geometry. The first three projects are designed for 10–12 year old pupils, and the remaining projects for youngsters aged 13–14.

0. Introduction

An important reason for teaching mathematics in schools is to promote pupils' thought processes and powers of observation. Pupils should be given the opportunity to discover facts and connections between phenomena on their own whenever this is possible.

One way of introducing youngsters to independent study is to get them involved in work on projects. Especially beneficial for all types of pupils are projects centered around open ended problems: Feeling under no particular stress, even the weakest in the class are usually capable of solving easier tasks, and advanced pupils often get carried away, producing original ideas.

The aim of this paper is to describe problems selected for five projects on topics from geometry. The first three projects, designed for 10–12 years old pupils rely on the use of cardboard models and linkages. The remaining projects, for youngsters aged 13–14, are based on phenomena from physics: the laws of reflection for light and the location of the centre of gravity.

1. Projects for 10–12 year old pupils

PROJECT 1. *Construction of polyhedra with congruent equilateral faces. Comparison of the constructed solids.*

This project was carried out on a number of occasions, as follows: The pupils were given a large number of congruent, equilateral triangles, cut out of cardboard, and were asked to stick some of them together along their edges, to form various solids. They had to comment in writing on their observations at work. Also, they had to answer one specific question: Which of the solids they preferred to the rest, and for what reason?

The outcome of this project work was usually satisfactory. The pupils enjoyed employing their imagination, creating a wide variety of shapes. Apart from convex bodies, they usually made also some stellated polyhedra. There were always some “mathematically minded” who spotted that the solids with 4, 8 and 20 faces were “most pleasing” (apart from some of the stellated polyhedra) because “you could turn them around in different ways, but they still looked the same”. Some pupils remarked that there are some natural numbers n (e.g. $n = 5$) for which it is not possible to construct solids with n triangular faces.

The completion of the project was followed by a general discussion. The notion of a convex solid was clarified, and reasons for the importance of platonic solids (referring to their symmetries) were underlined. — In conclusion it can be said that the aims of the project were achieved: By producing shapes instead of looking at ready made ones, and by trying out different possibilities, the youngsters extended their knowledge of spatial figures. At the same time their sense of aesthetical beauty was enhanced.

PROJECT 2. *Identifying: a) nets of tetrahedra and*
b) nets of cubes.

The pupils were familiar with the following notions: a regular tetrahedron, a cube, and a net of a solid. They were given the following instructions.

1) a) On a sheet of paper draw shapes of various forms, made of four congruent equilateral triangles each of which has a side in common with at least one of the remaining three triangles. Draw as many different shapes as you can think of.
   b) Draw on a sheet of paper as many different “hexominoes” as you can think of. — A hexomino is a shape made of six congruent squares, each of which has a side in common with at least one of the remaining five squares.
2) Examine carefully your drawings, and before cutting them out answer the following questions:
   a) Which of the drawings, consisting of triangles, represent nets of tetrahedra?
   b) Which of the hexominoes represent nets of cubes?
3) Check your answers to Questions 2a) and 2b) by cutting out the shapes and folding them along the sides of the triangles, or of the squares.

This project has three major aims:

– The drawing of the shapes is a combinatorial problem. The pupils have to learn the benefits of a systematic approach for listing the various shapes. It is quite easy to produce all shapes, made of triangles, however the construction of all different hexominoes (there are 35 of them!) is a diffucult task at that age.
– Identifying nets of solids, without cutting them out is meant to train powers of visualization and to deepen understanding of threedimensional configurations.

– Asking the pupils to check their solutions to Questions 2a) and 2b) is an important step in creating the habits, necessary for independent work. The pupils should get accustomed to verify each detail during an investigation.

PROJECT 3. *A study of quadrilaterals with the help of linkages.*

The pupils were told that a quadrilateral is a shape, formed by four straight line segments: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$. They were given a large number of rods of various lengths, which could be fixed together to form linkages. The following instructions were given:

1) Construct a linkage by fitting together four rods in the shape of a quadrilateral and examine the geometrical properties of this shape.
2) By selecting rods of various lengths construct various types of linkages representing quadrilaterals. Describe some basic properties of these quadrilaterals.

The use of linkages had many advantages. First of all, the pupils noticed that a linkage, made in the shape of a quadrilateral, is not rigid. Thus they realized that there are infinitely many quadrilaterals $ABCD$ with given side lengths $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$. Some of these quadrilaterals could not be put on the table, with all of their sides lying on it; hence the existence of skew quadrilaterals was discovered. Any plane quadrilateral linkage, whose sides were not all equal, could be indented; this led to the discovery of concave quadrilaterals.

Task 2 led to the classification of plane quadrilaterals with respect to their side lengths and angles. Making all possible choices for the side lengths was a problem of combinatorics which required systematic treatment. By experimenting with rods of various lengths it was also discovered that in any quadrilateral each side has to have a smaller length than the lengths of the remaining three sides added together.

At the discussion after the completion of the project the pupils were given elastic rubber bands. By fixing the ends of the rubber bands to two opposite vertices of the quadrilateral represented by linkages, they could find out how the changing shape of the quadrilateral affected the position of the diagonals. The pupils discovered the law: A necessary and sufficient condition for a quadrilateral to be skew is that the lines carrying its diagonals do not belong to a common plane. This discovery reinforced the significance of skew lines in geometry.

2. Projects for 13–14 year old pupils

PROJECT 4. *Geometrical patterns created by reflected light rays.*

The pupils were familiar with the laws of reflection of light:

1. The incident ray, the reflected ray and the normal at the point of incidence all lie in the same plane.
2. The angle of incidence is equal to the angle of reflection.

The following problems were stated:

Four strips of plane mirrors of equal length l are the sides of a square $ABCD$. A light source S is situated on $\overline{AB}$ at distance x from B. A light ray emerges from S, hitting $\overline{BC}$ at a point T at distance y from B (fig. 1). The light ray gets reflected from $\overline{BC}$ and continues its travel.

Fig. 1

Problem 1. Take $l = 12$, $x = 1$ and $y \in \{1, 2, 3, \ldots, 11\}$. Study the path of the light ray emerging from S and proceeding through T, for all integer values of y from 1 to 11 inclusive. Write down all your observations concerning the path of the light ray, comparing your results for various values of y.

Problem 2. Change the position of the light source S on $\overline{AB}$ by taking $x = 2$, 3, 4, ... , 11, but keeping the slope y/x of the path's first leg from S to the mirror $\overline{BC}$ unchanged. Find out, how this affects the path of the light ray for any fixed value of y/x. (If the ray hits a corner of the square, it gets absorbed and the path terminates.)

Problem 3. In Problems 1 and 2 the slopes y/x were natural numbers. Consider now cases, where the slopes are rational numbers (for example take $x = 2$ and $y = 3$). Comment on the behaviour of the light ray.

This project had various aims, such as: 1) To introduce the notion of periodicity: The path of the light ray is periodical for all rational values of the slope y/x. This fact, verified for special cases, could not be verified in general at this stage, but it was pointed out to the pupils after the completion of their work. 2) To contribute to the training of pupils in analysing diagrams, and discovering connections between changing numerical data and the details of corresponding shapes. 3) To point out that mathematics is not an isolated science; the geometrical properties of the light rays' path, summarized in the laws of reflection, were known already to the ancient Greeks.

In the discussion after the work has been completed, it was underlined that: If $y/x = 1$ then the path of the light ray from S to S has the same lengths as the straight line segment $\overline{SS'''}$ obtained by three reflections of $ABCD$, as shown in fig. 2.

After that, using the same method, the youngsters were able to prove the following statement concerning all possible quadrilaterals $S\tilde{T}\tilde{U}\tilde{V}$ with given vertex S, which can be inscribed into the square $ABCD$: Among all quadrilaterals $S\tilde{T}\tilde{U}\tilde{V}$ inscribed into the square $ABCD$ with fixed vertex S the rectangle with $\overline{ST}$ as one of its sides has the smallest perimeter (equal to $2\,|AC|$). The proof is illustrated in fig. 3.

Fig. 2

$$|S\tilde{T}| + |\tilde{T}\tilde{U}| + |\tilde{U}\tilde{V}| + |\tilde{V}S| = |S\tilde{T}| + |\tilde{T}\tilde{U}'| + |\tilde{U}'\tilde{V}''| + |\tilde{V}''S'''| > |SS'''|$$

Fig. 3

The pupils were also told (without proof) that the path of the light ray is periodical if and only if the slope y/x is a rational number. If y/x is irrational, then the path is ergodic: it runs arbitrarily near to any point of the square.

PROJECT 5. *Properties of polygons and polyhedra deduced from knowledge about centre of gravity.*

The pupils knew the following fact from Physics:

If two masses m_A and m_B are attached to the endpoints A and B of a weightless rod $\overline{AB}$ then the centre of gravity of the system, consisting of m_A and m_B is at the point G of $\overline{AB}$ such that

$$m_A \cdot |AG| = m_B \cdot |GB|. \tag{1}$$

Before starting the project, the following problem was solved in class:

Prove, by using this property of the centre of gravity, that the medians of an arbitrary triangle meet in a common point.

The proof can be carried out as follows:

An arbitrary triangle will be considered as a weightless plate, with unit massses m_A, m_B and m_C attached to the vertices A, B and C respectively. There are three ways of grouping these masses into disjoint subsets, at least one of which contains more than one mass:

$$\begin{array}{ll} \text{(I)} & \{m_A, m_B\} \cup \{m_C\}; \\ \text{(II)} & \{m_B, m_C\} \cup \{m_A\}; \\ \text{(III)} & \{m_C, m_A\} \cup \{m_B\}. \end{array}$$

In case (I) one first constructs the centre of gravity for the system, consisting of the masses in $\{m_A, m_B\}$. Since $m_A = m_B$, this is the midpoint C' of $\overline{AB}$. The next step is to relocate both masses m_A and m_B to C'. In this way the centre of gravity of the initial system, with all masses at the vertices of the triangle, coincides with the centre of gravity of the weightless median $\overline{CC'}$ of the triangle ABC, with a unit mass at C and a mass of two units at C'. According to (1), the centre of gravity is the point G of $\overline{CC'}$ dividing it in the ratio

$$|CG| : |GC'| = 2 : 1.$$

By similar arguments, applied to cases (II) and (III), one shows that the centre of gravity G must belong to the median $\overline{AA'}$ and to the median $\overline{BB'}$, dividing them in ratio

$$|AG| : |GA'| = 2 : 1$$

and

$$|BG| : |GB'| = 2 : 1.$$

The above proportions lead to the geometrical statement:

In an arbitrary triangle all medians pass through a common point G, dividing each median in ratio $2 : 1$, starting from the vertex of the triangle on that median. G is called the centroid of the triangle.

The project which the pupils had to carry out had as its aim to discover geometrical properties of various polygons and polyhedra by considering them as weightless shapes with unit masses at their vertices, and by locating the centres of gravity of the corresponding system of masses.

We shall describe here results obtained by studying plane quadrilaterals, tetrahedra, and hexagons.

(a) Let $ABCD$ be an arbitrary quadrilateral in a plane. Consider it as a weightless plate with unit masses m_A, m_B, m_C and m_D attached to the vertices A, B, C and D respectively.

There are two types of grouping the four masses in disjoint subsets, which will be useful for locating the centre of gravity of the system:

(i) To divide $\{m_A, m_B, m_C, m_D\}$ into two disjoint subsets of two elements each. This can be done in three different ways: $\{m_A, m_B\} \cup \{m_C, m_D\}$, $\{m_A, m_C\} \cup \{m_B, m_D\}$, $\{m_A, m_D\} \cup \{m_B, m_C\}$.

Each grouping can be used to locate the centre of gravity G of the original system of masses. It is left to the reader to verify that the location of G leads to the geometrical statement:

(S_1): In an arbitrary plane quadrilateral the two straight line segments, joining the midpoints of the opposite sides of the quadrilateral, and the straight line segment, joining the midpoints of the diagonals have a common midpoint (see fig. 4).

Fig. 4

(ii) To divide $\{m_A, m_B, m_C, m_D\}$ into two disjoint subsets, one of which contains three elements, and the other the remaining fourth element. This can be done in four different ways.

The system of the three masses in the first subset has its centre of gravity G' at the centroid of the triangle whose vertices carry these masses. Thus the centre of gravity of the whole system lies on the line segment, connecting G' with the vertex, carrying the fourth mass—the element of the second subset in our division.

Fig. 5

This way of location reveals an intriguing property of the quadrilateral $ABCD$:

(S_2) Denote by D', C', B' and A' the centroids of the triangles ABC, ABD, ACD and BCD respectively. Then the straight line segments, connecting these centroids with the fourth vertex of the quadrilateral $ABCD$ (that is $\overline{D'D}$, $\overline{C'C}$, $\overline{B'B}$ and $\overline{A'A}$) meet in a common point G. The point G divides each of the segments in ratio $3:1$ (fig. 5).

It is interesting to note that the quadrilateral $A'B'C'D'$ is an enlargement of $ABCD$ with scale factor $-1/3$ and centre of enlargement G. (Prove it.)

(b) Four points A, B, C, D, carrying unit masses m_A, m_B, m_C and m_D do not have to belong to a common plane. In that case they can be considered as the vertices of a wieghtless tetrahedron, and the centre of gravity G of the system is the centroid of the tetrahedron. There are the same types of groupings of the four masses as in (a) (types (i) and (ii)) and the processes of locating G are the same. However the geometrical statements, deduced from the location of G have to be rephrased, referring to elements of a solid, instead of a plane shape $ABCD$: Instead of (S_1) we have

(S_1') The straight line segments, connecting the midpoints of the three pairs of opposite edges of an arbitrary tetrahedron, have a common midpoint.

(S_2) is replaced by

(S_2') The straight line segments, connecting the vertices of an arbitrary tetrahedron with the centroids of the opposite faces meet in a common point, which divides each segment in ratio $3:1$, starting from the vertex of the tetrahedron, which belongs to the segment.

Grouping the masses into two subsets of two elements each leads apart from (S_1') to a statement about planes meeting at G instead of segments:

(S_1'') The six planes, each of them containing an edge of an arbitrary tetrahedron and the midpoint of the opposite edge, meet in a common point—the centre G of the tetrahedron.

(c) Finally, two statements about an arbitrary (not necessarily plane) hexagon:

(S_3) Denote by M_1, M_2, M_3, M_4, M_5 and M_6 the midpoints of the sides $\overline{A_1A_2}$, $\overline{A_2A_3}$, $\overline{A_3A_4}$, $\overline{A_4A_5}$, $\overline{A_5A_6}$ and $\overline{A_6A_1}$ of an arbitrary (not necessarily plane) hexagon $A_1A_2A_3A_4A_5A_6$. Regardless of the shape of the hexagon the triangles $M_1M_3M_5$ and $M_2M_4M_6$ have a common centroid.

(S_4) Let A_i, A_j, A_k be any three vertices of an arbitrary (not necessarily plane) hexagon $A_1A_2A_3A_4A_5A_6$ and $A_{i'}$, $A_{j'}$, $A_{k'}$ the remaining vertices of the hexagon. Denote by T_{ijk} and $T_{i'j'k'}$ the centroids of the triangles formed by the above triples of vertices. Then the straight line segments $\overline{T_{ijk}T_{i'j'k'}}$ have a common midpoint for all possible pairs of disjoint subsets $\{A_i, A_j, A_k\}$ and $\{A_{i'}, A_{j'}, A_{k'}\}$ of the set $\{A_1, A_2, \ldots, A_6\}$. (There are 10 such pairs.)

Concluding remarks. The benefits from Project 5 are many sided. One of them is that it makes pupils aware: Mathematics is not a rigid, ready-made science. By experimenting with data, altering assumptions, and paying attention to all details at work, new problems can emerge.

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