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PERIODIC WAVE PROPAGATION IN NONLOCAL BEAMS RESTING ON A BILINEAR FOUNDATION

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Abstract. The free wave propagation of periodic flexural waves on an infinite elastic Euler–Bernoulli nonlocal beam embedded in bilinear Winkler-type foundation is investigated. A general formulation of the elastic potential energy leads to a nonlinear nonlocal model with spatial derivatives up to the sixth order. The effect of the nonlocal parameters and of the different soil stiffnesses on the dynamical characteristics of the system is critically discussed. An enrichment of the system response with respect to the local beam is unveiled, and the crucial role played by the sixth-order nonlocal term is highlighted.

1. Introduction

The topic of the dynamic response in beams resting on elastic foundation has been widely studied in the literature for several decades [12, 14, 25, 28]. Besides numerous papers focused on linear and unilateral foundations, the bilinear case has received less attention. Investigations were carried by considering the bilinear foundation as a perturbation of the linear case [4], or under the effect of moving loads [9, 11, 20, 21], while few papers were devoted to the problem of the free wave propagation. Recently, Lenci and Demeio studied the wave propagation in taut cables [2] and flexural beams [15]. For taut cables, the problem results to be governed by a second-order spatial derivative nonlinear (piecewise linear) system, while for classical flexural beams the system has fourth-order spatial derivative. It has been observed that increasing the order of the mathematical model leads to the interesting phenomena, with complex interactions between single wave and multiple wave solutions found for large values of the two foundation stiffnesses.

In addition to the classical models, in the last few years, nonlocal theories for beams and plates have been diffusely proposed in the literature, mainly with the aim of describing the behavior of structures at the nanoscale, for which the

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classical continuum theories have been found to be incapable of providing accurate predictions [7].

Indeed, a large amount of experimental evidence shows that the mechanical properties of nanostructures have a significant size-dependent effect. Among them, nanobeams, nanorods, nanoplates and nanotubes are widely used to model resonators, oscillators, charge detectors, sensors or actuators, with practical applications that range from the aeronautical (nanoturbines, nanogears, nanoshafts), to the biomedical (biosensors, tissue engineering, rug delivery), and the civil engineering (high strength, ductility, durability of materials) field [1, 3, 16, 19].

Focusing on micro- and nano-beams embedded in an elastic foundation, Li et al. considered an elastic medium according to the Eringen's nonlocal elasticity theory [17], while Eptaimeros et al. used a nonlocal integral stress model [8]. The Pasternak elastic foundation - which is also a nonlocal substrate - was investigated by Wu et al. [30] and Togun et al. [26, 27], the latter applying an analytical asymptotic approach. There are several papers devoted to the study on carbon nanotubes, embedded in elastic foundation [10, 23, 24], two-parameter elastic foundation [18, 22], and Pasternak-type medium [29, 31]. Other models of microbeams were investigated, too, like nanobeams made by functionally graded materials [5] and temperature-dependent functionally graded materials [6]. All these models lead to governing equations with derivatives up to the fourth order with respect to the space variable.

However, to the best of the authors' knowledge, the dynamic response of a nonlocal beam resting on bilinear elastic foundation has not been studied yet. The interest in this kind of study is twofold. From a practical point of view, the aim is to describe the response of nanobeams (e.g. carbon nanotubes) which are possibly embedded in elastic media with different compression and tensile stiffnesses, like ceramic or concrete matrices. From a theoretical point of view, the goal is to select a nonlocal elastic model able to increase the derivative order of the governing equations, in order to analyze the possible enriched behavior of propagating waves, and to verify the effect of nonlocal parameters on the system response.

The paper is organized as follows. In Section 2, the nonlocal model of an infinite beam resting on a bilinear elastic foundation is presented, moving from a very general formulation of the elastic potential energy which is able to increase the order of the system derivatives to the sixth order. In Section 3, the behavior of the linear model obtained in case of uniform medium is critically analyzed, while the general case of bilinear foundation is discussed in Section 4. Eventually, concluding remarks are reported in Section 5.

2. Model formulation

In this paper, the flexural undamped free wave propagation in an infinite uniform beam resting on a uniform bilinear substrate is investigated. The wavelengths are assumed to be large enough (i.e. much larger than the beam radius of gyration) for the Euler–Bernoulli beam theory to be applicable, and rotatory inertia and shear deformation to be neglected. This permits to overcome the problem of infinite velocity occurring in the case of small wavelengths. As concerns the modeling of nonlocal elasticity, a general form for the density of elastic potential energy is assumed:

$$\varphi\Big(\kappa, \frac{\partial \kappa}{\partial x}\Big) = \frac{\tilde{c}_1}{2}\kappa^2 + \tilde{c}_2\kappa\frac{\partial \kappa}{\partial x} + \frac{\tilde{c}_3}{2}\frac{\partial \kappa}{\partial x}^2, \tag{2.1}$$

where $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$ account for local and nonlocal elastic modulus. κ is the bending curvature defined as $\kappa = v''$, with v being the vertical displacement of the beam and prime denoting derivative with respect to x. Applying the variational approach, the extended Hamiltonian for the system is:

$$\begin{aligned}
&\int_{t_1}^{t_2} (\delta T - \delta\Psi + \delta W)dt = 0,\\
&\delta T = \int_0^L \rho A \dot{v}\delta\dot{v}\, dx,\\
&\delta\Psi = \int_0^L (\tilde{c}_1 v''\delta v'' + \tilde{c}_2 v''\delta v''' + \tilde{c}_2 v'''\delta v'' + \tilde{c}_3 v'''\delta v''')dx,\\
&\delta W = \int_0^L b\delta v\, dx,
\end{aligned} \tag{2.2}$$

where T is the kinetic energy, Ψ is the elastic potential energy, and W is the load energy; δ is the variational operator, t_1, t_2 are generic time instants, and L is the wavelenght of the considered periodic wave, so that $[0, L]$ is the spatial domain in which the solution is sought for. ρ and A are mass per unit volume and cross-section area of the Euler–Bernoulli beam, respectively. Dot represents the derivative with respect to time t. $b(x, t)$ is the general external load. Assuming $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ to be independent of x and t, the equation of motion for the nonlocal beam results in

$$\rho A\ddot{v} + \tilde{c}_1 v^{IV} - \tilde{c}_3 v^{VI} = b. \tag{2.3}$$

The following boundary conditions complete the problem formulation:

$$\begin{aligned}
&(\tilde{c}_2 v'' + \tilde{c}_3 v''')\ |_{x=0,L} = 0 \quad \text{or} \quad v''\ |_{x=0,L} = 0,\\
&(\tilde{c}_1 v'' - \tilde{c}_3 v^{IV})\ |_{x=0,L} = 0 \quad \text{or} \quad v'\ |_{x=0,L} = 0,\\
&(\tilde{c}_3 v^{V} - \tilde{c}_1 v''')\ |_{x=0,L} = 0 \quad \text{or} \quad v\ |_{x=0,L} = 0.
\end{aligned} \tag{2.4}$$

The proposed formulation is very general because the expression of the elastic potential energy overlooks any physical and mechanical consideration. Nevertheless, it is of interest to furnish a physical interpretation of the problem by comparing the obtained equations with those proposed in the literature. Indeed, by looking at the strain gradient theory [13], it is straightforward to obtain a correspondence between the coefficients of the two formulations:

$$\tilde{c}_1 = D_1 = EJ + \rho A\Big(2l_0^2 + \frac{8}{15}l_1^2 + l_2^2\Big), \quad \tilde{c}_3 = D_2 = \rho J\Big(2l_0^2 + \frac{4}{5}l_1^2\Big),$$

where EJ is the bending stiffness, and l_0, l_1, l_2 are nonlocal parameters. For the following analyses it is thus useful to highlight the local and nonlocal contributions

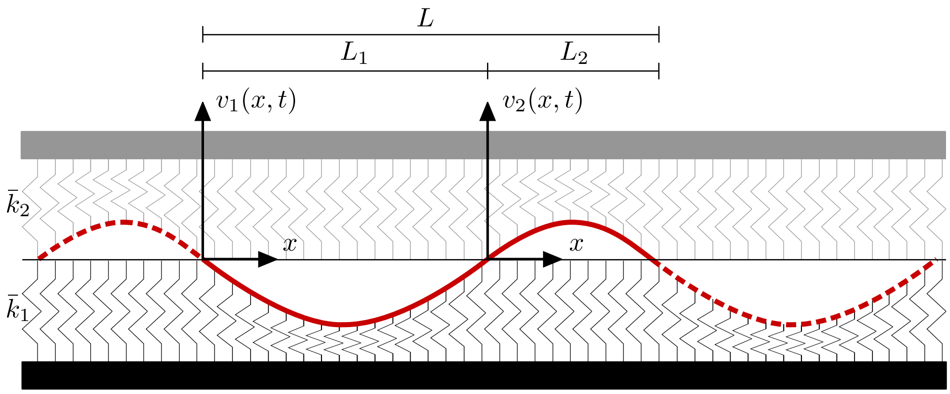

FIGURE 1. Single wave periodic response propagating in the bilinear elastic foundation. k_1, k_2 are linear stiffnesses of the compression and tension regions, respectively; v_1, v_2 are the transversal displacements of the compression and tension responses, respectively; L is the spatial extension of the single wave periodic response (i.e. the wavelength), while L_1 and L_2 are the spatial extension of the compression and tension solutions, respectively.

inside the $\tilde{c}_1$ coefficient by expressing it as $\tilde{c}_1 = EJ + \hat{c}_1$, where $\hat{c}_1$ is the nonlocal part. Note that $\tilde{c}_3$ is a purely nonlocal parameter, while the $\tilde{c}_2$ coefficient does not find correspondence in literature formulations. Actually, it results to be irrelevant also in the present problem.

The beam is assumed to be embedded in an elastic bilinear medium, which is described by means of a bilinear Winkler-type model:

$$b(x,t) = -f(v)v, \qquad f(v) = \begin{cases} \tilde{k}_1, v < 0 \\ \tilde{k}_2, v \geqslant 0 \end{cases}, \tag{2.5}$$

where $\tilde{k}_1$ and $\tilde{k}_2$ are the stiffnesses per unit length of the substrate, which are assumed to be constant, but different from each other. Denoting v_1 and v_2 the vertical displacements when $v < 0$ and $v \geqslant 0$, respectively, the equation (2.3) can be alternatively written in the form:

$$\begin{aligned} &\rho A\ddot{v}_1 + \tilde{k}_1 v_1 + \tilde{c}_1 v_1^{IV} - \tilde{c}_3 v_1^{VI} = 0, \quad v_1 < 0, \\ &\rho A\ddot{v}_2 + \tilde{k}_2 v_2 + \tilde{c}_1 v_2^{IV} - \tilde{c}_3 v_2^{VI} = 0, \quad v_2 \geqslant 0. \end{aligned} \tag{2.6}$$

The solution of the problem can be found in the form of a periodic traveling wave, where a single periodic wave shows both compression ($v_1 < 0$) and traction ($v_2 > 0$) within the wavelength L, as shown in Figure 1:

$$\begin{aligned} v_1(x,t) &= V_1(x - \tilde{c}t), \\ v_2(x,t) &= V_2(x - \tilde{c}t), \end{aligned} \tag{2.7}$$

with $\tilde{c}$ being the unknown phase velocity. By recalling the definition of $\tilde{c}_1$ and by defining $z = x - \tilde{c}t$ and

$$\eta = \frac{z}{L}, \quad c = \tilde{c}\frac{L\sqrt{\rho A}}{\sqrt{EJ}}, \quad \beta = \frac{\hat{c}_1}{EJ}, \quad \delta = \frac{\tilde{c}_3}{EJL^2}, \quad k_{1,2} = \tilde{k}_{1,2}\frac{L^4}{EJ}, \tag{2.8}$$

equations (2.6) become

$$\begin{aligned} &-\delta\frac{d^6V_1}{d\eta^6} + (1+\beta)\frac{d^4V_1}{d\eta^4} + c^2\frac{d^2V_1}{d\eta^2} + k_1V_1 = 0, \quad V_1(\eta) < 0, \\ &-\delta\frac{d^6V_2}{d\eta^6} + (1+\beta)\frac{d^4V_2}{d\eta^4} + c^2\frac{d^2V_2}{d\eta^2} + k_2V_2 = 0, \quad V_2(\eta) \geqslant 0. \end{aligned} \tag{2.9}$$

The sixth-order term is governed by the purely nonlocal parameter δ, while in the fourth-order term, the classical local part is corrected by the β-dependent nonlocal contribution.

The relevant eigenvalue problem is composed of two sixth-degree quadratic polynomials, which furnish three couples of eigenvalues for each equation of (2.9), i.e., λ_i, $i = 1, \ldots, 6$ for the first equation, and λ_j, $j = 7, \ldots, 12$ for the second one. In particular, for each set of eigenvalues one couple is real, while the other two are complex conjugates, i.e., $\lambda_1 = -\lambda_2$, $\lambda_7 = -\lambda_8$, $\lambda_3 = \bar{\lambda}_4$, $\lambda_9 = \bar{\lambda}_{10}$, $\lambda_5 = \bar{\lambda}_6$, $\lambda_{11} = \bar{\lambda}_{12}$, with bar indicating the complex conjugate. By separating the real and imaginary parts of the complex eigenvalues ($\lambda_3 = \lambda_{3r} + i\lambda_{3i}$, $\lambda_5 = \lambda_{5r} + i\lambda_{5i}$, $\lambda_9 = \lambda_{9r} + i\lambda_{9i}$, $\lambda_{11} = \lambda_{11r} + i\lambda_{11i}$), the general form of the solutions of (2.9) is

$$\begin{aligned} V_1(\eta) = a_1e^{-\lambda_1\eta} + a_2e^{\lambda_1\eta} &+ e^{\lambda_{3r}\eta}(a_3\cos(\lambda_{3i}\eta) + a_4\sin(\lambda_{3i}\eta)) \\ &+ e^{\lambda_{5r}\eta}(a_5\cos(\lambda_{5i}\eta) + a_6\sin(\lambda_{5i}\eta)), \end{aligned} \tag{2.10}$$

$$\begin{aligned} V_2(\eta) = a_7e^{-\lambda_7\eta} + a_8e^{\lambda_7\eta} &+ e^{\lambda_{9r}\eta}(a_9\cos(\lambda_{9i}\eta) + a_{10}\sin(\lambda_{9i}\eta)) \\ &+ e^{\lambda_{11r}\eta}(a_{11}\cos(\lambda_{11i}\eta) + a_{12}\sin(\lambda_{11i}\eta)), \end{aligned} \tag{2.11}$$

where a_i, $i = 1, \ldots, 12$ are the unknown amplitudes to be determined.

Due to the complex nature of some eigenvalues, the coefficients a_i could also be complex, notwithstanding the fact that $V_1(\eta)$ and $V_2(\eta)$ must be real. In order to determine the unknown amplitudes, boundary and continuity conditions for a single wave periodic solution are imposed:

$$\begin{aligned} &V_1(0) = 0, && V_2(1-\alpha) = 0, \\ &V_1(\alpha) = 0, && V_2(0) = 0, \\ &V_1'(0) = V_2'(1-\alpha), && V_1'(\alpha) = V_2'(0), \\ &V_1''(0) = V_2''(1-\alpha), && V_1''(\alpha) = V_2''(0), \\ &V_1'''(0) = V_2'''(1-\alpha), && V_1'''(\alpha) = V_2'''(0), \\ &V_1^{IV}(0) = V_2^{IV}(1-\alpha), && V_1^{IV}(\alpha) = V_2^{IV}(0), \\ &V_1^{V}(0) = V_2^{V}(1-\alpha), && V_1^{V}(\alpha) = V_2^{V}(0), \end{aligned} \tag{2.12}$$

where the spatial extension of the compression solution V_1 is $L_1 = \alpha L$ and that of the tension solution V_2 is $L_2 = (1-\alpha)L$. The α parameter is unknown and must be determined as a part of the solution. Boundary conditions (2.12) can be written in matrix form, i.e., $\mathbf{Ma} = \mathbf{0}$, where $\mathbf{a}$ is a (12×1) vector collecting the a_i coefficients, and $\mathbf{M}$ is a (14×12) matrix, not reported here for the sake of brevity.

It is worth remembering here that two extra conditions included in matrix $\mathbf{M}$ are necessary in order to determine the unknown phase velocity c and parameter α, which cannot be expressed in an explicit form, as done for the a_i amplitudes. The solvability conditions of the problem require that $\det(\mathbf{M}_n) = M_n = 0$ for each (12×12) squared sub-matrix $\mathbf{M}_n$ of $\mathbf{M}$, with $n = 1, \ldots, 91$.

Once the solution $c(k_1, k_2)$ has been obtained, we have that the physical frequency of the wave is

$$f = \frac{\tilde{c}}{L} = \frac{\sqrt{EJ}}{L^2\sqrt{\rho A}}\, c\Big(\frac{\tilde{k}_1 L^4}{EJ}, \frac{\tilde{k}_2 L^4}{EJ}\Big), \tag{2.13}$$

which gives $f = f(L)$, i.e. the dispersion relation of the considered problem.

3. Linear problem

If $k_1 = k_2 = k$, the problem (2.9) becomes linear, and the expression of the phase velocity c_l can be obtained in closed form:

$$c_l = 2\pi\sqrt{1 + \beta + (2\pi)^2\delta + \frac{k}{(2\pi)^4}} \tag{3.1}$$

for a single wave periodic solution, for which of course $\alpha = 1/2$. It can be noted that the phase velocity depends on both the nonlocal parameters β and δ, differently from what occurs when a local model is considered, where the expression of the phase velocity c_l reduces to $c_{\text{loc}} = 2\pi\sqrt{1 + k/(2\pi)^4}$.

The effects of the nonlocal parameters on the phase velocity are shown in Figure 2. As expected, they produce a qualitatively similar effect in increasing the phase velocity, even if the δ parameter has a major impact, meaning that the nonlocal sixth-order term is crucial in modifying the propagation characteristics of the beam.

Due to the homogeneous substrate embedding the beam, it is of interest also to study multiple wave periodic solutions, by conveniently referring to a reduced system, starting from a linear Winkler-type model, which is described by the first equation of (2.9), with $V_1(\eta) = V_l(\eta)$. The relevant solution takes the form proposed in (2.10), where the six eigenvalues are the same of the bilinear problem. The corresponding boundary problem $\mathbf{M_l a_l} = \mathbf{0}$ leads to the definition of the matrix $\mathbf{M_l}$, which is (7×6) rectangular matrix, and the (6×1) vector $\mathbf{a_l}$ of unknown amplitudes. Note that, when dealing with homogeneous substrate, the α parameter describing the portion of solution subjected to compression and that subjected to tension is no longer significant, so that the only one extra-condition of the problem is necessary to determine the phase velocity c_l. By imposing the zeroing of the determinants of the seven squared sub-matrices of $\mathbf{M_l}$ (i.e. $M_{ln} = 0$, $n = 1, \ldots, 7$), several responses can be detected. Figure 3 displays the zero-value curves of the determinants as a function of the wavelength L and of the phase velocity c_l, when assuming $k = 10$, $\beta = \delta = 0.01$. For the sake of readability of the figure, the curves of only three determinants are reported here, i.e. M_{l3}, M_{l5} and M_{l7}, which indeed are sufficient to assess the solvability of the problem, except for pathological cases. Of course, in the following analyses, the nullification of all seven determinants has been always verified.

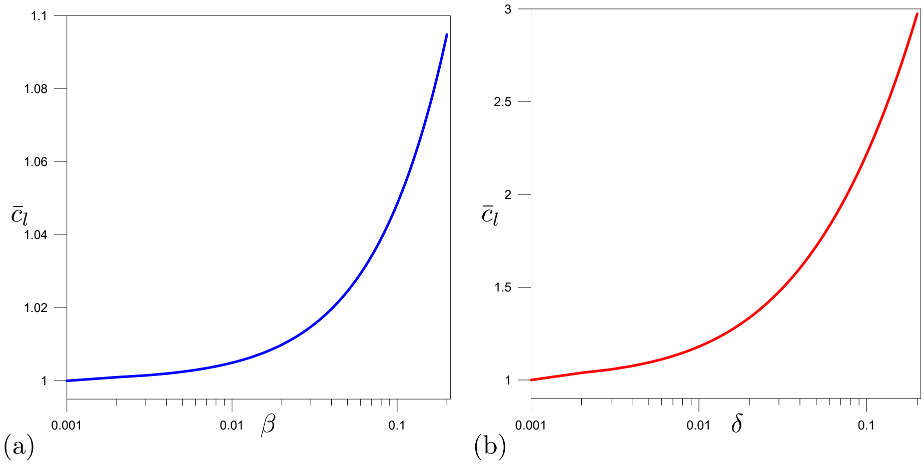

FIGURE 2. Effect of the nonlocal parameters on the nondimensional phase velocity $\bar{c}_l = c_l/c_{\text{loc}}$ for $k_1 = k_2 = 10$. (a) $\bar{c}_l$ as a function of the fourth-order nonlocal parameter β and (b) of the sixth-order nonlocal parameter δ.

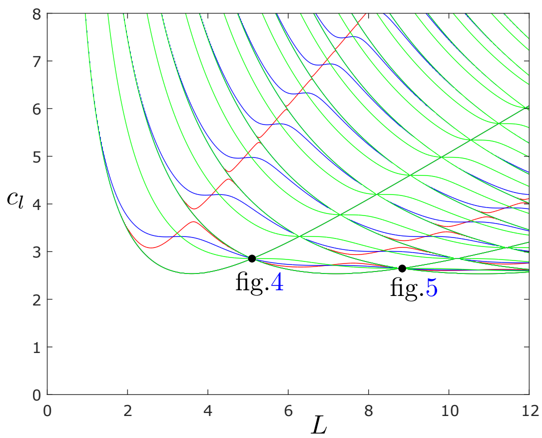

FIGURE 3. For $k_1 = k_2 = 10$, zero-value curves of the determinants of the squared sub-matrices of $\mathbf{M_l}$ as a function of wavelength L and phase velocity c_l. Red curve = M_{l3} determinant; blue curve = M_{l5} determinant; green curve = M_{l7} determinant.

As shown by the graphics, there are several points corresponding to the cross of all the curves, thus being solutions of the problem, some of them being investigated in Figures 4–5, by way of example.

One scenario is reported in Figures 4(a)–(c), for $c_l = 2.8541$ and $L = 5.101$. Interestingly, for a single value of the phase velocity, three different eigenvectors $\mathbf{a_l}$ satisfy the problem, leading to three different spatial shapes of the propagatinge-wave:

$$\mathbf{a_{l1}} = (0, 0, 0.7071, 0, -0.7071, 0),$$
$$\mathbf{a_{l2}} = (0, 0, 0, 0.8068, 0, 0.5908),$$
$$\mathbf{a_{l3}} = (0, 0, 0, 0.9854, 0, 0.1700).$$

A more in-depth discussion about the results can be carried out if a classical local model is also analyzed. It is worth remembering that the sixth-order term is governed by the purely nonlocal parameter δ, so that moving from a nonlocal to a local model implies the need to reformulate the problem, by imposing that

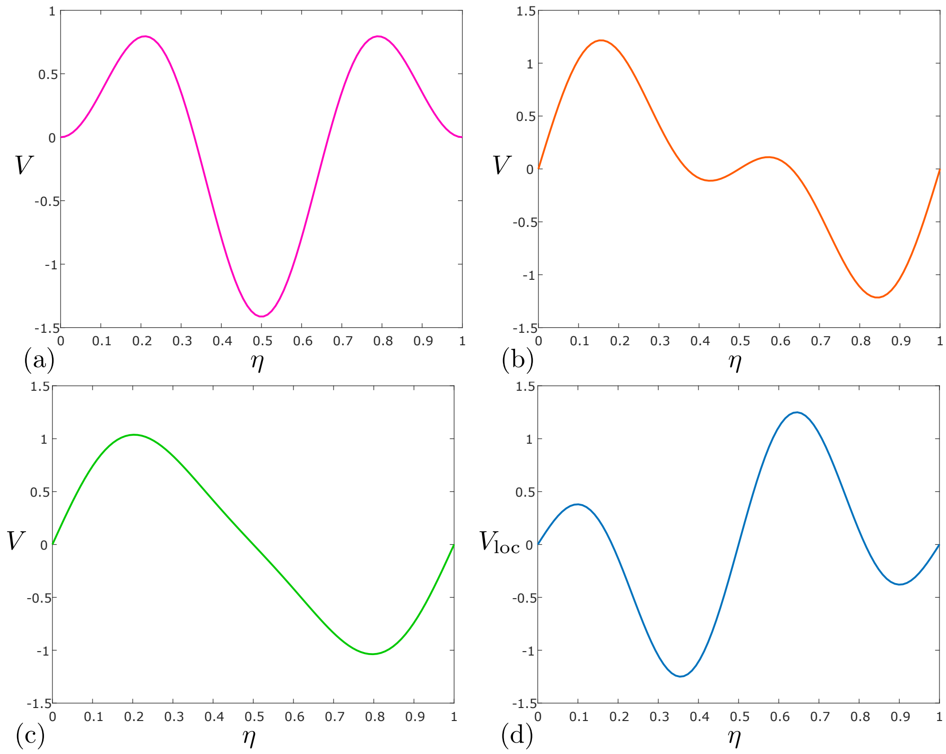

FIGURE 4. For $k_1 = k_2 = 10$, spatial shapes of the solutions for the nonlocal model at $c_l = 2.8541$ and $L = 5.1008$ (a)-(c), and for the local model at $c_{\text{loc}} = 2.8117$ and $L = 4.9969$ (d). The relevant eigenvectors are $\mathbf{a_{l1}}$ (a), $\mathbf{a_{l2}}$ (b) , $\mathbf{a_{l3}}$ (c), $\mathbf{a}_{\text{loc}}$ (d).

$\tilde{c}_2 = \tilde{c}_3 = 0$ in equation (2.1). The relevant problem equation sees the sixth-order term disappear in equation (2.9) (the nonlocal parameters β and δ are both null). The eigenvalue problem is now governed by a quadratic fourth-order polynomial whose solution furnishes two couples of complex eigenvalues. Thus, the solution can be sought in the form of (2.10) $V_1(\eta) = V_{\rm loc}(\eta)$, where the terms related to the real-valued eigenvalue λ_1 must be deleted. The matrix form of the boundary problem is composed of a (5×4) matrix $\mathbf{M}_{\rm loc}$, and a (4×1) vector of the unknown amplitudes $\mathbf{a}_{\rm loc} = (a_3, a_4, a_5, a_6)$. As for the nonlocal model, the solution is sought by imposing the contemporary annulment of the determinants of the five squared sub-matrices of $\mathbf{M}_{\rm loc}$. For the same set of mechanical values of the nonlocal case, a solution is detected for $c_{\rm loc} = 2.8117$. At this phase velocity, the spatial shape of the solution is shown in Figure 4(d), with $\mathbf{a}_{\rm loc} = (0, 0.7834, 0, -0.6216)$.

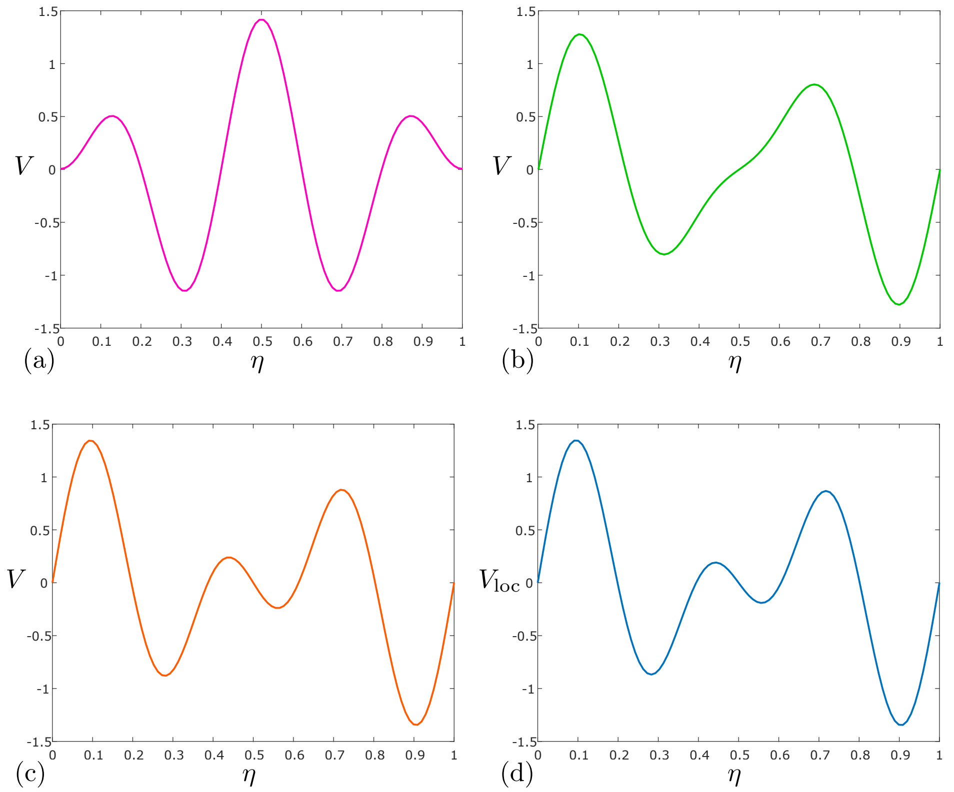

FIGURE 5. For $k_1 = k_2 = 10$, spatial shapes of the solutions for the nonlocal model at $c_l = 2.8541$ and $L = 8.8148$ (a)-(c), and for the local model at $c_{\rm loc} = 2.8117$ and $L = 8.6547$ (d). The relevant eigenvectors are $\mathbf{a_{l1}} = (0, 0, 0.7071, 0, -0.7071, 0)$ (a), $\mathbf{a_{l2}} = (0, 0, 0, 0.8921, 0, 0.4519)$ (b) , $\mathbf{a_{l3}} = (0, 0, 0, 0.6516, 0, 0.7586)$ (c), $\mathbf{a}_{\rm loc} = (0, 0.5973, 0, 0.8019)$ (d).

From the results, it is seen that including nonlocality into the model allows for an enrichment of the response with the peculiar possibility to have various wave shapes at the same phase velocity. Indeed, by looking at another scenario described in Figure 5, it can be observed that for the same set of mechanical parameters the system displays various responses at different phase velocities (and wavelengths), with the presence of periodic multiple wave solutions as the wavelength L increases.

4. Nonlinear problem

When considering different substrates embedding the beam, $k_1 \neq k_2$, the nonlocal bilinear problem must be referred to. Focusing on the study of single wave periodic solutions, the system (2.9)–(2.12) can be numerically solved, although here the computational effort is considerably increased by the nonlinear nature of the problem and by the need to investigate the determinants of all the 91 squared sub-matrices of the system matrix $\mathbf{M}$.

By applying continuation techniques it is possible to discuss the effect of the nonlocal parameters on the phase velocity by considering different values of the soil stiffness, as reported in Figure 6(a). Moving from the local bilinear model (black curve), the β parameter is shown to be capable of decreasing the slope of the curve (blue curve), although the greatest effect can be observed if the parameter δ is taken into account (red curve). This is likely due to the fact that the nonlocal β parameter is added to the local part (with unitary dimensionless coefficient) governing the fourth-order term in the system equation (2.9). As a consequence,

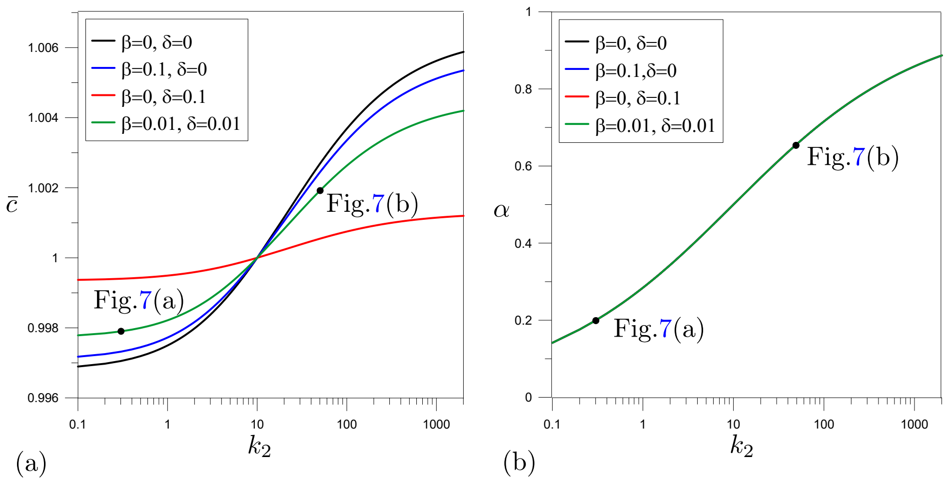

FIGURE 6. For $k_1 = 10$, effect of varying k_2 on the normalized phase velocity $\bar{c}$ (a) and the α parameter (b). Black curve = local model; blue curve = nonlocal fourth-order model; red curve = nonlocal sixth-order model; green curve = complete nonlocal sixth-order model.

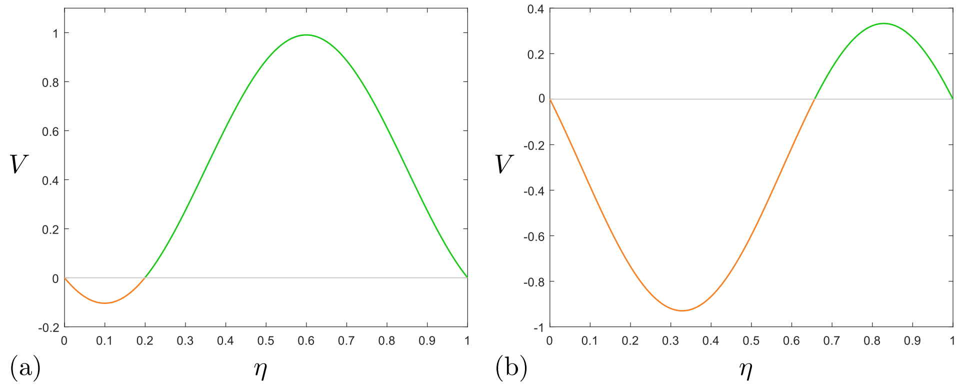

FIGURE 7. Single wave response for $k_1 = 10, k_2 = 0.3$ ($c = 7.4484$, $\alpha = 01994$) (a), $k_1 = 10, k_2 = 50$ ($c = 7.4783$, $\alpha = 0.6567$) (b). The orange part is subjected to compression ($V_1(\eta)$) while the green part is subjected to tension ($V_2(\eta)$).

the β parameter furnishes an incremental contribution to the system response, also in consideration of the fact that the acceptable values for the nonlocal parameters are order of magnitude smaller than unity. Differently, the sixth-order term is exclusively governed by the δ parameter, so that its presence, although with small values, is able to significantly change the system response. Indeed, black and blue curves are obtained by solving the fourth-order bilinear problem, while red and green curves require the investigation of the sixth-order model.

Note that in Figure 6(a) the values of the phase velocity are nondimenzionalized with respect to the relevant linear problem (i.e., when $k_1 = k_2$), so that all curves pass through the point ($k_2 = 10, \bar{c} = 1$), even if the dimensional phase velocities at this point are different. In particular, for $k_2 = k_1 = 10$, $c = 6.3033$ for the local problem (black curve), $c = 6.6091$ when considering the β parameter (blue curve), $c = 13.9852$ when including the δ parameter, and $c = 7.4640$ when both the nonlocal terms are considered (green curve). This means that the sixth-order term is able to substantially increase the value of the phase velocity, as already discussed in Figure 2, even if the latter undergoes very small variations as the substrate stiffness changes.

In Figure 6(b), the effect of the k_2 stiffness variation on the compression-to-tension ratio α is shown. As expected, the nonlocal parameters do not influence the response, since the α parameter is strictly related to the substrate stiffnesses. As an example, two single wave periodic responses for the nonlocal bilinear system at $k_2 = 0.3$ and $k_2 = 50$, with fixed $k_1 = 10$, are shown in Figure 7.

5. Concluding remarks

In the paper, a dynamical model for a nonlocal infinite elastic beam embedded in bilinear Winkler-type foundation is proposed. The general form of the elastic potential energy leads to a system of nonlinear equations with derivatives up to the sixth order. With the focus on investigating the single wave periodic responses, an analytical solution strategy is proposed. Since an exact mathematical solution is obtained, it can be used to validate numerical solutions that can be developed to solve this problem, or similar ones.

In order to critically discuss the effects of the different characteristics of the model on the free wave propagation, various simplified systems are defined and investigated. In particular, the influence of the nonlocal parameters is discussed by referring to the relevant linear model, obtained when a homogeneous medium is considered, and to the simplified fourth-order local model, ensuing when nonlocality is neglected. Successively, the effect of varying stiffnesses of the bilinear foundation is studied by analyzing the complete nonlocal nonlinear model.

The results highlight the crucial role played by the sixth-order term, exclusively governed by a nonlocal parameter, in quantitatively modifying the phase velocity. Indeed, as this term gains importance, the phase velocity significantly increases, as compared to the other nonlocal parameter included in the fourth-order term. From the mathematical viewpoint, this result is somehow expected, since nonlocality plays a substantially different role in the two terms, being able to activate the sixth-order term while having just an incremental effect of the fourth-order one.

More interestingly, nonlocality seems to substantially enrich the dynamical behavior of the model with respect to the local model, with the possibility to have waves with different spatial shapes at the same phase velocity.

On the other hand, when dealing with a bilinear foundation, the phase velocity is shown to have a monotone trend in relation to the change in ratio between the two stiffnesses, with again the sixth-order nonlocal term being able to significantly reduce the slope of the curve.

From a practical viewpoint, these outcomes suggest the potential interest in considering simple nonlocal formulations which are able to increase the order of the mathematical model.

There are many possible developments of the proposed research. Among them, we mention the comparison of the proposed theoretical results with experimental results, that are not yet available to the best of the authors' knowledge.

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ПЕРИОДИЧНО ПРОСТИРАЊЕ ТАЛАСА У НЕЛОКАЛНОЈ БИЛИНЕАРНОЈ ГРЕДИ

Резиме. Истражује се слободно простирање периодичних таласа на бесконачно еластичној Ојлер-Бернулијевој нелокалној греди уграђеној у билинеарни темељ Винклеровог типа. Општа формулација еластичне потенцијалне енергије доводи до нелинеарног нелокалног модела са просторним изводима до шестог реда. Разматра се утицај нелокалних параметара и различитих крутости тла на динамичке карактеристике система. Објашњено је обогаћивање одзива система у односу на стандардни локални штап и наглашена је кључна улога коју игра нелокални термин шестог реда.

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