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THE PROBLEM OF ACCELERATION IN THE DYNAMICS OF A DOUBLE-LINK WHEELED VEHICLE WITH ARBITRARILY DIRECTED PERIODIC EXCITATION

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Abstract. This study investigates the motion of a nonholonomic mechanical system that consists of two wheeled carriages articulated by a rigid frame. There is a point mass which oscillates at a given angle α to the main axis of one of the carriages. As a result, periodic excitation occurs in the system. The equations of motion in quasi-velocities are obtained. Eventually, the dynamics of a double-link wheeled vehicle is modeled by a system that defines a non-autonomous flow on a three-dimensional phase space. The behavior of integral curves at large velocities depending on the angle α is investigated. We use the generalized Poincaré transformation and reduce the original problem to the stability problem for the system with a degenerate linear part. The proof of stability uses the restriction of the system to the central manifold and averaging by normal forms up to order 4. The range of values of α for which one of the velocity components increases indefinitely is found and asymptotics for the solutions of the initial dynamical system is determined.

1. Introduction

We study the dynamics of a double-link wheeled vehicle (leading carriage and trailer) moving along a plane with an arbitrarily directed periodic excitation. The periodic excitation occurs due to oscillations of the point mass along a straight line located at a given angle α to the main axis of the leading carriage. One wheelset is rigidly fixed on each carriage. Nonholonomic constraints on wheels prohibit the motion of each carriage in a direction perpendicular to its main axis.

The first scientific papers on the dynamics of vehicles, including wheel systems, are dated to the middle of the XX century. For example, these are the papers of Y. Rocard [1], B. Stückler [2,3] and O. Bottema [4]. The results of modern studies of free dynamics of wheeled vehicles were published in [5–9]. The dynamics and control of mobile wheeled robots were considered and investigated in [10, 11].

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A well-studied model imitating the behavior of a simple two-wheeled robot on the horizontal plane is Chaplygin sleigh model [12–15]. The equivalence of the problem of any wheeled carriage and the problem of a similar carriage with a sharp blade instead of a wheel was proved in [13]. The dynamics of Chaplygin sleigh with parametric excitation was studied in [14, 15]. In [15], it was shown that, in some cases, the mass oscillating perpendicular to the main axis of Chaplygin sleigh is capable of indefinitely increasing the linear velocity of the sleigh.

More complex systems are vehicles with two bodies. One of these systems consists of two articulated Chaplygin sleighs and is called as Roller Racer. Free dynamics of Roller Racer was studied, for example, in [16]. The problem of the existence of regimes with unlimited energy growth (non-conservative Fermi acceleration) in the dynamics of Roller Racer was considered in [17].

We have published several papers on this sphere [18–20]. The free motion of a multibody wheeled system was investigated in [18] and the controlled motion of a multibody wheeled system was investigated in [19]. The results of a qualitative analysis of the motion of a two-link vehicle were presented in [20]. In [20], periodic excitation occurs due to an oscillation of a pair of point masses only along the main axis of one of the carriages. The center of mass of this carriage does not change. It is analytically proved that all velocities are bounded functions of time.

In this study, we investigate the dynamics of the double-link vehicle with periodic excitation, that occurs when there is a point mass, which oscillates in an *arbitrary* direction. We derive the equations of motion of this system based on the equations of motion in quasi-velocities with indefinite multipliers [21]. Thereafter, we investigate the problem of an unbounded increase of one of the velocity components (linear velocity). We call this fact as acceleration. To study the problem of speedup at large velocities, we use the Poincaré transformation [22–24], reduce the original problem to the stability problem, and to prove the stability we use the restriction of the system to the central manifold and normal forms up to order 4. The Poincaré transformation reflects the phase space of a dynamical system to the so-called "Poincaré sphere". The integral curves of the phase space pass into the corresponding curves on the sphere, and fixed points of higher orders corresponding to infinitely distant points of the phase space appear on the sphere. In [24], the Poincaré transformation was generalized for a non-autonomous system of differential equations and speedup effect in the Chaplygin sleigh problem was shown. It should be borne in mind that the method in our study is somewhat different from the method proposed in [24]. Firstly, the dimension of the dynamical system in our study is higher. Secondly, we do not provide the rigorous estimates of the region of the initial conditions for which there exist unbounded trajectories. Also, we determine the values of mechanical parameters for which speedup of the double-link wheeled vehicle takes place. To illustrate the results, graphs of velocities and trajectories of the first carriage are constructed.

In general, a deep understanding of the dynamics of Chaplygin sleigh and the simplest wheeled carriages allows to create control algorithms [25, 26] and extend them to simple (toys) and more complex mechanisms (garden wheeled equipment, transport wheeled equipment and others) with a variable center of mass.

The authors often limit themselves to the derivation of the equations of motion, because further investigation of the obtained dynamical systems turns out to be quite difficult. We hope that the results presented in this paper will be useful in solving various mechanical problems, including the problems of multibody systems described, for example, in [**27**, **28**].

2. Mathematical model and equations of motion

2.1. Schematic design and key assumptions. A bundle of two carriages (first link $\mathcal{L}_1$, second link $\mathcal{L}_2$) moves along the plane (See Fig. 1).

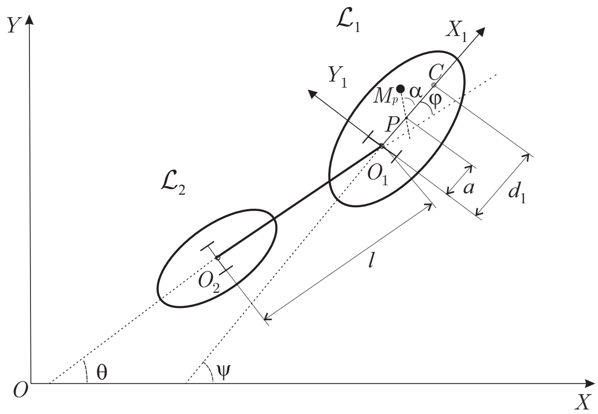

FIGURE 1. Schematic design of the double-link wheeled vehicle

We introduce the assumptions that we need to build a mathematical model.

- One balanced wheelset is rigidly attached to each carriage in the points O_i, $i = 1, 2$.
- Each wheel rolls without slipping and contacts with the reference plane only at one point.
- The carriages are connected at the points O_1 and O_2 by a rigid frame of length l. The second carriage (trailer) $\mathcal{L}_2$ is rigidly articulated with the frame. The frame connects to the first (leading) carriage $\mathcal{L}_1$ by a hinge and can freely rotate around a vertical axis passing through O_1.
- The center of mass C of the carriage $\mathcal{L}_1$ is shifted as illustrated in the Fig. 1 and $CO_1 = d_1$. The center of mass of the carriage $\mathcal{L}_2$ coincides with the point O_2.
- The point mass m_p moves along the first carriage $\mathcal{L}_1$. It performs periodic oscillations according to the law $b \sin \Omega t$ along a straight line that intersects the main axis of the carriage at an angle α at some point P and $O_1P = a$. We assume that $\Omega > 0$, $b > 0$. These conditions can always be fulfilled if the angle α, $\alpha \in [0, 2\pi)$ is chosen appropriately.

We introduce a fixed coordinate system OXY and a moving coordinate system $O_1X_1Y_1$ associated with the center of the wheelset O_1 on the plane (See Fig. 1).

Table 1. Key notations

$\mathbf{r}_1$	radius-vector of the point C
$\mathbf{r}_2$	radius-vector of the point O_2
$\mathbf{r}_p$	radius-vector of the point mass m_p
$\mathbf{v} = (v_1, v_2)$	velocity vector of the center of the first wheelset O_1
ω	angular velocity of the carriage $\mathcal{L}_1$
$\mathbf{u}_i$	velocity vectors of the center of mass of the carriage $\mathcal{L}_i$, $i = 1, 2$
$\mathbf{u}_p$	velocity vector of the point mass m_p

The axis O_1X_1 coincides with the main axis of the carriage $\mathcal{L}_1$. The axis O_1Y_1 passes through the wheel centers.

The orientations of the carriages $\mathcal{L}_1$ and $\mathcal{L}_2$ on the plane OXY are set by the angles ψ and θ respectively. The angle φ is the angle between the main axes of the carriages and

$$\varphi = \theta - \psi, \quad \dot{\psi} = \omega, \quad \dot{\theta} = \omega + \dot{\varphi}.$$

The angle φ takes positive value when the carriage $\mathcal{L}_2$ rotates around the point O_1 counterclockwise, otherwise it takes negative value.

We work in the $O_1X_1Y_1$ coordinate system. Key notations in this coordinate system are given in Tab. 1.

2.2. Nonholonomic constraints. The coordinates of the radius-vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_p$ are

$$\mathbf{r}_1 = (d_1, 0),\ \mathbf{r}_2 = (-l\cos\varphi, -l\sin\varphi),\ \mathbf{r}_p = (a + b\sin\Omega t\cos\alpha, b\sin\Omega t\sin\alpha).$$

The velocity vectors $\mathbf{u}_1$, $\mathbf{u}_2, \mathbf{u}_p$ are

$$\begin{aligned}
\mathbf{u}_1 &= (v_1, v_2 + d_1\omega),\\
\mathbf{u}_2 &= (v_1 + l(\omega + \dot{\varphi})\sin\varphi, v_2 - l(\omega + \dot{\varphi})\cos\varphi),\\
\mathbf{u}_p &= (v_1 + \Omega b\cos\Omega t\cos\alpha - \omega b\sin\Omega t\sin\alpha, v_2\\
&\qquad + \Omega b\cos\Omega t\sin\alpha + \omega(a + b\sin\Omega t\cos\alpha)).
\end{aligned}$$

Because velocity vector of the wheelset is always directed along the main axis of the carriage, the nonholonomic constraints are

$$\begin{gathered}
v_2 = 0,\\
v_1\sin\varphi - v_2\cos\varphi + l(\omega + \dot{\varphi}) = 0.
\end{gathered} \tag{2.1}$$

2.3. Kinetic energy and equations of motion. The kinetic energy of each carriage (T_1, T_2) and the kinetic energy of the moving point mass (T_p) are

$$\begin{aligned}
T_1 &= \tfrac{1}{2}I_1\omega^2 + \tfrac{1}{2}m_1(v_1^2+v_2^2) + m_1 v_2 d_1\omega,\\
T_2 &= \tfrac{1}{2}(I_2+m_2l^2)(\omega+\dot\varphi)^2 + \frac{1}{2}m_2(v_1^2+v_2^2) + m_2 l(\omega+\dot\varphi)(v_1\sin\varphi - v_2\cos\varphi),\\
T_p &= \tfrac{1}{2}m_p[v_1^2+v_2^2+\omega^2(a^2+2ab\sin\Omega t\cos\alpha + b^2\sin^2\Omega t)\\
&\qquad + 2\omega b\sin\Omega t(-v_1\sin\alpha + v_2\cos\alpha) + 2a\omega v_2\\
&\qquad + 2\Omega b\cos\Omega t(v_1\cos\alpha + (v_2+a\omega)\sin\alpha) + \Omega^2 b^2\cos^2\Omega t],
\end{aligned}$$

where m_i is the mass and I_i is the moment of inertia of the carriage $\mathcal{L}_i$ relative to the geometric center of its wheelset.

The kinetic energy of the whole system is

$$T = T_1 + T_2 + T_p. \tag{2.2}$$

We do not give the full expression of kinetic energy here, because it is volumetric.

The general equations of motion in quasi-velocities with indefinite multipliers λ_1, λ_2 is in [20]. We write down those equations for kinetic energy (2.2), solve them together with time derivatives of nonholonomic constraints (2.1) and obtain the equations of motion in the variables φ, v_1, ω in dimensionless quantities, counting $b = 1, m_p = 1$:

$$\begin{gathered}
\dot\varphi = -\tfrac{v_1}{\tilde l}\sin\varphi - \omega,\\
\begin{pmatrix}\dot v_1\\ \dot\omega\end{pmatrix} = \mathbf{\Sigma}\cdot\begin{pmatrix}-\frac{1}{2}\dot J_2(\varphi)v_1 + \omega^2(\tilde c + g_1) + 2\omega\Omega f_2 + \Omega^2 g_1\\ -v_1\omega(\tilde c+g_1) - \omega\dot J_1(t) + \Omega^2\tilde a f_1\end{pmatrix},
\end{gathered} \tag{2.3}$$

where

$$\begin{gathered}
\mathbf{\Sigma} = \begin{pmatrix}\frac{J_1(t)}{\Delta} & \frac{f_1(t)}{\Delta}\\ \frac{f_1(t)}{\Delta} & \frac{J_2(\varphi)}{\Delta}\end{pmatrix},\\
f_1 = f_1(t) = \sin\alpha\sin\Omega t,\quad f_2 = f_2(t) = \sin\alpha\cos\Omega t,\\
g_1 = g_1(t) = \cos\alpha\sin\Omega t,\quad g_2 = g_2(t) = \cos\alpha\cos\Omega t,\\
\tilde I_1 = \frac{I_1 + m_p a^2}{m_p b^2},\ \ \tilde I_2 = \frac{I_2 - m_2 l^2}{m_p l^2},\ \ \tilde c = \frac{d_1 m_1 + a m_p}{b m_p},\\
m = \frac{m_1+m_2+m_p}{m_p},\ \ \tilde a = \frac{a}{b},\ \ \tilde l = \frac{l}{b},\\
J_1(t) = \tilde I_1 + 2\tilde a g_1 + f_1^2 + g_1^2,\ \ \dot J_1(t) = 2\Omega(\tilde a g_2 + f_1 f_2 + g_1 g_2),\\
J_2(\varphi) = \tilde I_2\sin^2\varphi + m,\ \ \dot J_2(\varphi) = 2\tilde I_2\sin\varphi\cos\varphi\dot\varphi,\ \ \Delta = J_1(t)J_2(\varphi) - f_1^2.
\end{gathered}$$

The equations (2.3) define the non-autonomous flow on a three-dimensional phase space

$$\mathcal{M}^3 = \{(\varphi, v_1, \omega)\mid\ \varphi \bmod 2\pi, (v_1,\omega)\in\mathbb{R}^2\}.$$

We assume that

$$\tilde c + g_1 > 0$$

or

$$\tilde{c} > |\cos\alpha|.$$

As a rule, this case is realized in nonholonomic mechanics [24].

REMARK 2.1. In general case, the system (2.3) has neither symmetries nor involutions. However, for example, for $\alpha \in \{0, \pi\}$, the equations are invariant with respect to substitution

$$t \to t, \quad \varphi \to -\varphi, \quad v_1 \to v_1, \quad \omega \to -\omega.$$

For $\alpha \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ the equations are invariant with respect to substitution

$$t \to -t, \quad \varphi \to -\varphi, \quad v_1 \to -v_1, \quad \omega \to \omega,$$

that is, the system is reversible. Under the influence of this transformation, any trajectory of the system becomes the trajectory with the opposite direction of motion.

The trajectory of motion. For a complete analysis of the dynamics, we can construct the trajectory of the carriage $\mathcal{L}_1$ (the trajectory of the point O_1). To do this, we need to supplement the equations (2.3) with the following equations

$$\begin{pmatrix}\dot{X}\\ \dot{Y}\end{pmatrix} = v_1 \begin{pmatrix}[1.3]\cos\psi\\ \sin\psi\end{pmatrix}, \quad \dot{\psi} = \omega. \tag{2.4}$$

Next, we will determine the behavior of integral curves at large velocities for arbitrary parameters $\tilde{a}, \alpha$ under the early assumptions. For this purpose, we will use the Poincaré transformation [22], which was generalized for non-autonomous systems in [24].

3. Poincaré transformation and reduction to the stability problem

3.1. Poincaré transformation. In this section, we perform the Poincaré transformation and reduce the problem of investigating the behavior of integral curves of the system (2.3) at large velocities to the stability problem for a dynamical system with a degenerate linear part. In order to do this, we introduce an angular coordinate

$$\tau = t \bmod \frac{2\pi}{\Omega}$$

and rewrite the system (2.3) in the autonomous form

$$\begin{gathered}\frac{d\varphi}{dt} = -\frac{v_1}{\tilde{l}}\sin\varphi - \omega, \quad \frac{d\tau}{dt} = 1,\\ \begin{pmatrix}\frac{dv_1}{dt}\\ \frac{d\omega}{dt}\end{pmatrix} = \mathbf{\Sigma}\cdot\begin{pmatrix}-\tilde{I}_2\sin\varphi\cos\varphi\frac{d\varphi}{dt}v_1 + \omega^2(\tilde{c}+g_1) + 2\omega\Omega f_2 + \Omega^2 g_1\\ -v_1\omega(\tilde{c}+g_1) - 2\omega\Omega(2\tilde{a}g_2 + f_1f_2 + g_1g_2) + \Omega^2\tilde{a}f_1\end{pmatrix},\\ \mathbf{\Sigma} = \begin{pmatrix}\frac{J_1}{\Delta} & \frac{f_1}{\Delta}\\ \frac{f_1}{\Delta} & \frac{J_2}{\Delta}\end{pmatrix},\end{gathered} \tag{3.1}$$

where functions $f_1 = f_1(\tau)$, $f_2 = f_2(\tau)$, $g_1 = g_1(\tau)$, $g_2 = g_2(\tau)$, $J_1 = J_1(\tau)$, $J_2 = J_2(\varphi)$, $\Delta = \Delta(\varphi, \tau)$ are written with new variables. The system is defined on $\mathbb{R}^2\{v_1, \omega\} \times \mathbb{T}^2\{\varphi, \tau\}$.

Then, we make the Poincaré transformation:

$$v_1 = \frac{1}{x}, \quad \omega = \frac{y}{x}$$

and rescale the time (for $v_1 > 0$)

$$ds = v_1 dt.$$

The system (3.1) is rewritten with the new variables as

$$\frac{d\varphi}{ds} = -\Big[\frac{\sin\varphi}{\tilde{l}} + y\Big], \quad \frac{d\tau}{ds} = x,$$

$$\begin{aligned}
\frac{dx}{ds} &= -\Delta^{-1}\Big[J_1\Big(-\tilde{I}_2\sin\varphi\cos\varphi\frac{d\varphi}{ds}x + (\tilde{c}+g_1)xy^2 + 2\Omega f_2x^2y + \Omega^2 g_1x^3\Big)\\
&\quad + f_1(-(\tilde{c}+g_1)xy - 2\Omega(\tilde{a}g_2 + f_1f_2 + g_1g_2)x^2y + \Omega^2\tilde{a}f_1x^3)\Big],\\
\frac{dy}{ds} &= \Delta^{-1}\Big[f_1\Big(-\tilde{I}_2\sin\varphi\cos\varphi\frac{d\varphi}{ds} + (\tilde{c}+g_1)y^2 + 2\Omega f_2xy + \Omega^2 g_1x^2\Big)\\
&\quad + J_2(-(\tilde{c}+g_1)y - 2\Omega(\tilde{a}g_2 + f_1f_2 + g_1g_2)xy + \Omega^2\tilde{a}f_1x^2)\\
&\quad - J_1\Big(-\tilde{I}_2\sin\varphi\cos\varphi\frac{d\varphi}{ds}y + (\tilde{c}+g_1)y^3 + 2\Omega f_2xy^2 + \Omega^2 g_1x^2y\Big)\\
&\quad - f_1(-(\tilde{c}+g_1)y^2 - 2\Omega(\tilde{a}g_2 + f_1f_2 + g_1g_2)xy^2 + \Omega^2\tilde{a}f_1x^2y)\Big].
\end{aligned} \tag{3.2}$$

The phase space of the system (3.2) is four-dimensional:

$$\mathcal{M}^4 = \Big\{(\varphi, x, y, \tau) \mid x \in [0, +\infty),\ y \in (-\infty, +\infty),\ \varphi \bmod 2\pi,\ \tau \bmod \frac{2\pi}{\Omega}\Big\}.$$

Thus, the infinitely distant points ($v_1 \to +\infty$) of the phase space of the system (3.1) are transformed to the points of the submanifold $x = 0$ of the system (3.2).

3.2. Fixed points. The system (3.2) has a one-parameter family of fixed points

$$\varphi = 0,\ x = 0,\ y = 0, \tau = \tau_0. \tag{3.3}$$

This family forms a one-dimensional invariant submanifold of the system. We will find the conditions at which it is asymptotically stable and thereby we will obtain an acceleration criterion for the initial dynamical system.

The system (3.2) can be represented as two systems:

$$\frac{d}{ds}\begin{pmatrix}\varphi\\ y\end{pmatrix} = \mathbf{A}\begin{pmatrix}\varphi\\ y\end{pmatrix} + \mathbf{\Psi}(\varphi, x, y, \tau), \quad \mathbf{A} = \begin{pmatrix}-\frac{1}{\tilde{l}} & -1\\ 0 & -\frac{J_2(0)(\tilde{c}+g_1(\tau))}{\Delta(0,\tau)}\end{pmatrix},$$

$$\frac{d}{ds}\begin{pmatrix}x\\ \tau\end{pmatrix} = \mathbf{B}\begin{pmatrix}x\\ \tau\end{pmatrix} + \mathbf{\Phi}(\varphi, x, y, \tau), \quad \mathbf{B} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

in a neighborhood of the points (3.3). Both eigenvalues of the matrix $\mathbf{A}$ have negative real part and both eigenvalues of the matrix $\mathbf{B}$ have zero real parts for any τ. The linearization of the equations (3.2) does not accurately determine the nature of fixed points. Therefore, we reduce the stability problem to the analysis of the behavior of the system on the central manifold [29, 30].

4. Restriction of the system to the central manifold and asymptotic expansion

4.1. Restriction of the system to the central manifold. According to [29], in the neighborhood of (3.3) there is a two-dimensional central invariant manifold

$$\mathcal{M}_0^2 = \Big\{(\varphi, x, y, \tau) \mid 0 < x < \varepsilon, y = y(\tau, x) = O(x^n), \varphi = \varphi(\tau, x) = O(x^n), \tau \bmod \frac{2\pi}{\Omega}\Big\},$$

$n \geqslant 2$, on which we approximate the functions φ and y using series in power of x:

$$\begin{aligned} y &= \xi_2(\tau)x^2 + \xi_3(\tau)x^3 + \xi_4(\tau)x^4 + O(x^5), \\ \varphi &= \eta_2(\tau)x^2 + \eta_3(\tau)x^3 + \eta_4(\tau)x^4 + O(x^5). \end{aligned} \tag{4.1}$$

According to the reduction theorem (Theorem 5.5 in [29]), the trajectories are exponentially rapidly attracted to the central manifold $\mathcal{M}_0^2$. Therefore, the asymptotic stability of the original family (3.3) will follow from the asymptotic stability of the set $x = 0$ when the system is restricted to the central manifold.

So, we substitute (4.1) in the first and last equation of (3.2). The coefficients in the series (4.1) are equal

$$\begin{gathered} \xi_2 = \frac{\Omega^2(g_1 + \tilde{a}J_2(0))f_1}{J_2(0)(\tilde{c} + g_1)}, \\ \xi_3 = -\frac{1}{J_2(0)(\tilde{c} + g_1)} \cdot \frac{d}{d\tau}(\Delta(0,\tau)\cdot\xi_2), \\ \xi_4 = -\frac{1}{J_2(0)(\tilde{c} + g_1)} \cdot \frac{d}{d\tau}(\Delta(0,\tau)\cdot\xi_3) + \frac{f_1\xi_2^2}{J_2(0)} - \frac{\Omega^2\Delta(0,\tau)g_1\xi_2}{J_2(0)^2(\tilde{c} + g_1)}, \\ \eta_2 = -\xi_2\tilde{l}, \\ \eta_3 = -\tilde{l}\Big(\xi_3 + \frac{d\eta_2}{d\tau}\Big), \\ \eta_4 = -\tilde{l}\Big(\xi_4 + \frac{d\eta_3}{d\tau}\Big). \end{gathered}$$

The restriction of the system (3.2) has the following form:

$$\frac{dx}{ds} = \zeta_3x^3 + \zeta_4x^4 + \zeta_5x^5 + O(x^6), \quad \frac{d\tau}{ds} = x, \tag{4.2}$$

where

$$\begin{gathered} \zeta_3 = -\frac{\Omega^2 g_1}{J_2(0)}, \\ \zeta_4 = -\frac{1}{J_2(0)}\Big(2\Omega f_2\xi_2 + f_1\frac{d\xi_2}{d\tau}\Big), \\ \zeta_5 = -\frac{1}{J_2(0)}\Big(\frac{f_1\Omega^2(2g_1 + J_2(0)\tilde{a})}{J_2(0)}\xi_2 + 2\Omega f_2\xi_3 + f_1\frac{d\xi_3}{d\tau}\Big). \end{gathered}$$

We have the following cases depending on the parameters α and $\tilde{a}$.

If $\alpha \in \{0, \pi\}$ and $\tilde{a} \in \mathbb{R}$, then

$$y \equiv 0, \varphi \equiv 0.$$

We obtain the restriction of the system (3.2):

$$\frac{dx}{ds} = \zeta_3 x^3, \quad \frac{d\tau}{ds} = x, \tag{4.3}$$

where $\zeta_3 = -\frac{\Omega^2 g_1(\tau)}{J_2(0)}$.

Let $x \ll 1$. Dividing the first equation (4.3) by the second equation, after averaging over τ and moving to the initial time t, we obtain the equation

$$\frac{dx}{dt} = \langle \zeta_3 \rangle x^2.$$

REMARK 4.1. Here and further, the average value of the T-periodic function $f(t)$ is the value of the integral

$$\langle f \rangle = \frac{1}{T} \int_0^T f(t) dt.$$

Because $\langle \zeta_3 \rangle = 0$, we could not make an unambiguous conclusion referring to the Theorem 2 in [30] and have analyzed the behavior of integral curves in numerical experiments. The integral curves of the system (2.3) were bounded in numerical experiments. Similar conclusions can be drawn if $\alpha \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ and $\tilde{a} = 0$. The projections of the typical phase curves of the system (2.3) with $\alpha = 0$ on the plane (v_1, ω) are shown in Fig. 2.

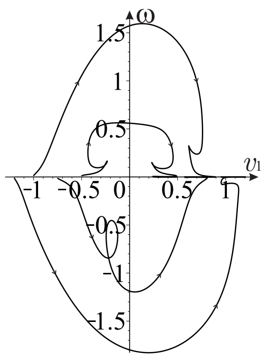

FIGURE 2. Projections of the typical phase curves on the plane (v_1, ω) with $\alpha = 0$

Then, we consider all the other values of the parameters $\alpha, \tilde{a}$, with the exception of those that have been described earlier.

Dividing the first equation (4.2) by the second equation we obtain the equation

$$\frac{dx}{d\tau} = \zeta_3 x^2 + \zeta_4 x^3 + \zeta_5 x^4 + O(x^5) \tag{4.4}$$

and $\langle \zeta_3 \rangle = \langle \zeta_4 \rangle = 0$.

We perform the transformation for averaging of the coefficients. Averaging the coefficients using normal forms up to order 4 and moving to the initial time t, we bring the equation (4.4) to the form

$$\frac{dx}{dt} = Z_3x^2 + Z_4x^3 + Z_5x^4 + O(x^5) \tag{4.5}$$

where Z_3, Z_4, Z_5 are constants.

4.2. Averaging up to order 4. We make the following change of variable in the equation (4.4)

$$x = R + s_2R^2 + s_3R^3 + s_4R^4 + O(R^5).$$

This gives the equation

$$\frac{dR}{d\tau} = (\zeta_3 - s_2')R^2 + (\zeta_4 + 2s_2s_2' - s_3')R^3 + (\zeta_5 + 2(s_3s_2)' + s_3's_2 - 5s_2^2s_2' - s_4')R^4 + O(R^5).$$

We choose s_1, s_2, s_3 as the periodic antiderivatives with zero averages from the equations

$$s_2' = \zeta_3,$$
$$s_3' = \zeta_4 + (s_2^2)',$$
$$s_4' = \zeta_5 + 2(s_3s_2)' + \zeta_4s_2 - (s_2^3)' - \langle\zeta_5\rangle - \langle\zeta_4s_2\rangle.$$

The equation describing the evolution of R is

$$\frac{dR}{d\tau} = (\langle\zeta_5\rangle + \langle\zeta_4s_2\rangle)R^4 + O(R^5).$$

So we obtain the coefficients in the expantion (4.5) for x:

$$Z_3 = 0, \quad Z_4 = 0, \quad Z_5 = \langle\zeta_5\rangle + \langle\zeta_4s_2\rangle.$$

4.3. Conclusions about the stability of fixed points. If the first non-zero coefficient Z_5 takes a negative value, according to [30] the family of fixed points (3.3) is asymptotically stable and the asymptotics for x as $t \to +\infty$ is obtained by solving the equation (4.5). We analytically find Z_5. Let $\alpha \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ and $\tilde{a} \neq 0$. In this case $\zeta_3 = 0$, then $s_2 = 0$ and $\langle\zeta_4s_2\rangle = 0$. Calculating $\langle\zeta_5\rangle$ we find the coefficient

$$Z_5 = Z^{(1)} = \frac{\tilde{a}\Omega^4}{8m^2c^2}(4\tilde{I}_1m + 3(m-1) - 4\tilde{a}\tilde{c}m).$$

The coefficient $Z^{(1)}$ takes negative value when

$$\tilde{a} \in (-\infty, 0) \cup \left(\frac{4\tilde{I}_1m + 3(m-1)}{4\tilde{c}m}, +\infty\right).$$

Then, according to the Theorem 2 in [30], the family of fixed points (3.3) is asymptotically stable and one can obtain the asymptotics for x, y, φ. Now we can formulate the following statement.

PROPOSITION 4.1. *The one-parameter family of fixed points*

$$\varphi = 0,\ x = 0,\ y = 0,\ \tau = \tau_0$$

of the system (3.2) *with the numeric parameters*

$$\alpha \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}, \quad \tilde{a} \in (-\infty, 0) \cup \left(\frac{4\tilde{I}_1 m + 3(m-1)}{4\tilde{c}m}, +\infty\right)$$

is asymptotically stable and on the $\mathcal{M}_0^2$ *we have the following asymptotics as* $t \to +\infty$*:*

$$\begin{aligned} x(t) &= (-3Z^{(1)}t)^{-1/3} + o(t^{-1/3}),\\ y(t) &= \frac{\tilde{a} f_1 \Omega^2}{\tilde{c}} \cdot (-3Z^{(1)}t)^{-2/3} + o(t^{-2/3}), \\ \varphi(t) &= -\frac{\tilde{a} f_1 \Omega^2 \tilde{l}}{\tilde{c}} \cdot (-3Z^{(1)}t)^{-2/3} + o(t^{-2/3}). \end{aligned} \tag{4.6}$$

Let $\alpha \neq \frac{\pi k}{2}, k \in \mathbb{Z}$, and $\tilde{a} \in \mathbb{R}$. In this case $\langle \zeta_4 s_2 \rangle \neq 0$. Calculating $\langle \zeta_5 \rangle$ and $\langle \zeta_4 s_2 \rangle$ we find the coefficient

$$Z_5 = Z^{(2)} = -\frac{\Omega^4 \sin^2\alpha}{m^3} \cdot G$$

where

$$\begin{gathered} G = \frac{F_1 + F_2}{r^3(\tilde{c}+r)^2} + \frac{(\tilde{c}-r)(\tilde{c}-\tilde{a}m)}{r}, \\ F_1 = (\tilde{c}-r)\left(\tilde{I}_1 m(\tilde{c}+r)^3 + \tilde{I}_1 m r^2(\tilde{c}+r) + (m-1)\tilde{c}k_1 + \tilde{c}(\tilde{c}+r)k_2\right), \\ F_2 = 2\tilde{a}^2 m^2(\tilde{c}+r)k_3 - \tilde{a}m\left(\tilde{I}_1 m\tilde{c}(\tilde{c}+r)^2 + (m-1)k_4 + 3(\tilde{c}^2 - r^2)k_5\right), \\ k_1 = \tilde{c}^2 + 3\tilde{c}r + 5r^2 > 0, \\ k_2 = \tilde{c}^3 + 2\tilde{c}^2 r + 6\tilde{c}r^2 - 3r^3 > 0, \\ k_3 = \tilde{c}^3 + \tilde{c}^2 r + \tilde{c}r^2 - 2r^3 > 0, \\ k_4 = \tilde{c}^3 + 2\tilde{c}^2 r + 2\tilde{c}r^2 - 2r^3 > 0, \\ k_5 = \tilde{c}^3 + 2\tilde{c}^2 r + 4\tilde{c}r^2 - r^3 > 0, \\ 0 < r = \sqrt{\tilde{c}^2 - \cos^2\alpha} < \tilde{c}. \end{gathered}$$

The coefficient $Z^{(2)}$ takes negative value when $G > 0$.

Then, according to the Theorem 2 in [30], the family of fixed points (3.3) is asymptotically stable and one can obtain the asymptotics for x, y, φ. Now, we can formulate the following statement:

PROPOSITION 4.2. *The one-parameter family of fixed points*

$$\varphi = 0, \quad x = 0, \quad y = 0, \quad \tau = \tau_0$$

of the system (3.2) *with the numeric parameters* $\alpha \neq \frac{\pi k}{2}, k \in \mathbb{Z}$, *and* $\tilde{a} \in \mathbb{R}$, *for which* $G > 0$, *is asymptotically stable and on the* $\mathcal{M}_0^2$ *we have the following* *asymptotics as* $t \to +\infty$*:*

$$
\begin{aligned}
x(t) &= (-3Z^{(2)}t)^{-1/3} + o(t^{-1/3}),\\
y(t) &= \frac{(g_1 + \tilde{a}m)\, f_1 \Omega^2}{m(\tilde{c} + g_1)} \cdot (-3Z^{(2)}t)^{-2/3} + o(t^{-2/3}),\\
\varphi(t) &= -\frac{(g_1 + \tilde{a}m)\, f_1 \Omega^2 \tilde{l}}{m(\tilde{c} + g_1)} \cdot (-3Z^{(2)}t)^{-2/3} + o(t^{-2/3}).
\end{aligned}
\tag{4.7}
$$

REMARK 4.2. Using the methods of normal forms [24], it is possible to show not only the asymptotic stability of the equilibrium points, but also to estimate the initial conditions, for which the trajectories of the system (3.2) have the asymptotic behavior by the formulas (4.1) or (4.2). According to Fenichel's theorem [31], in the neighborhood of the central invariant manifold, all trajectories exhibit similar asymptotic behavior.

5. Acceleration criteria

5.1. Acceleration criteria. Returning to the initial variables v_1, ω, φ, we formulate acceleration criteria that are given without rigorous proof, but will be verified with numerical experiments.

There exist a sufficiently small $\varepsilon > 0$ *and* ω_0, φ_0 *are some constants, so that, for the trajectories of the system* (2.3) *with the numeric parameters*

$$
\alpha \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}, \quad \tilde{a} \in (-\infty, 0) \cup \left(\frac{4\tilde{I}_1 m + 3(m-1)}{4\tilde{c}m}, +\infty\right)
$$

and the initial conditions $v_1 > \varepsilon^{-1}, |\omega| < \omega_0\varepsilon^2, |\varphi| < \varphi_0\varepsilon^2$ *we have* $v_1 \to +\infty$, $\omega \to 0$, $\varphi \to 0$ *as* $t \to +\infty$ *and*

$$
\begin{aligned}
v_1(t) &= (-3Z^{(1)}t)^{1/3} + o(t^{1/3}),\\
\omega(t) &= \frac{\tilde{a} f_1 \Omega^2}{\tilde{c}}(-3Z^{(1)}t)^{-1/3} + o(t^{-1/3}),\\
\varphi(t) &= -\frac{\tilde{a} f_1 \Omega^2 \tilde{l}}{\tilde{c}}(-3Z^{(1)}t)^{-2/3} + o(t^{-2/3}).
\end{aligned}
$$

There exist a sufficiently small $\varepsilon > 0$ *and* ω_0, φ_0 *are some constants, so that, for the trajectories of the system* (2.3) *with the numeric parameters* $\alpha \neq \frac{\pi k}{2}, k \in \mathbb{Z}$*, and* $\tilde{a} \in \mathbb{R}$*, for which* $G > 0$*, and the initial conditions* $v_1 > \varepsilon^{-1}, |\omega| < \omega_0\varepsilon^2, |\varphi| < \varphi_0\varepsilon^2$*, we have* $v_1 \to +\infty$, $\omega \to 0$, $\varphi \to 0$ *as* $t \to +\infty$ *and*

$$
\begin{aligned}
v_1(t) &= (-3Z^{(2)}t)^{1/3} + o(t^{1/3}),\\
\omega(t) &= \frac{(g_1 + \tilde{a}m)\, f_1 \Omega^2}{m(\tilde{c} + g_1)}(-3Z^{(2)}t)^{-1/3} + o(t^{-1/3}),\\
\varphi(t) &= -\frac{(g_1 + \tilde{a}m)\, f_1 \Omega^2 \tilde{l}}{m(\tilde{c} + g_1)}(-3Z^{(2)}t)^{-2/3} + o(t^{-2/3}).
\end{aligned}
$$

Thus, we have indicated cases when nonlinear speedup at large velocities takes place. Now, we verify the obtained results using numerical experiments.

5.2. Numerical experiments. To illustrate the conclusions, graphs of solutions of the equations (2.3) are constructed. Graphs of $v_1(t)$, $\varphi(t)$, $\omega(t)$ and the trajectory of the point O_1 with

$$\tilde{I}_1 = 0.8, \quad \tilde{I}_2 = 0.2, \quad m = 4, \quad \tilde{l} = 3, \quad \tilde{c} = 0.5, \quad \Omega = 0.5, \quad \tilde{a} = -0.5, \quad \alpha = \frac{\pi}{2}$$

are shown in Fig. 3. These parameters correspond to the case when there is acceleration in the system ($Z^{(1)} < 0$). The asymptotic behavior of the function $\frac{v_1}{t^{1/3}}$ is shown.

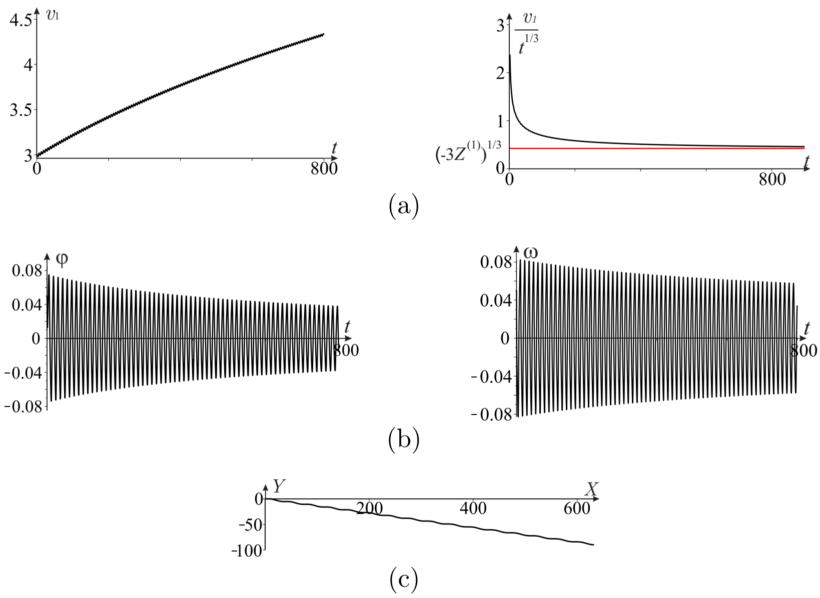

FIGURE 3. Illustration of the numerical experiment with $\tilde{a} = -0.5$, $\alpha = \frac{\pi}{2}$ and the initial conditions $v_1(0) = 3$, $\varphi(0) = 0.05$, $\omega(0) = 0.05$: (a) graph of $v_1(t)$ and graph of $\frac{v_1(t)}{t^{1/3}}$ (the red curve sets the asymptotics for function $\frac{v_1(t)}{t^{1/3}}$); (b) graphs of $\varphi(t)$, $\omega(t)$; (c) the trajectory of the point O_1 on the plane OXY

The results of the numerical experiment and the trajectory of the point O_1 on the plane OXY with $\alpha = 0.785$, $\tilde{a} = 0$ ($Z^{(2)} < 0$) corresponding to the unbounded increase of the velocity v_1 are shown in Fig. 4. The results of the numerical experiment and the trajectory of the point O_1 on the plane OXY with $\alpha = 0.785$, $\tilde{a} = 0.3$ ($Z^{(2)} > 0$) corresponding to the bounded velocity v_1 are shown in Fig. 5. Other mechanical parameters are

$$\tilde{I}_1 = 0.8, \quad \tilde{I}_2 = 0.2, \quad m = 4, \quad \tilde{l} = 3, \quad \tilde{c} = 0.8, \quad \Omega = 0.5.$$

The results of numerical experiments are consistent with analytical calculations.

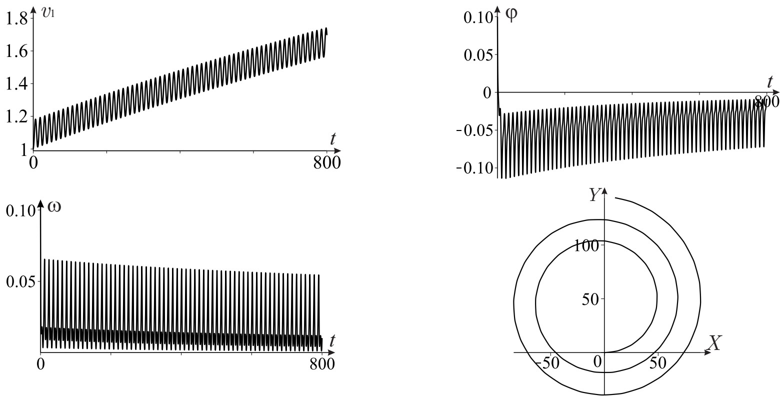

FIGURE 4. Illustration of the numerical experiment with $\alpha = 0.785$, $\tilde{a} = 0$ and the initial conditions $v_1(0) = 1$, $\varphi(0) = 0.1$, $\omega(0) = 0.1$

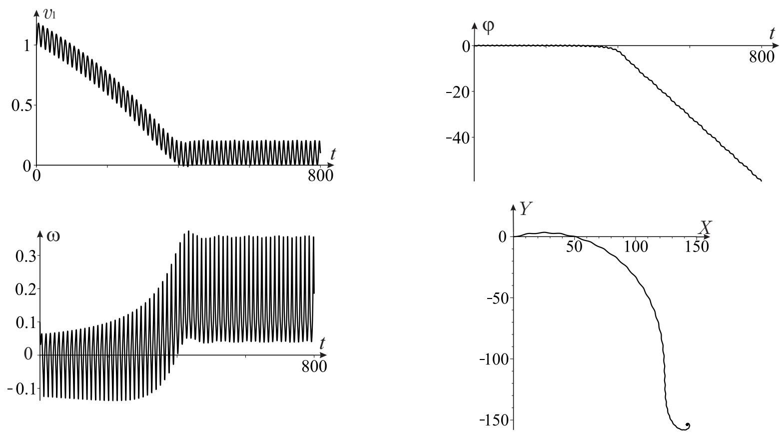

FIGURE 5. Illustration of the numerical experiment with $\alpha = 0.785$, $\tilde{a} = 0.3$ and the initial conditions $v_1(0) = 1$, $\varphi(0) = 0.1$, $\omega(0) = 0.1$

6. Conclusion

The dynamics of the double-link wheeled vehicle with arbitrarily directed periodic excitation was investigated in this paper. The equations of motion were derived. The values of the mechanical parameters, for which velocity v_1 increases indefinitely, were found, that is, there is acceleration in the system.

Some results obtained during this study may be consistent with the results obtained in [24]. For example, the case $\alpha = \frac{\pi}{2}$ or $\alpha = \frac{3\pi}{2}$ and $\tilde{a} = 0$ corresponds to the case $f_0 \equiv 0, g \equiv 0$ in [24]. In this case, the linear velocity v is bounded in [24]. We have observed similar results in numerical experiments for v_1. There is an interesting fact. Let the point mass oscillates perpendicular to the main axis of the carriage along a straight line passing through the geometric center of the wheelset or along the main axis of the carriage and there is no acceleration in the system. But if we rotate the axis of oscillation by any small angle or shift it by certain distance, speedup appears at large velocities.

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ПРОБЛЕМ УБРЗАЊА У ДИНАМИЦИ ПАРА ВАГОНА СПОЈЕНИХ КРУТОМ ВЕЗОМ СА ПРОИЗВОЉНО УСМЕРЕНОМ ПЕРИОДИЧНОМ ПОБУДОМ

Резиме. Ова студија истражује кретање нехолономног механичког система који чини пар вагона на точковима спојених крутом везом. При томе је постављена материјална тачка која осцилује под датим углом α у односу на главну осу једног од вагона. Као резултат, у систему се јавља периодична побуда. Добијене су једначине кретања у квази-брзинама. Испоставља се да је динамика возила на точковима са двоструком кариком моделована неаутономним током у тродимензионалном фазном простору. Истражује се понашање интегралних кривих при великим брзинама у зависности од угла α. Помоћу генерализоване Поенкареове трансформације оригинални проблем је сведен на проблем стабилности система са дегенерисаним линеарним делом. Доказ стабилности користи ограничавање система на централну многострукост и усредњавање нормалним формама до реда 4. Нађен је опсег вредности α за који се једна од компоненти брзине неограничено повећава и одређена је асимптотика за решења иницијалног динамичког система.

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