THEORETICAL AND APPLIED MECHANICS
Volume 51 (2024) Issue 1, 13–25 DOI: https://doi.org/10.2298/TAM230927001V

SYNCHRONIZATION CONDITIONS FOR STOCHASTIC LANDSLIDE CHAIN MODEL WITH DELAYED COUPLING

Nebojša Vasović, Srđan Kostić, Kristina Todorović, and Dragoslav Kuzmanović

Abstract. We examine the conditions for synchronization of landslide stochastic chain model with delayed coupling. Firstly, a new chain model for landslide dynamics is proposed, with the included effect of delayed coupling and background noise. The model is of the microscopic type, where the state of each block in the chain is influenced by the previous state of the same block and its neighbors as well as by noise. Secondly, we examine the stochastic synchronization of such a system of stochastic delay-differential equations. A sufficient condition for the exponential mean square stability of the synchronization is obtained. The sufficient condition indicates that the uni-directional asymmetric coupling induces the synchronization much more efficiently than the bi-directionally symmetric one. From the practical viewpoint, the results obtained confirm that different parts of the large unstable slope could exhibit synchronized activity under certain conditions, which indicates their possible larger influence on the structures (and generation of corresponding deformation) compared to the individual effect of unsynchronized activities.

1. Introduction

Slow-moving, the so-called creeping landslides are usually represented as large earth masses moving permanently and slowly down the slope, with variable rate of movement, depending on the influence of precipitation, groundwater, oscillation of surface water levels and impact of river flow and dynamic conditions [1]. In Serbia, creeping landslides are abundant in the Neogene silty-clay sedimentary formations, like Mramor, Ražanj, Begaljičko brdo, Umka, Duboko, etc., whose impact on the urban environment and infrastructure is at high financial rate, requiring complex and expensive mitigation and remediation measures [2], some of which include cutting-edge early warning methods [3] or anti-sliding piles [4]. However, as a rule, if applied, these measures do not secure the absolute stability of the unstable slope, since, besides the influence of constant moving of the large creeping

2020 *Mathematics Subject Classification:* 37N99.
Key words and phrases: synchronization, time delay, noise, landslides.

landslide on the occurrence of structural deformation, there is a significant impact of many shallow landslides which are commonly activated within the same unstable slope (large creep landslide) and which exhibit more frequent activity with more rapid movement compared to the activity of the whole unstable slope. Activities of these "isolated" landslides are usually not correlated. However, it would be of great practical significance to provide the conditions for their synchronization, since synchronized activity of many smaller landslides within the same unstable slope indicates the conditions for the activity of the whole large landslide, thus, presenting greater threat to the nearby structures. From the viewpoint of nonlinear dynamics, the problem of synchronization has been in the focus for many decades. Although synchronization in natural systems could lead to unwanted states, including neurological diseases, involuntary muscle contraction, or networks of routers to pedestrians on the Millenium bridge, in some cases, synchronization represents a favorable state of a dynamical system. In particular, in the case of synchronization many connected (or isolated) units start to behave in the same way, which makes the system controllable and easily described. Therefore, there is a great interest in the analysis of conditions necessary for the synchronization to occur. For instance, Xie et al. [5] confirmed the conditions for finite-time synchronization in complex delayed networks with Markovian jumping parameters and stochastic perturbations. Liu et al. [6] designed an adaptive controller for cluster synchronization in complex network models, influenced by coupling delays and perturbed by noise, also by applying Ito's formula, as in the present case. Cong et al. [7] used the Lyapunov stability theory and the Ito formula in order to obtain new sufficient conditions for the synchronization of two chaotic systems, with included influence of stochastic noise and time-varying delay. In our previous papers, we investigated the conditions for synchronization of noisy excitable systems with coupling and internal delays [8]. For the pair of interacting units, it is shown that the external/internal character of noise primarily influences frequency synchronization and the competition between the noise-induced and delay-driven oscillatory modes, while coherence of firing and phase synchronization substantially depend on internal delay. Ren et al. [9] examined the synchronization stability of stochastic linearly coupled differential equation systems, with signal-dependent noise perturbation. In order to examine the possible conditions for synchronization of activity of different slope parts, one needs to represent the landslide activity in a form of convenient dynamical system. Such model for landslides has been initially suggested by Davis [10], who proposed a spring-block model (composed of two blocks connected with damped spring) as a dynamical system of an accumulation slide, where two blocks represent feeder and accumulation part of the slope. Soon afterwards, Chau [11] also suggested a spring-block model as a convenient dynamical system for dynamics of a creeping land-slide. Helmestter et al. [12] examined landslide as a slider-block model, under the assumption of state and velocity dependent friction law. They established four different dynamical regimes, two of which are ascribed to the Vaiont landslide (velocity weakening unstable regime) and the La Clapiere (velocity strengthening stable regime). In our previous paper [13], we examined the dynamics of the model initially suggested by Chau [11], with included delay coupling. Morales et al. [14] examined the landslide dynamics by studying dynamics of a mechanical block-spring slider model with three different friction laws.

To the best of our knowledge, although we previously examined the synchronization of neurons [15], there were no previous attempts in defining the conditions of synchronization for any of the previously suggested landslide mechanical models. Moreover, previously suggested landslide spring-block models have not taken into account the delayed coupling, as suggested by Davis [10], nor did they consider the influence of the background noise. Concerning the important impact of time delay on the generation of complex dynamics and possibly significant effect of noise, especially in the case where the dynamical system under study is near the transition between different dynamical regimes, or in the case where the system exhibits excitable behavior (which could be the case for landslide dynamics), we feel that the new model proposed in this paper addresses the landslide dynamics properly and enables revealing the background mechanism of complex displacements of unstable slope.

One should note that two properties of delay-differential systems make the study of the influence of noise on such systems nontrivial. Firstly, a deterministic delay-differential system has a nonzero memory, i.e., it does not satisfy the Markov property. Consequently, some of the well-established methods to study stochastic systems which are valid for Markov processes, like Fokker-Planck equations, cannot be used [16]. Secondly, a single nonlinear scalar deterministic delay-differential equation with a single fixed time-lag τ gives an infinite dimensional dynamical system on the phase space $C(-\tau, 0)$ of continuous functions on the interval $(-\tau, 0)$ [17]. Large τ usually implies high-dimensional chaotic attractor, first studied in [18]. Apart from hyperchaos, an important effect of large delay is also the multistability of periodic attractors, see e.g. the classical reference [19].

In this paper we firstly suggest a new landslide mechanical model, with delayed coupling and background noise. Such model could be considered as microscopic because the structure of the landslide flow is considered discrete with the main ingredients being the individual blocks (several landslides within the same slope or parts of the same unstable slope). Secondly, we provide a sufficient condition for synchronization of a land-slide chain model, with different time lags and background random noise.

Paper is structured as follows. In Section 2 we provide a brief description of the examined model. In Section 3 we postulate the theorem, which is proved in Section 4. Conclusions are provided in the final section, together with the directions for further research.

2. Model Description

We shall examine the general open chain model of spring-block elements as the phenomenological model for landslide dynamics, expressed by scalar delay-differential equations (DDE) of the following form:

$$\dot{x} = \gamma g[x(t - \tau_1)] - \{\alpha x(t) - \beta f[x(t)]\}. \tag{2.1}$$

This model describes the local dynamics of units. One should note that f and g are quite general nonlinear functions, which satisfy Lipschitz conditions, as given further in the text, Eq. (2.3), and α, β, γ are parameters. Parameters α and β represent general friction parameters, qualitatively the same as parameters a and b in Dieterich-Ruina friction law. In particular, parameters a and b represent material properties, e.g. the direct velocity effect and evolutionary friction effect, which depend on the temperature and size of the normal stress [20]. Hence, the whole term $\alpha x(t) - \beta f[x(t)]$ represents a general expression for the nonlinear friction term. On the other hand, parameter γ denotes the strength of the effect of the delayed term. The delayed self-feedback τ_1 qualitatively describes the effect of the state of the sliding surface on the dynamics of the landslide model ("memory effect").

We consider N dynamical systems (2.1) with bi-directional coupling between the neighboring units of the form $c_1(x_i - x_{i-1}) + c_2(x_i - x_{i+1})$. In real applications, the influence of the neighboring systems on the system $x_i(t)$ involves a nonzero time-lag.

In this specific landslide model, the state (velocity) $x_i(t)$ depends on the differences between the states (velocities) $x_i(t-\tau_1)$ and $x_{i+1}(t-\tau_1)$ and the states (velocities) $x_i(t-\tau_2)$ and $x_{i-1}(t-\tau_2)$. In other words, the previous coupling becomes $c_1[x_i(t-\tau_2) - x_{i-1}(t-\tau_2)] + c_2[x_i(t-\tau_2) - x_{i-1}(t-\tau_2)]$. Feedback and transmission time lags τ_1 and τ_2 are not related in general.

It is also natural to assume that many features of real systems have been neglected in making the model, but that they can be retaken into consideration as different types of random perturbations of the deterministic model.

We thus arrive at the systems of Ito stochastic delay-differential equations (SDDEs) of the following form:

$$\begin{aligned} \mathrm{d}x_i(t) = {} & \{-\alpha x_i(t) + \beta f[x_i(t)] + \gamma g[x_i(t-\tau_1)]\}\mathrm{d}t \\ & - \{c_1[x_i(t-\tau_2) - x_{i-1}(t-\tau_2)] + c_2[x_i(t-\tau_2) - x_{i+1}(t-\tau_2)]\}\mathrm{d}t \\ & + x_i(t)\sqrt{2D}\,\mathrm{d}W, \quad \text{where} \quad i = 2, 3, \ldots, N-1. \end{aligned} \tag{2.2}$$

In further analysis we shall consider an open chain of spring-block elements with moving boundaries, i.e., the system (2.2).

We shall always assume that $\mathrm{d}W$, formally written as $\mathrm{d}W = \xi(t)\mathrm{d}t$, is the stochastic increment of the Wiener process $\xi(t)$ for which:

$$E(\xi) = 0, \quad E[\xi(t)\xi(t^{'})] = \delta(t - t^{'}),$$

where $E(\cdot\,)$ denotes the mean with respect to the stochastic process. The increments satisfy:

$$E(\mathrm{d}W) = 0, \quad \mathrm{d}W\mathrm{d}W = \mathrm{d}t.$$

We repeat that $f(x)$ and $g(x)$ are piece-wise continuous functions, α, β, γ, c_1 and c_2 are parameters, and τ_1 and τ_2 are the feedback and transmission time-lags, respectively.

In order to ensure the existence of solutions for (2.2), we shall require that the functions f and g satisfy the usual uniform Lipschitz condition, i.e.

$$|\beta f[x_i(t)] + \gamma g[x_i(t-\tau_1)] - \beta f[x_j(t)] - \gamma g[x_j(t-\tau_1)]| \tag{2.3}$$

$$\leqslant k[|x_i(t) - x_j(t)| + |x_i(t-\tau_1) - x_j(t-\tau_1)|].$$

The Lipschitz condition (2.3) will prove to be crucial in obtaining the conditions that guarantee the global stability of the exact synchronization in systems of the form (2.2). Let

$$\Delta_i(t) = x_i(t) - x_{i-1}(t), \quad i = 2, 3, \dots, N.$$

denote the difference (synchronization error) between the solution $x_i(t)$ and the nearest neighbor solution $x_{i-1}(t)$, where $i = 2, 3, \dots, N$. Then, $\mathrm{d}\Delta i(t)$ can be estimated using the Lipschitz condition (2.3) as follows:

$$\begin{aligned} \mathrm{d}\Delta_i(t) \leqslant [&-\alpha\Delta_i(t) + k|\Delta_i(t)| + k|\Delta_i(t-\tau_1)| \\ &- (c_1+c_2)\Delta_i(t-\tau_2) + c_2\Delta_{i+1}(t-\tau_2) \\ &+ c_1\Delta_{i-1}(t-\tau_2)]\mathrm{d}t + \Delta_i(t)\sqrt{2D}\,\mathrm{d}W, \end{aligned} \tag{2.4}$$

where $\Delta_i(t-\tau) = x_i(t-\tau) - x_{i-1}(t-\tau)$ and $i = 3, 4, \dots, N-1$. For $i = 2$ we obtain:

$$\begin{aligned} \mathrm{d}\Delta_2(t) \leqslant [&-\alpha\Delta_2(t) + k|\Delta_2(t)| + k|\Delta_2(t-\tau_1)| \\ &- (c_1+c_2)\Delta_2(t-\tau_2) + c_2\Delta_3(t-\tau_2)]\mathrm{d}t + \Delta_2(t)\sqrt{2D}\,\mathrm{d}W, \end{aligned} \tag{2.5}$$

and for $i = N$:

$$\begin{aligned} \mathrm{d}\Delta_N(t) \leqslant [&-\alpha\Delta_N(t) + k|\Delta_N(t)| + k|\Delta_N(t-\tau_1)| - \\ &- (c_1+c_2)\Delta_N(t-\tau_2) + c_2\Delta_{N-1}(t-\tau_2)]\mathrm{d}t + \Delta_N(t)\sqrt{2D}\,\mathrm{d}W. \end{aligned} \tag{2.6}$$

In the case of deterministic DDE global asymptotic stability of $\Sigma_{i=2}^{N}\Delta_i^2(t) = 0$ which implies that the global attractor of the deterministic part of (2.2) satisfies $x_1 = \dots = x_N$ and, in that case, sufficient condition for the global asymptotic stability of $\Sigma_{i=2}^{N}\Delta_i^2(t) = 0$ could be found by applying a generalization of the Lyapunov first method.

In the case of stochastic DDEs, the global exponential stability is replaced by the analogous stability in the mean value with respect to the distribution given by the stochastic process. The stability in the mean value for the given model (2.2) is given by the Theorem in Section 3.

3. Theorem

If constants α, k, c_1, c_2 and D (where k, c_1, c_2, α, D are real numbers, and k, c_1, c_2, α, $D \geqslant 0$) satisfy:

$$2k + 2c_1 + 2c_2 + D - \alpha < 0, \tag{3.1}$$

then the system (2.4), (2.5), (2.6) is exponentially stable in the mean value of

$$\sum_{i=2}^{N} \Delta_i^2(t),$$

i.e., exponentially stable in mean square.

Explicitly, if (3.1) is satisfied then the following inequality holds:

$$E\left[\sum_{i=2}^{N}\Delta_i^2(t)\right] \leqslant CE\left[\sum_{i=2}^{N}\Delta_i^2(0)\right]e^{-\lambda t}, \quad t \geqslant 0,$$

where $\lambda > 0$ is a constant and C is independent of t but might depend on c_1, c_2, k, α, τ, D and λ.

4. Generalized Ito formula

Suppose that $\mathrm{d}X^i = F^i\mathrm{d}t + G^i\mathrm{d}W$, with $F^i \in \mathbb{L}^1(0,T)$, $G^i \in \mathbb{L}^2(0,T)$, for $i = 1,\dots,n$. If $u\colon R_n \times [0,T] \to R$ is continuous, with continuous partial derivatives [21]: $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $(i,j = 1,\dots,n)$, then $\mathrm{d}[u(X^1,\dots,X^n,t)] = \frac{\partial u}{\partial t}\mathrm{d}t + \sum_{i=1}^{N}\frac{\partial u}{\partial x_i}\mathrm{d}X^i + \frac{1}{2}\sum_{i,j=1}^{N}\frac{\partial^2 u}{\partial x_i\partial x_j}G^iG^j\mathrm{d}t$, where we denote by $\mathbb{L}^2(0,T)$ the space of all real-valued progressively measurable stochastic processes $G(\cdot\,)$ such that $E\big(\int_0^T G^2\,\mathrm{d}t\big) < \infty$. Likewise, $\mathbb{L}^1(0,T)$ is the space of all real-valued progressively measurable processes $F(\cdot\,)$ such that $E\big(\int_0^T |F|\,\mathrm{d}t\big) < \infty$.

5. Proof of the theorem

Applying the Ito formula to $\sum_{i=2}^{N}\Delta_i^2(t)$, we obtain:

$$\mathrm{d}\left[\sum_{i=2}^{N}\Delta_i^2(t)\right] = 2\sum_{i=2}^{N}\Delta_i(t)\mathrm{d}\Delta_i(t) + \sum_{i=2}^{N}\Delta_i^2(t)2D\,\mathrm{d}t, \tag{5.1}$$

and

$$\begin{aligned}
2\sum_{i=2}^{N}&\Delta_i(t)\mathrm{d}\Delta_i(t) + \sum_{i=2}^{N}\Delta_i^2(t)2D\,\mathrm{d}t \\
&\leqslant 2\Delta_2(t)\{[-\alpha\Delta_2(t) + k|\Delta_2(t)| + k|\Delta_2(t-\tau_1)| \\
&- (c_1+c_2)\Delta_2(t-\tau_2) + c_2\Delta_3(t-\tau_2)]\mathrm{d}t + \Delta_2(t)\sqrt{2D}\,\mathrm{d}W\} \\
&+ 2\sum_{i=3}^{N-1}\Delta_i(t)\{[-\alpha\Delta_i(t) + k|\Delta_i(t)| + k|\Delta_i(t-\tau_1)| \\
&- (c_1+c_2)\Delta_i(t-\tau_2) + c_2\Delta_{i+1}(t-\tau_2) + c_1\Delta_{i-1}(t-\tau_2)]\mathrm{d}t \\
&+ \Delta_i(t)\sqrt{2D}\,\mathrm{d}W\} + 2\Delta_N(t)\{[-\alpha\Delta_N(t) \\
&+ k|\Delta_N(t)| + k|\Delta_N(t-\tau_1)| - (c_1+c_2)\Delta_N(t-\tau_2) \\
&+ c_1\Delta_{N-1}(t-\tau_2)]\mathrm{d}t + \Delta_N(t)\sqrt{2D}\,\mathrm{d}W\} + \sum_{i=2}^{N}\Delta_i^2(t)2D\,\mathrm{d}t,
\end{aligned} \tag{5.2}$$

i.e., after some rearrangements, (5.1) and (5.2) become:

$$\mathrm{d}\left[\sum_{i=2}^{N}\Delta_i^2(t)\right] \leqslant \sum_{i=2}^{N}\{[-2\alpha\Delta_i^2(t) + 2k\Delta_i(t)|\Delta_i(t)| + 2k\Delta_i(t)|\Delta_i(t-\tau_1)|$$

$$- 2(c_1 + c_2)\Delta_i(t)\Delta_i(t - \tau_2)]\mathrm{d}t + 2\Delta_i^2(t)\sqrt{2D}\,\mathrm{d}W\}$$
$$+ \sum_{i=3}^{N} 2c_1\Delta_i(t)\Delta_{i-1}(t-\tau_2)\mathrm{d}t + \sum_{i=2}^{N-1} 2c_2\Delta_i(t)\Delta_{i+1}(t-\tau_2)\mathrm{d}t + \sum_{i=2}^{N}\Delta_i^2(t)2D\,\mathrm{d}t,$$

where k, c_1, c_2, α, $D \geqslant 0$.

Let us introduce the new variable U:

$$U(\Delta_2^2(s), \dots, \Delta_N^2(s), s) = \sum_{i=2}^{N} e^{-2\alpha(t-s)}\Delta_i^2(s)$$

and, integrating with respect to s, we get:

$$\int_0^t \mathrm{d}[e^{-2\alpha(t-s)}\Delta_i^2(s)] = \int_0^t \sum_{i=2}^{N}\Big[\frac{\partial U}{\partial s}\mathrm{d}s + \frac{\partial U}{\partial x_i}\mathrm{d}X_i + \frac{1}{2}\frac{\partial^2 U}{\partial x_i^2}G_i^2\mathrm{d}s\Big], \tag{5.3}$$

where $X_i = \Delta_i(s)$ and $\mathrm{d}X_i = \mathrm{d}\Delta_i(s) = F_i\mathrm{d}s + G_i\mathrm{d}W(s)$.

After the integration in (5.3), one gets:

$$e^{-2\alpha(t-t)}\sum_{i=2}^{N}\Delta_i^2(t) - e^{-2\alpha(t-0)}\sum_{i=2}^{N}\Delta_i^2(0) \leqslant \int_0^t 2\alpha e^{-2\alpha(t-s)}\sum_{i=2}^{N}\Delta_i^2(s)\mathrm{d}s$$
$$+ \int_0^t e^{-2\alpha(t-s)}\Big\{\sum_{i=2}^{N}\{[-2\alpha\Delta_i^2(s) + 2k\Delta_i(s)|\Delta_i(s)| + 2k\Delta_i(s)|\Delta_i(s-\tau_1)|$$
$$- 2(c_1+c_2)\Delta_i(s)\Delta_i(s-\tau_2)]\mathrm{d}s + 2\Delta_i^2(s)\sqrt{2D}\,\mathrm{d}W(s) + \Delta_i^2(s)2D\,\mathrm{d}s\}$$
$$+ \sum_{i=3}^{N} 2c_1\Delta_i(s)\Delta_{i-1}(s-\tau_2)\mathrm{d}s + \sum_{i=2}^{N-1} 2c_2\Delta_i(s)\Delta_{i+1}(s-\tau_2)\mathrm{d}s\Big\}.$$

From the preceding expression, after some rearrangements, we obtain:

$$\sum_{i=2}^{N}\Delta_i^2(t) \leqslant e^{-2\alpha t}\sum_{i=2}^{N}\Delta_i^2(0) \tag{5.4}$$
$$+ \int_0^t e^{-2\alpha(t-s)}\Big\{\sum_{i=2}^{N}\{[2k\Delta_i^2(s) + 2k|\Delta_i(s)||\Delta_i(s-\tau_1)|$$
$$+ 2(c_1+c_2)|\Delta_i(s)||\Delta_i(s-\tau_2)|]\mathrm{d}s + 2\sqrt{2D}\Delta_i^2(s)\mathrm{d}W(s) + 2D\Delta_i^2(s)\mathrm{d}s\}$$
$$+ \sum_{i=3}^{N} 2c_1|\Delta_i(s)||\Delta_{i-1}(s-\tau_2)|\mathrm{d}s + \sum_{i=2}^{N-1} 2c_2|\Delta_i(s)||\Delta_{i+1}(s-\tau_2)|\mathrm{d}s\Big\}.$$

From (3.1) there exists some sufficiently small positive constant λ, $\alpha > \lambda > 0$, such that:

$$\alpha - \lambda - k - ke^{\lambda\tau} - 2(c_1+c_2)e^{\lambda\tau} - D > 0, \tag{5.5}$$

where $\tau = \max(\tau_1, \tau_2)$.

Considering (5.4) and (5.5) and since the following holds:

$$E\left[\int_0^t e^{[2\lambda t-2\alpha(t-s)]}2\sqrt{2D}\sum_{i=2}^{N}\Delta_i^2(s)\mathrm{d}W\right]=0,$$

one obtains:

$$\begin{aligned}(5.6)\quad E\left[\sum_{i=2}^{N}\Delta_i^2(t)\right]e^{2\lambda t}\leqslant{}& e^{(2\lambda-2\alpha)t}E\left[\sum_{i=2}^{N}\Delta_i^2(0)\right]\\
&+\int_0^t e^{(2\lambda-2\alpha)(t-s)}\Bigg\{2k\sum_{i=2}^{N}E[\Delta_i^2(s)]e^{2\lambda s}\\
&+2ke^{\lambda\tau_1}\sum_{i=2}^{N}E[|\Delta_i(s)||\Delta_i(s-\tau_1)|]e^{\lambda s}e^{\lambda(s-\tau_1)}\\
&+2(c_1+c_2)e^{\lambda\tau_2}\sum_{i=2}^{N}E[|\Delta_i(s)||\Delta_i(s-\tau_2)|]e^{\lambda s}e^{\lambda(s-\tau_2)}\\
&+2D\sum_{i=2}^{N}E[\Delta_i^2(s)]e^{2\lambda s}\\
&+2c_1e^{\lambda\tau_2}\sum_{i=3}^{N}E[|\Delta_i(s)||\Delta_{i-1}(s-\tau_2)|]e^{\lambda s}e^{\lambda(s-\tau_2)}\\
&+2c_2e^{\lambda\tau_2}\sum_{i=2}^{N-1}E[|\Delta_i(s)||\Delta_{i+1}(s-\tau_2)|]e^{\lambda s}e^{\lambda(s-\tau_2)}\Bigg\}\mathrm{d}s.\end{aligned}$$

Let us denote by:

$$G(t)=\sup_{\substack{-\tau\leqslant\theta\leqslant t\\-\tau\leqslant\psi\leqslant t\\2\leqslant i,j\leqslant N}}E[|\Delta_i(\theta)||\Delta_j(\psi)|]e^{\lambda\theta}e^{\lambda\psi}. \quad (5.7)$$

Then, considering (5.6), we obtain:

$$\begin{aligned}E\left[\sum_{i=2}^{N}\Delta_i^2(t)\right]e^{2\lambda t}\leqslant{}& e^{[2\lambda-2\alpha]t}E\left[\sum_{i=2}^{N}\Delta_i^2(0)\right]\\
&+\int_0^t e^{[2\lambda-2\alpha](t-s)}\mathrm{d}s[2k(N-1)G(t)+2ke^{\lambda\tau}(N-1)G(t)\\
&+2(c_1+c_2)e^{\lambda\tau}(N-1)G(t)+2D(N-1)G(t)\\
&+2c_1e^{\lambda\tau}(N-2)G(t)+2c_2e^{\lambda\tau}(N-2)G(t)],\end{aligned}$$

and after the integration in (5.7), one obtains:

$$(5.8)\quad E\left[\sum_{i=2}^{N}\Delta_i^2(t)\right]e^{2\lambda t}\leqslant E\left[\sum_{i=2}^{N}\Delta_i^2(0)\right]$$

$$+\frac{1}{(2\alpha-2\lambda)}\{[2k+2ke^{\lambda\tau}+2(c_1+c_2)e^{\lambda\tau}+2D](N-1)G(t) \\ +2(c_1+c_2)e^{\lambda\tau}(N-2)G(t)\}.$$

Considering (5.7), it follows from (5.8):

$$(2\alpha-2\lambda)(N-1)G(t)\leqslant(2\alpha-2\lambda)E\Big[\sum_{i=2}^{N}\Delta_i^2(0)\Big] \\ +\{[2k+2ke^{\lambda\tau}+2(c_1+c_2)e^{\lambda\tau}+2D](N-1)G(t) \\ +2(c_1+c_2)e^{\lambda\tau}(N-1)G(t)\}.$$

and, after some rearrangements:

$$[2\alpha-2\lambda-2k-2ke^{\lambda\tau}-4(c_1+c_2)e^{\lambda\tau}-2D](N-1)G(t)\leqslant(2\alpha-2\lambda)E\Big[\sum_{i=2}^{N}\Delta_t^2(0)\Big].$$

Further, we can write

$$[2\alpha-2\lambda-2k-2ke^{\lambda\tau}-4(c_1+c_2)e^{\lambda\tau}-2D]E\Big[\sum_{i=2}^{N}\Delta_i^2(t)\Big]e^{2\lambda t} \\ \leqslant[2\alpha-2\lambda-2k-2ke^{\lambda\tau}-4(c_1+c_2)e^{\lambda\tau}-2D](N-1)G(t) \\ \leqslant(2\alpha-2\lambda)E\Big[\sum_{i=2}^{N}\Delta_t^2(0)\Big]\leqslant 2\alpha E\Big[\sum_{i=2}^{N}\Delta_t^2(0)\Big],$$

which finally gives:

$$E\Big[\sum_{i=2}^{N}\Delta_i^2(t)\Big]\leqslant\frac{2\alpha E\big[\sum_{i=2}^{N}\Delta_i^2(0)\big]e^{-2\lambda t}}{[2\alpha-2\lambda-2k-2ke^{\lambda\tau}-4(c_1+c_2)e^{\lambda\tau}-2D]}$$

The proof is completed.

6. Example

We show that our theorem is valid for the stochastic landslide chain model with delayed coupling of the general form (2.2), composed of 96 units, where functions $f(v)$ and $g(v)$ are of the following general forms: $f(v)=\arctan v(t)$ and $g(v)=\sin v(t)$, respectively. Inverse trigonometric function could be considered as a qualitative representation of the plastic soil behavior, commonly referred to normally consolidated clays. Both functions satisfy Lipschitz conditions ($k=1$). For the sake of numerical simulation, we adopted the following parameters: $\beta=1$, $\tau=1$, $c_1=1$, $c_2=1$, $D=0.1$, $\tau_1=0.23$, $\tau_2=0.42$. In the case when conditions of the theorem are not satisfied, we adopted the value of $\alpha=1$ (according to our theorem, parameter α needs to be higher than 6.1 to satisfy the theorem conditions). This case is shown in Figure 1a. In Figure 1b we show the results of the computation when theorem conditions are satisfied ($\alpha=6.2$). Evidently, one observes synchronization of units when theorem conditions are satisfied.

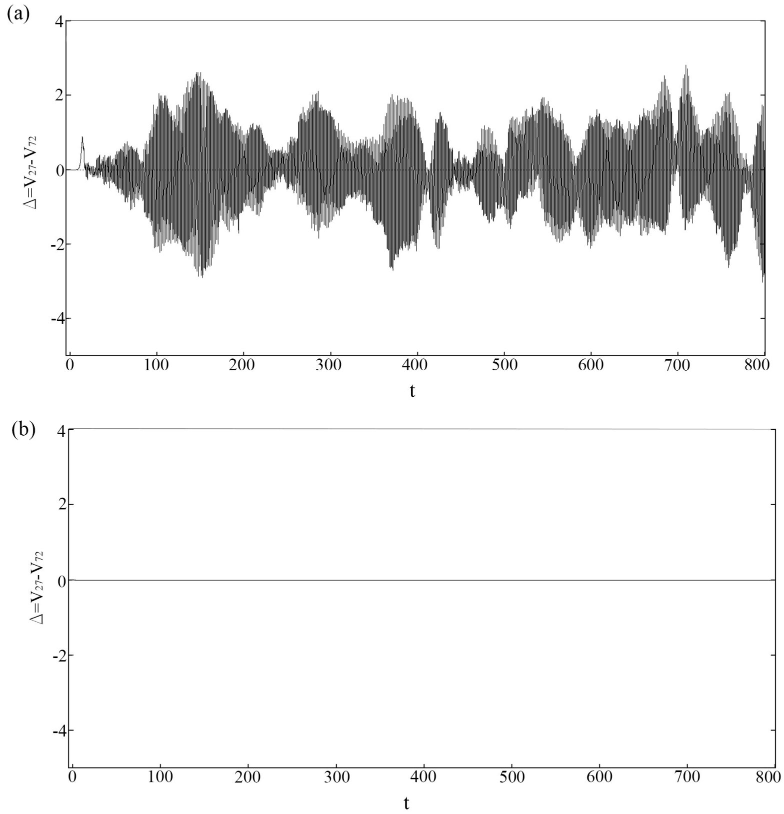

FIGURE 1. Temporal evolution of the velocity difference between the two randomly chosen units 27 and 72 in the array of 96 units of the stochastic landslide chain model with delayed coupling of the general form (2): (a) $\alpha = 1$ (theorem condition is not satisfied), (b) $\alpha = 6.2$ (theorem condition is satisfied). Other parameters of the model are set as follows: $\beta = 1$, $\gamma = 1$, $c_1 = 1$, $c_2 = 1$, $D = 0.1$, $\tau_1 = 0.23$, $\tau_2 = 0.42$.

7. Conclusions

We have studied the stochastic stability of exact synchronization in a novel model of one-dimensional landslide dynamics. The model is formulated in terms of the block's velocities so that the acceleration of each block is determined by the block's velocity and the difference of the velocities of the nearest neighbors in front and behind. The model explicitly considers the effect of noise and different types of feedback delays. Quite a general form of nonlinear feedback is considered, and the number of units in the chain is arbitrary. Mathematically, the system is described by a set of N scalar nonlinear stochastic DDEs. The nonlinear functions characterizing each unit in the chain are piece-wise continuous and satisfy the uniform

Lipschitz condition but are otherwise arbitrary. Thus, the dynamics of each unit when decoupled from the chain can be quite complex depending on the internal parameters and the feedback delay. We have proved that exact synchronization in the mean square can be achieved if an explicit sufficient condition is satisfied. The sufficient condition for the global asymptotic in the mean square of the exact synchronization involves the coupling constants c_2, c_1 and the Lipschitz constant k. The latter can be estimated in terms of the internal parameters and the feedback delay τ_1. The sufficient condition does not depend at all on N and coupling delay τ_2. The sufficient condition indicates that the uni-directional asymmetric coupling induces the synchronization much more efficiently than the bi-directionally symmetric one. It means that by setting $c_1 = 0$ or $c_2 = 0$, there is a larger range of noise intensities where the synchronized solution is stable with other parameters fixed. Also, the mean-square stability of the synchronized solution is independent of the system size.

Acknowledgments. We devote this paper to the late professor Nikola Burić whose original ideas in the study of synchronization in different dynamical systems still inspire us.

References

1. M. G. Angeli, E. Bromhead, P. Gasparetto, F. Marabini, F. Pontoni, *Stop-start landslides and the creep phenomena*, In: S. Aversa, I. Cascini, L. Picarelli, C. Scavia (eds.), Landslides and Engineered Slopes. Experience, Theory and Practice, 325–332, CRC press, 2016.
2. D. Sunarić, *Engineering Geodynamics: theory and practice*, Jaroslav Černi Water Institute and the Academy of Engineering Sciences of Serbia, Belgrade, 2017. (in Serbian)
3. Z.-X. Zhang, L. Wang, Y.-M. Wang, *A novel early warning method for handling non-homogeneous information*, Mathematics **10** (2022), 3016.
4. X. Gao, L. Wang, Q. Wang, X. Hu, Y. Wang, Y. Zhang, *Stability analysis and the random response of anti-sliding pile for Erdaogou landslide considering spatial variability* Mathematics **11** (2023), 2318.
5. Q. Xie, G. Si, Y. Zhang, Y. Yuan, R. Yao, *Finite-time synchronization and identification of complex delayed networks with Markovian jumping parameters and stochastic perturbations*, Chaos Solitons Fractals **86** (2016), 35–49.
6. Q. Liu, B. Kan, Q. Wang, Y. Fang, *Cluster synchronization of Markovian switching complex networks with hybrid couplings and stochastic perturbations*, Phys. A **526** (2019), 120937.
7. X. Cong, D. Tong, Q. Chen, W. Zhou, Y. Xu, *Exponential synchronization of chaotic systems with stochastic noise via periodically intermittent control*, Int. J. Robust Nonlinear Control **30** (2020), 2611–2624.
8. I. Franović, K. Todorović, N. Vasović, N. Burić, *Stability, coherent spiking and synchronization in noisy excitable systems with coupling and internal delays*, Commun. Nonlinear Sci. Numer. Simul. **19** (2014), 3202–3219.
9. Y. Ren, Q. Luo, W. Lu, *Synchronization analysis of linearly coupled systems with signal-dependent noises*, Mathematics **11** (2023), 2328.
10. R. O. Davis, *Modelling stability and surging in accumulation slides*, Engineering geology **33** (1992), 1–9.
11. K. T. Chau, *Landslides modeled as bifurcations of creeping slopes with nonlinear friction law*, Int. J. Solids Struct. **32** (1995), 3451–3464.
12. A. Helmstetter, D. Sornette, J.-R. Grasso, J. V. Andersen, S. Gluzman, V. Pisarenko, *Slider-block friction model for landslides: application to Vaiont and La Clapiere landslides*, JGR: Solid Earth **109** (2004), B02409.

13. S. Kostić, N. Vasović, I. Franović, D. Jevremović, D. Mitrinović, K. Todorović, *Dynamics of landslide model with time delay and periodic parameter perturbations*, Commun. Nonlinear Sci. Numer. Simul. **19** (2014), 3346–3361.
14. J. E. M. Morales, G. James, A. Tonnelier, *Travelling waves in a spring-block chain sliding down a slope*, Phys. Rev. E (3) **96** (2017), 012227.
15. N. Burić, K. Todorović, N. Vasović, *Exact synchronization of noisy bursting neurons with coupling delays*, Chaos Solitons Fractals **40** (2009), 1127–1135.
16. D. Goldobin, M. Rosenblum, A. Pikovsky, *Coherence of noisy oscillators with delayed feedback*, Physica A **327** (2003), 134.
17. J. Hale, S. V. Lunel, *Introduction to Functional Differntial Equations*, Springer, New York, 1993.
18. J. D. Farmer, *Chaotic attractors of an infinite-dimensional dynamical system*, Physica D **4** (1982), 366–393.
19. M. Wolfrum, S. Yanchuk, *Eckhaus instability in systems with large delay*, Phys. Rev. Lett. **96** (2006), 220201.
20. C. Scholz, *Earthquakes and friction laws*, Nature **391** (1998), 37–42.
21. C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, Springer-Verlag, Berlin, 2004.

УСЛОВИ СИНХРОНИЗАЦИЈЕ У ЛАНЧАНОМ МОДЕЛУ КЛИЗИШТА СА ШУМОМ И КАШЊЕЊЕМ У ВЕЗИ

Резиме. У раду испитујемо услове за синхронизацију ланачног модела клизишта са шумом и кашњењем у вези. Као прво, предлаже се нови ланчани модел динамике клизишта, са укљученим ефектом кашњења у вези и позадинског шума. Модел је микроскопског типа, где на стање сваког блока у ланцу утиче претходно стање истог блока, као и и суседних блокова, укључујући и утицај шума. Као друго, испитујемо стохастичку синхронизацију таквог система стохастичких диференцијалних једначина са кашњењем. Као резултат, добија се довољан услов за експоненцијалну средње квадратну стабилност синхронизације, који указује на то да једносмерно асиметрично повезивање блокова доводи до синхронизације много ефикасније од двосмерно симетричног. Са практичне тачке гледишта, добијени резултати потврђују да би различити делови велике нестабилне падине могли да испољавају синхронизовану активност под одређеним условима, што указује на њихов могући већи утицај на конструкције (и генерисање одговарајућих деформација) у односу на индивидуални ефекат несинхронизованих активности различитих делова падине.

(Received 27.09.2023)
(Revised 22.11.2023)
(Available online 15.01.2024)

Department of Applied Mathematics and Informatics
Faculty of Mining and Geology
University of Belgrade
Belgrade
Serbia
nebojsa.vasovic@rgf.bg.ac.rs
https://orcid.org/0000-0002-8294-5117

Geology Department
Jaroslav Černi Water Institute
Belgrade

Faculty of Technical Sciences
University of Novi Sad
Novi Sad
Serbia
srdjan.kostic@jcerni.rs; srdjan.kostic@uns.ac.rs
https://orcid.org/0000-0002-3705-3080

Department of Mathematics and Physics
Faculty of Pharmacy
University of Belgrade
Belgrade
Serbia
kristina.todorovic@pharmacy.bg.ac.rs
https://orcid.org/0000-0002-9960-022X

Faculty of Transport and Traffic Engineering
University of Belgrade
Belgrade
Serbia
kuzmanovic.dragoslav@gmail.com