THEORETICAL AND APPLIED MECHANICS
Volume 50 (2023) Issue 2, 103–115 DOI: https://doi.org/10.2298/TAM230719008G

PROPERTIES OF OPERATOR CONSTITUTIVE RELATIONS IN MECHANICS OF DEFORMABLE SOLID

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Abstract. The constitutive relations between stresses and strains in the mechanics of a deformable solid, including their operator connection, are considered. Some important and frequently occurring properties of tangent modulus and tangent pliability as rank four tensors are described. Depending on this, a possible classification of continuous media is proposed. Scleronomous and rheonomic media, homogeneous and inhomogeneous media (in particular, composites), media with memory, spatially nonlocal media, materials with hard or soft characteristics are distinguished. For non-linearly elastic isotropic media, the apparatus of tensor nonlinear isotropic functions of one argument is developed. Particular attention is paid to the three-term representation of power tensor series in three-dimensional space, reversibility of tensor functions, Taylor tensor series, tensor linearity (quasilinearity) and nonlinearity.

1. Introduction and the simplest examples

In closed systems of equations used in the formulation of initial-boundary-value problems of continuum mechanics, all equations can be divided by their physical meaning and by their origin into certain groups. One of them necessarily includes differential consequences of conservation laws or continuum mechanics postulates, whereas the other one consists of kinematic relations and/or, as a consequence, equations of compatibility of components of certain vectors and tensors. In more complex formulations, taking into account, for example, phase transitions or the structural state of substances, there are other groups of equalities. But, it is mandatory for the closure of the system, that a mechanician specifies the physical relations connecting some kinematic and some force characteristics of the deformation process that highlight the selected class of media. These relations in a broad sense are called constitutive relations. In mechanics of deformable solid they are most often set in the form of operator communication of stresses and strains. Such an

2020 *Mathematics Subject Classification:* 74A20; 74B20.

Key words and phrases: constitutive relation, operator, material function, tangent module, tangent pliability, soft (hard) characteristics, rheonomous and scleronomous media, composite, non-local medium, isotropic tensor function.

operator relationship, which sets a mathematical model of the medium, certainly exists, since a change in the deformed state, as any observation shows, entails a change in the stressed state and vice versa [1, 10, 11, 14, 15, 17, 18].

In other words, the constitutive relations reflect, generally speaking, the operator relationship of the parameters of a process implemented in a continuous medium with the reaction of the medium to this process. In the simplest case, constants, and in more complex functions and even functionals included in the definition of this operator and specifying exactly the selected medium from all others of the same class, are called material functions of constitutive relations. Material functions differ in that they are not found as a result of calculations or the solution of any equations, but only from special experiments called setup. Mathematical model of the medium, i.e. the set of selected constitutive relations is suitable for research, or, as they say, is adequate if there is a set of setup experiments that allows to find (at least theoretically) all the material functions of the model. Note that the theory of the setup experiment is an important component of modern experimental mechanics of deformable solid.

We give two typical examples of constitutive relations in mechanical systems.

1.1. Deformable spring. Let the spring, which was in an undeformed state at $t < t_0$, begin from the moment $t = t_0$ to stretch by the force of $F(t)$. The displacement of the right end caused by the action of force, in comparison with its initial position, we denote $u(t)$. Operator connection of functions $F(t)$ and $u(t)$ in the form

$$u(t) = \check{\mathcal{A}}_{t_0}\big[F(\tau)\big]_{\tau=t_0}^{t} \tag{1.1}$$

is exactly what is called the constitutive relations of the spring. The operator $\check{\mathcal{A}}_{t_0}$, generally speaking, may depend on the initial moment t_0, which is emphasized by the subscript in its notation. The very displacement u at the moment of t can be determined by the entire loading history from t_0 to t. This history also includes information about the state (active loading or unloading) of the system at time t.

Let us write the inverse relation to (1.1)

$$F(t) = \check{\mathcal{B}}_{t_0}\big[u(\tau)\big]_{\tau=t_0}^{t}$$

which is also naturally called constitutive. The question of connection of the operators $\check{\mathcal{A}}$ and $\check{\mathcal{B}}$ and the establishment of their reciprocity connections is not easy when it comes to the complex behavior of the spring under load.

If, on the basis of the experiments carried out, it is possible to establish that the spring is characterized by a rather particular type of behavior, namely, the displacement u at any moment $t > t_0$ is completely determined only by the value of F at the same time t and does not depend on t_0:

$$u = A(F), \quad F = B(u), \qquad t > t_0$$

where $A(F)$ and $B(u)$ are reverse functions of one argument, $A(B(u)) = u$, then it is said that the spring has the fundamental property of elasticity. If the functions A and B are nonlinear, for example,

$$B(u) = b_1 u + b_2 u^3 \tag{1.2}$$

then there is a nonlinear elasticity. In the constitutive relation (1.2), there are the material constants b_1 and b_2. They can also be functions if, for example, they depend on temperature or some other external parameter. The spring behaves linearly elastic if we put $b_2 = 0$ in (1.2).

1.2. Heat-conducting medium. Let a homogeneous heat-conducting medium occupy the entire three-dimensional space and at all its points a scalar temperature field $T(z,t)$, $t > t_0$ is set from the outside, depending on time and only on one spatial coordinate z. Another, conjugate to T, measurable physical quantity is the heat flux vector $\mathbf{q}\,(z,t)$, which is different from zero due to the uneven distribution of temperature over space. The greater the temperature drop, the greater the modulus $|\mathbf{q}\,(z,t)|$ of the heat flux vector directed, as suggested by the simplest experiment, from more heated areas to less heated ones. And vice versa, no matter how high or low the temperature is, if T does not depend on z, but, for example, only on t, then there are no heat flows in the medium ($\mathbf{q} = \mathbf{0}$).

These arguments lead to the fact that the constitutive relations in this example should connect $\mathbf{q}$ and $\partial T/\partial z$. If the medium is isotropic in the sense of its heat-conducting properties, then the only nonzero component of $\mathbf{q}$ will be $q_z(z,t)$. Thus, the operator relations have the form

$$q_z(z,t) = \check{\mathcal{C}}_{t_0}\Big[\frac{\partial T}{\partial z}(z,\tau)\Big]_{\tau=t_0}^{t} \tag{1.3}$$

Their simplest special case is the Fourier law known in thermodynamics

$$q_z(z,t) = -\lambda\,\frac{\partial T}{\partial z}(z,t) \tag{1.4}$$

where $\lambda > 0$ is the thermal conductivity coefficient of the medium, which is the only one in (1.4) a material constant.

2. Operator connections of stresses and strains

It follows from the above examples that the constitutive relations in various fields of mechanics should connect a pair of so-called conjugate physical quantities, and regardless of their tensor nature. This conjugacy is understood in the sense that one of the values of the pair describes some independent, or externally carried out, process in the medium, while the other reflects the reaction of the medium to this process.

In mechanics of deformable solid, a pair of second-rank tensors is most often chosen as conjugate physical quantities. One of them, $\boldsymbol{\varepsilon}(\mathbf{x},t)$, describes the kinematics of the deformation of the medium, the other, $\boldsymbol{\sigma}(\mathbf{x},t)$ characterizes the distribution of internal forces as a response to the deformation that occurred. These two tensors will be generically called the kinematic tensor, or strain tensor, and the force tensor, or stress tensor, respectively. Then the constitutive relations in the direct and inverse form can be represented as

$$\boldsymbol{\sigma} = \check{\mathcal{F}}(\boldsymbol{\varepsilon}), \quad \boldsymbol{\varepsilon} = \check{\mathcal{G}}(\boldsymbol{\sigma}) \tag{2.1}$$

where $\check{\mathcal{F}}$ and $\check{\mathcal{G}}$ are inverse to each other tensor operators of the second rank, each of which depends on its tensor argument:

$$\check{\mathcal{G}}\big(\check{\mathcal{F}}(\boldsymbol{\varepsilon})\big) = \boldsymbol{\varepsilon}, \quad \check{\mathcal{F}}\big(\check{\mathcal{G}}(\boldsymbol{\sigma})\big) = \boldsymbol{\sigma} \tag{2.2}$$

From the whole variety of possible operators $\check{\mathcal{F}}$, we immediately distinguish a class of physically linear operators for which the superposition principle holds, namely, equality

$$\check{\mathcal{F}}(c_1\boldsymbol{\varepsilon}_1 + c_2\boldsymbol{\varepsilon}_2) = c_1\check{\mathcal{F}}(\boldsymbol{\varepsilon}_1) + c_2\check{\mathcal{F}}(\boldsymbol{\varepsilon}_2) \tag{2.3}$$

is fulfilled for any strain tensors $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ which are included in the domain of $\check{\mathcal{F}}$, and any numbers c_1 and c_2. We will call a continuous medium physically linear if the operator of its constitutive relations (2.1) is physically linear in the sense of (2.3).

We will further take into consideration consideration two tensor operators of the fourth rank, namely tangent modulus $(\partial\check{\mathcal{F}}/\partial\boldsymbol{\varepsilon})(\boldsymbol{\varepsilon})$ and tangent pliability $(\partial\check{\mathcal{G}}/\partial\boldsymbol{\sigma})(\boldsymbol{\sigma})$. We will determine them by the results of the action on any guide (or trial) strain tensor $\boldsymbol{\varepsilon}'$ and stress tensor $\boldsymbol{\sigma}'$, respectively:

$$\frac{\partial\check{\mathcal{F}}}{\partial\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}' \equiv D\check{\mathcal{F}}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}') = \frac{d}{d\xi}\Big[\check{\mathcal{F}}(\boldsymbol{\varepsilon}+\xi\boldsymbol{\varepsilon}')\Big]_{\xi=0} \equiv \lim_{\xi\to 0}\ \frac{1}{\xi}\big(\check{\mathcal{F}}(\boldsymbol{\varepsilon}+\xi\boldsymbol{\varepsilon}') - \check{\mathcal{F}}(\boldsymbol{\varepsilon})\big)$$

$$\frac{\partial\check{\mathcal{G}}}{\partial\boldsymbol{\sigma}} : \boldsymbol{\sigma}' \equiv D\check{\mathcal{G}}(\boldsymbol{\sigma},\boldsymbol{\sigma}') = \frac{d}{d\xi}\Big[\check{\mathcal{G}}(\boldsymbol{\sigma}+\xi\boldsymbol{\sigma}')\Big]_{\xi=0} \equiv \lim_{\xi\to 0}\ \frac{1}{\xi}\big(\check{\mathcal{G}}(\boldsymbol{\sigma}+\xi\boldsymbol{\sigma}') - \check{\mathcal{G}}(\boldsymbol{\sigma})\big)$$

The expressions $D\check{\mathcal{F}}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}')$ and $D\check{\mathcal{G}}(\boldsymbol{\sigma},\boldsymbol{\sigma}')$ are called Gâteaux differentials [12], or weak differentials, in relation to the mappings of $\check{\mathcal{F}}$ and $\check{\mathcal{G}}$ at the "points" of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ in increments in the "directions" of $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\sigma}'$. The Gâteaux differential has the property of uniformity. So, if $D\check{\mathcal{F}}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}')$ exists, then for any number c, $D\check{\mathcal{F}}(\boldsymbol{\varepsilon},c\boldsymbol{\varepsilon}')$ will also exist and besides

$$D\check{\mathcal{F}}(\boldsymbol{\varepsilon},c\boldsymbol{\varepsilon}') = cD\check{\mathcal{F}}(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}')$$

3. Properties of tangent modulus and tangent pliability

We will give below a number of definitions [16] concerning the properties of the introduced tensors of the second rank $\check{\mathcal{F}}$ and $\check{\mathcal{G}}$, as well as tensors of the fourth rank $(\partial\check{\mathcal{F}}/\partial\boldsymbol{\varepsilon})(\boldsymbol{\varepsilon})$ and $(\partial\check{\mathcal{G}}/\partial\boldsymbol{\sigma})(\boldsymbol{\sigma})$.

- Tangent modulus and tangent pliability are called *bounded from above* if there are such constants M and N that inequalities

$$\boldsymbol{\varepsilon}' : \frac{\partial\check{\mathcal{F}}}{\partial\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}' \leqslant M\boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}', \quad \boldsymbol{\sigma}' : \frac{\partial\check{\mathcal{G}}}{\partial\boldsymbol{\sigma}} : \boldsymbol{\sigma}' \leqslant N\boldsymbol{\sigma}' : \boldsymbol{\sigma}' \tag{3.1}$$

are satisfied for any guiding tensors $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\sigma}'$.

- Tangent modulus and tangent pliability are called *non-negative* if for any guiding tensors $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\sigma}'$

$$\boldsymbol{\varepsilon}' : \frac{\partial\check{\mathcal{F}}}{\partial\boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}' \geqslant 0, \quad \boldsymbol{\sigma}' : \frac{\partial\check{\mathcal{G}}}{\partial\boldsymbol{\sigma}} : \boldsymbol{\sigma}' \geqslant 0 \tag{3.2}$$

• It is said that the tangent modulus and tangent pliability are *positively defined* if for any guiding tensors $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\sigma}'$ there are constants $m(\boldsymbol{\varepsilon}')$ and $n(\boldsymbol{\sigma}')$, such that

$$\boldsymbol{\varepsilon}' : \frac{\partial \check{\mathcal{F}}}{\partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}' \geqslant m\boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}', \quad \boldsymbol{\sigma}' : \frac{\partial \check{\mathcal{G}}}{\partial \boldsymbol{\sigma}} : \boldsymbol{\sigma}' \geqslant n\boldsymbol{\sigma}' : \boldsymbol{\sigma}' \tag{3.3}$$

Inequalities (3.3) can be conditionally written without using tensors $\boldsymbol{\varepsilon}'$ and $\boldsymbol{\sigma}'$:

$$\frac{\partial \check{\mathcal{F}}}{\partial \boldsymbol{\varepsilon}} \geqslant m\boldsymbol{\Delta}, \quad \frac{\partial \check{\mathcal{G}}}{\partial \boldsymbol{\sigma}} \geqslant n\boldsymbol{\Delta} \tag{3.4}$$

where $\boldsymbol{\Delta}$ is the unit tensor of the fourth rank, i.e., one for which

$$\boldsymbol{\Delta} : \boldsymbol{h} = \boldsymbol{h} : \boldsymbol{\Delta} = \boldsymbol{h}, \quad \forall \boldsymbol{h}$$

Note that the constants M and m in (3.1), (3.3) and (3.4) have the dimension of stresses, and the constants N and n have the dimension inverse to the stresses, so that the products of MN and mn are dimensionless.

If in (3.3) and (3.4) m and n can be chosen such that $mn < 1$, then the following two-sided estimates

$$m\boldsymbol{\Delta} \leqslant \frac{\partial \check{\mathcal{F}}}{\partial \boldsymbol{\varepsilon}} \leqslant \frac{1}{n}\boldsymbol{\Delta}, \quad n\boldsymbol{\Delta} \leqslant \frac{\partial \check{\mathcal{G}}}{\partial \boldsymbol{\sigma}} \leqslant \frac{1}{m}\boldsymbol{\Delta}$$

are valid for tangent modulus and tangent pliability.

• It is said that the tangent modulus has *a soft characteristic* if for any strain tensor $\boldsymbol{\varepsilon}$, which is included in the domain of definition of the operator $\check{\mathcal{F}}$, the following tensor of the fourth rank

$$\left.\frac{\partial \check{\mathcal{F}}}{\partial \varepsilon}\right|_{\boldsymbol{\varepsilon}=\mathbf{0}} - \frac{\partial \check{\mathcal{F}}}{\partial \boldsymbol{\varepsilon}} \tag{3.5}$$

is non-negative in the sense of definition (3.2). The tangent modulus has *a rigid characteristic* if the tensor opposite in sign to (3.5) is non-negative. Similar definitions of soft and rigid characteristics can be given with respect to the tangent pliability $\partial\check{\mathcal{G}}/\partial\boldsymbol{\sigma}$.

• The conditions of *reversibility* of the tangent modulus and the tangent pliability as two tensor operators of the fourth rank are represented as

$$\frac{\partial \check{\mathcal{F}}}{\partial \varepsilon} : \frac{\partial \check{\mathcal{G}}}{\partial \boldsymbol{\sigma}} = \frac{\partial \check{\mathcal{G}}}{\partial \boldsymbol{\sigma}} : \frac{\partial \check{\mathcal{F}}}{\partial \varepsilon} = \boldsymbol{\Delta}$$

• The constitutive relations (2.1) and the media modeled by them are called *rheonomic* if the material functions of the operators $\check{\mathcal{F}}$ and $\check{\mathcal{G}}$ clearly depend on time. This fact is emphasized in the record as follows

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \check{\mathcal{F}}\big(\boldsymbol{\varepsilon}(\mathbf{x}, t), t\big), \quad \boldsymbol{\varepsilon}(\mathbf{x}, t) = \check{\mathcal{G}}\big(\boldsymbol{\sigma}(\mathbf{x}, t), t\big)$$

Examples of rheonomic media can be viscoelastic media, in which the constitutive relations, in particular, in the physically linear case

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \int_0^t \boldsymbol{\Gamma}(t, \tau) : \boldsymbol{\varepsilon}(\mathbf{x}, \tau)\, d\tau, \quad \boldsymbol{\varepsilon}(\mathbf{x}, t) = \int_0^t \mathbf{K}(t, \tau) : \boldsymbol{\sigma}(\mathbf{x}, \tau)\, d\tau \tag{3.6}$$

include as material functions tensors of the fourth rank of relaxation kernels $\boldsymbol{\Gamma}$ and creep kernels $\mathbf{K}$, depending on two time variables at once.

If the material functions of the operators $\check{\mathcal{F}}$ and $\check{\mathcal{G}}$ are clearly independent of time, then the medium is called *scleronomous*.

• Media with constitutive relations (2.1) in which material functions clearly depend on coordinates, i.e.

$$\boldsymbol{\sigma}(\mathbf{x},t) = \check{\mathcal{F}}\big(\boldsymbol{\varepsilon}(\mathbf{x},t),\mathbf{x}\big), \quad \boldsymbol{\varepsilon}(\mathbf{x},t) = \check{\mathcal{G}}\big(\boldsymbol{\sigma}(\mathbf{x},t),\mathbf{x}\big)$$

are called *inhomogeneous*, and otherwise *homogeneous*. There are continuously inhomogeneous, or continuously stratified media, in which the dependence of material functions on coordinates is continuous, and *composites*, in which material functions discontinuously depend on coordinates. Most often, this discontinuous dependence is modeled piecewise constant, i.e. the composite structurally consists of several pieces of homogeneous materials (components) with different physical and mechanical properties.

Special cases of composites that are widely used can be layered media (laminates), in which heterogeneity is realized along only one of the coordinates, fibrous media (fibrites) consisting of a light matrix and reinforcing it in a certain way bearing fibers, granular media. Composites whose typical sizes of different components differ greatly from each other are called *microcomposites* (the difference is 3–6 orders of magnitude) and *nanocomposites* (more than 6 orders of magnitude).

Due to the discontinuity in the coordinates of material functions, mathematical modeling of the behavior of composites is associated with the involvement of generalized functions apparatus. The mechanics of composites formed on the basis of this is a relatively new part of mechanics of deformable solid. Over the past decades it has become a separate and intensively developing discipline of great applied importance.

• Media with constitutive relations of type (3.6) are *non-local in time*, they are also called *media with memory*. In such media, the state at each point $\mathbf{x}$ at the current time t is determined by the entire history of the process that occurred at this point for some finite or even infinite period of time preceding t. In this case, the stress and strain relationships (2.1) can be represented in a form similar to (1.1) and (1.3):

$$\boldsymbol{\sigma}(\mathbf{x},t) = \check{\mathcal{F}}_{t_0}\big[\boldsymbol{\varepsilon}(\mathbf{x},\tau)\big]_{\tau=t_0}^{t}, \quad \boldsymbol{\varepsilon}(\mathbf{x},t) = \check{\mathcal{G}}_{t_0}\big[\boldsymbol{\sigma}(\mathbf{x},\tau)\big]_{\tau=t_0}^{t}$$

The initial moment t_0 can also be attributed to $-\infty$.

• Non-locality is possible not only in time, but also in coordinates. A medium is called *spatially nonlocal* if the stress state at each point $\mathbf{x}$ at time t depends on the strain state in the whole neighborhood $V_R(\mathbf{x}) = \{\mathbf{y} : |\mathbf{x}-\mathbf{y}| < R\}$ at the same time:

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{x},t) &= \int_{V_R(\mathbf{x})} \mathbf{A}(\mathbf{x},\mathbf{y}) : \boldsymbol{\varepsilon}(\mathbf{y},t)\, dV(\mathbf{y}), \\ \boldsymbol{\varepsilon}(\mathbf{x},t) &= \int_{V_R(\mathbf{x})} \mathbf{B}(\mathbf{x},\mathbf{y}) : \boldsymbol{\sigma}(\mathbf{y},t)\, dV(\mathbf{y}) \end{aligned} \tag{3.7}$$

where $\mathbf{A}(\mathbf{x},\mathbf{y})$ and $\mathbf{B}(\mathbf{x},\mathbf{y})$ are material tensor functions of the fourth rank characterizing given non-local medium.

If the radius R of the neighborhood $V_R(\mathbf{x})$ is finite (not infinitely small), then the nonlocality is strong, but if R in (3.7) can be made arbitrarily small, then they speak of weak nonlocality at the point $\mathbf{x}$. Weak nonlocality actually means that stresses at the point $\mathbf{x}$ depend not only on strains, but also on their derivatives in coordinates at the same point $\mathbf{x}$. Media with such constitutive relations are also called *gradient* of one order or another.

• *An elastic solid* is a continuous medium in which at each moment of time t the only independent state parameter is the strain tensor $\boldsymbol{\varepsilon}(\mathbf{x}, t)$ at the same moment t, i.e., the stress tensor $\boldsymbol{\sigma}(\mathbf{x}, t)$ at any point $\mathbf{x}$ at any time t is completely determined by setting the strain tensor $\boldsymbol{\varepsilon}(\mathbf{x}, t)$ at the same point at the same moment [13]. The definition will remain valid if the strain tensor and stress tensor are reversed in it:

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{f}\big(\boldsymbol{\varepsilon}(\mathbf{x}, t)\big), \quad \boldsymbol{\varepsilon}(\mathbf{x}, t) = \mathbf{g}\big(\boldsymbol{\sigma}(\mathbf{x}, t)\big) \tag{3.8}$$

where $\mathbf{f}$ and $\mathbf{g}$ are inverse to each other tensor functions of the second rank, each of which depends on its tensor argument:

$$\mathbf{g}\big(\mathbf{f}(\boldsymbol{\varepsilon})\big) = \boldsymbol{\varepsilon}, \quad \mathbf{f}\big(\mathbf{g}(\boldsymbol{\sigma})\big) = \boldsymbol{\sigma} \tag{3.9}$$

The constitutive relations (3.8) and (3.9) are naturally very special cases of the general operator relations (2.1) and (2.2).

From the above definition, in particular, it follows that the following statement is true for an elastic solid. Whatever the loading process may be in time at some point $\mathbf{x}$, but at the moment T of full unloading, i.e. when $\boldsymbol{\sigma}(\mathbf{x}, T) = \mathbf{0}$, there are no strains at this point: $\boldsymbol{\varepsilon}(\mathbf{x}, T) = \mathbf{0}$. If at the moment T full unloading occurred at all points of the elastic solid, then it completely returned to its original undeformed state. This property is also called the absence of residual deformations.

In addition to the strain tensor, independent state parameters may include fields of a various physical (not purely mechanical) nature, for example, temperature $T(\mathbf{x}, t)$, electric field strength $\mathbf{E}(\mathbf{x}, t)$, such that they can be changed externally independently of each other. Then they talk about *a thermoelastic solid*

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{f}\big(\boldsymbol{\varepsilon}(\mathbf{x}, t), T(\mathbf{x}, t)\big), \quad \boldsymbol{\varepsilon}(\mathbf{x}, t) = \mathbf{g}\big(\boldsymbol{\sigma}(\mathbf{x}, t), T(\mathbf{x}, t)\big) \tag{3.10}$$

or *electroelastic* (*electromagnetoelastic*) solid

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{f}\big(\boldsymbol{\varepsilon}(\mathbf{x}, t), \mathbf{E}(\mathbf{x}, t)\big), \quad \boldsymbol{\varepsilon}(\mathbf{x}, t) = \mathbf{g}\big(\boldsymbol{\sigma}(\mathbf{x}, t), \mathbf{E}(\mathbf{x}, t)\big) \tag{3.11}$$

or in the general case of *an electrothermoelastic* solid

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{f}\big(\boldsymbol{\varepsilon}(\mathbf{x}, t), T(\mathbf{x}, t), \mathbf{E}(\mathbf{x}, t)\big), \quad \boldsymbol{\varepsilon}(\mathbf{x}, t) = \mathbf{g}\big(\boldsymbol{\sigma}(\mathbf{x}, t), T(\mathbf{x}, t), \mathbf{E}(\mathbf{x}, t)\big) \tag{3.12}$$

Physical linearity of elastic models (3.8), (3.10)–(3.12), as in the general case, is checked by the implementation of the superposition principle (2.3).

4. Nonlinear isotropic tensor functions of one argument

The presence in the constitutive relations of the elastic medium (3.8) of the inverse to each other tensor functions $\mathbf{f}$ and $\mathbf{g}$ makes it necessary to study in more detail the mathematical properties of these objects. As can be seen from (3.8), the tensor function $\mathbf{f}$ matches the tensor strain field $\boldsymbol{\varepsilon}(\mathbf{x}, t)$ tensor stress field $\boldsymbol{\sigma}(\mathbf{x}, t)$ pointwise at the same time. In general, representations of physically nonlinear anisotropic tensor functions are quite complex and cumbersome. Therefore, we will limit ourselves to considering here nonlinear but isotropic models [2–9].

4.1. Tensor power series and three-term relations. A wide class of nonlinear isotropic tensor functions in three-dimensional space connecting two symmetric tensors of the second rank $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ is described by tensor power series of the form

$$\boldsymbol{\sigma} = A_0\,\mathbf{I} + \sum_{n=1}^{\infty} A_n\,\boldsymbol{\varepsilon}^n \tag{4.1}$$

where $\mathbf{I}$ as before is the unit tensor of the second rank. The material functions $A_0, A_1, A_2, \ldots$ of the constitutive relations (4.1) can depend only on the invariants of the tensor $\boldsymbol{\varepsilon}$. A symmetric tensor of the second rank has no more than three independent invariants in three-dimensional space. Let us choose as such, for example, $I_{\varepsilon 1}$, $I_{\varepsilon 2}$ and $I_{\varepsilon 3}$, where

$$I_{\varepsilon n} = \sqrt[n]{\operatorname{tr}\boldsymbol{\varepsilon}^n}, \quad n = 1, 2, \ldots \tag{4.2}$$

Note that as the norm $\|\boldsymbol{\varepsilon}\|$, we can take the quadratic invariant $I_{\varepsilon 2}$, for a symmetric tensor equal to the root of the sum of the squares of all its components. In terms of principal strains, the invariants $I_{\varepsilon 1}$, $I_{\varepsilon 2}$ and $I_{\varepsilon 3}$ look like this

$$I_{\varepsilon 1} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad I_{\varepsilon 2} = \sqrt{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2}, \quad I_{\varepsilon 3} = \sqrt[3]{\varepsilon_1^3 + \varepsilon_2^3 + \varepsilon_3^3}$$

The invariants $I_{\varepsilon n}$, $n \geqslant 4$, are algebraically expressed in terms of $I_{\varepsilon 1}$, $I_{\varepsilon 2}$ and $I_{\varepsilon 3}$. For example, for $n = 4, 5, 6$ we have [2]

$$6I_{\varepsilon 4}^4 = I_{\varepsilon 1}^4 - 6I_{\varepsilon 1}^2 I_{\varepsilon 2}^2 + 8I_{\varepsilon 1} I_{\varepsilon 3}^3 + 3I_{\varepsilon 2}^4 \tag{4.3}$$
$$6I_{\varepsilon 5}^5 = I_{\varepsilon 1}^5 - 5I_{\varepsilon 1}^3 I_{\varepsilon 2}^2 + 5I_{\varepsilon 1}^2 I_{\varepsilon 3}^3 + 5I_{\varepsilon 2}^2 I_{\varepsilon 3}^3$$

$$12I_{\varepsilon 6}^6 = I_{\varepsilon 1}^6 - 3I_{\varepsilon 1}^4 I_{\varepsilon 2}^2 + 4I_{\varepsilon 1}^3 I_{\varepsilon 3}^3 - 9I_{\varepsilon 1}^2 I_{\varepsilon 2}^4 + 12I_{\varepsilon 1} I_{\varepsilon 2}^2 I_{\varepsilon 3}^3 + 3I_{\varepsilon 2}^6 + 4I_{\varepsilon 3}^6 \tag{4.4}$$

In addition to $I_{\varepsilon 2}$ and $I_{\varepsilon 3}$, other quadratic and cubic invariants of the tensor $\boldsymbol{\varepsilon}$ are often used:

$$J_\varepsilon = \frac{1}{2}(I_{\varepsilon 1}^2 - I_{\varepsilon 2}^2), \quad \Delta_\varepsilon \equiv \det\boldsymbol{\varepsilon} = \frac{1}{6}I_{\varepsilon 1}^3 - \frac{1}{2}I_{\varepsilon 1} I_{\varepsilon 2}^2 + \frac{1}{3}I_{\varepsilon 3}^3$$

Let us now use the Hamilton–Cayley formula in three-dimensional space

$$\boldsymbol{\varepsilon}^3 = \Delta_\varepsilon\,\mathbf{I} - J_\varepsilon\,\boldsymbol{\varepsilon} + I_{\varepsilon 1}\,\boldsymbol{\varepsilon}^2 \tag{4.5}$$

It immediately follows that the n-th degree of the tensor ($n \geqslant 3$) is a linear combination of $(n-1)$-th, $(n-2)$-th and $(n-3)$-th of its powers, and therefore (after another application of the Hamilton–Cayley formula (4.5)), a linear combination of $(n-2)$-th, $(n-3)$-th and $(n-4)$-th degrees and so on. Finally, we come to the conclusion that the n-th degree of the tensor ($n \geqslant 3$) is a linear combination of its zero, first and second degrees:

$$\boldsymbol{\varepsilon}^n = K_{\varepsilon 0}^{(n)}\mathbf{I} + K_{\varepsilon 1}^{(n)}\boldsymbol{\varepsilon} + K_{\varepsilon 2}^{(n)}\boldsymbol{\varepsilon}^2, \quad n = 0, 1, 2, \ldots \tag{4.6}$$

where the coefficients $K_{\varepsilon 0}^{(n)}$, $K_{\varepsilon 1}^{(n)}$ and $K_{\varepsilon 2}^{(n)}$ are some still unknown functions of the invariants $I_{\varepsilon 1}$, $I_{\varepsilon 2}$ and $I_{\varepsilon 3}$.

According to (4.5) we have

$$\boldsymbol{\varepsilon}^{n+1} = K_{\varepsilon 0}^{(n)}\boldsymbol{\varepsilon} + K_{\varepsilon 1}^{(n)}\boldsymbol{\varepsilon}^2 + K_{\varepsilon 2}^{(n)}(\Delta_\varepsilon\,\mathbf{I} - J_\varepsilon\,\boldsymbol{\varepsilon} + I_{\varepsilon 1}\,\boldsymbol{\varepsilon}^2)$$

from where we get the following recurrent relationship of the coefficients

$$K_{\varepsilon 0}^{(n+1)} = \Delta_\varepsilon K_{\varepsilon 2}^{(n)}, \quad K_{\varepsilon 1}^{(n+1)} = K_{\varepsilon 0}^{(n)} - J_\varepsilon K_{\varepsilon 2}^{(n)}, \quad K_{\varepsilon 2}^{(n+1)} = K_{\varepsilon 1}^{(n)} + I_{\varepsilon 1} K_{\varepsilon 2}^{(n)}$$

It can also be written in matrix form

$$(K_{\varepsilon 0}^{(n+1)}, K_{\varepsilon 1}^{(n+1)}, K_{\varepsilon 2}^{(n+1)})^T = Q_\varepsilon \cdot (K_{\varepsilon 0}^{(n)}, K_{\varepsilon 1}^{(n)}, K_{\varepsilon 2}^{(n)})^T, \quad Q_\varepsilon = \begin{pmatrix} 0 & 0 & \Delta_\varepsilon \\ 1 & 0 & -J_\varepsilon \\ 0 & 1 & I_{\varepsilon 1} \end{pmatrix} \tag{4.7}$$

Therefore

$$(K_{\varepsilon 0}^{(n)}, K_{\varepsilon 1}^{(n)}, K_{\varepsilon 2}^{(n)})^T = Q_\varepsilon^n \cdot (1,0,0)^T, \quad n = 0,1,2,\dots \tag{4.8}$$

that is, the triple of coefficients $K_{\varepsilon 0}^{(n)}$, $K_{\varepsilon 1}^{(n)}$ and $K_{\varepsilon 2}^{(n)}$ are the first column of the matrix Q_ε^n. Technically, it is obviously difficult to write out the general form Q_ε^n for any n, so we leave expressions for invariants $K_{\varepsilon 0}^{(n)}$, $K_{\varepsilon 1}^{(n)}$ and $K_{\varepsilon 2}^{(n)}$, appearing in (4.6), in the form (4.8).

Thus, the power tensor series (4.1) is equivalent to the three-term relation

$$\boldsymbol{\sigma} = C_0\,\mathbf{I} + C_1\,\boldsymbol{\varepsilon} + C_2\,\boldsymbol{\varepsilon}^2 \tag{4.9}$$

with material functions calculated according to the procedure described above:

$$C_0 = \sum_{n=0}^{\infty} A_n K_{\varepsilon 0}^{(n)}, \quad C_1 = \sum_{n=1}^{\infty} A_n K_{\varepsilon 1}^{(n)}, \quad C_2 = \sum_{n=2}^{\infty} A_n K_{\varepsilon 2}^{(n)} \tag{4.10}$$

The lower limits of summation in (4.10) can be chosen exactly as follows, since $K_{\varepsilon 1}^{(0)} = 0$, $K_{\varepsilon 2}^{(0)} = 0$, $K_{\varepsilon 2}^{(1)} = 0$.

4.2. Inverse tensor functions. Suppose that the series (4.1) is invertible:

$$\boldsymbol{\varepsilon} = B_0\,\mathbf{I} + \sum_{n=1}^{\infty} B_n\,\boldsymbol{\sigma}^n \tag{4.11}$$

where the material functions $B_0, B_1, B_2, \dots$ depend on the invariants $I_{\sigma 1}$, $I_{\sigma 2}$ and $I_{\sigma 3}$ defined similarly (4.2). It is not difficult to deduce the following algebraic relations of the triples of invariants $I_{\sigma 1}$, $I_{\sigma 2}$, $I_{\sigma 3}$ and $I_{\varepsilon 1}$, $I_{\varepsilon 2}$, $I_{\varepsilon 3}$:

$$I_{\sigma 1} = 3C_0 + I_{\varepsilon 1} C_1 + I_{\varepsilon 2}^2 C_2 \tag{4.12}$$

$$I_{\sigma 2}^2 = 3C_0^2 + 2I_{\varepsilon 1} C_0 C_1 + I_{\varepsilon 2}^2 (C_1^2 + 2C_0 C_2) + 2I_{\varepsilon 3}^3 C_1 C_2 + I_{\varepsilon 4}^4 C_2^2 \tag{4.13}$$

$$\begin{aligned} I_{\sigma 3}^3 = 3C_0^3 + 3I_{\varepsilon 1} C_0^2 C_1 + 3I_{\varepsilon 2}^2 C_0 (C_1^2 + C_0 C_2) + I_{\varepsilon 3}^3 C_1 (C_1^2 + 6C_0 C_2) \\ + 3I_{\varepsilon 4}^4 C_2 (C_1^2 + C_0 C_2) + 3I_{\varepsilon 5}^5 C_1 C_2^2 + I_{\varepsilon 6}^6 C_2^3 \end{aligned} \tag{4.14}$$

where the expressions of $I_{\varepsilon 4}^4$, $I_{\varepsilon 5}^5$ and $I_{\varepsilon 6}^6$ should be substituted from (4.3)–(4.4). The inverse to (4.12)–(4.14) connections will be written in the general form

$$I_{\varepsilon n} = I_{\varepsilon n}(I_{\sigma 1}, I_{\sigma 2}, I_{\sigma 3}), \quad n = 1,2,3 \tag{4.15}$$

Let us now construct, as it was done in (4.7), (4.8), a sequence of the triples of invariants $K_{\sigma 0}^{(n)}$, $K_{\sigma 1}^{(n)}$ and $K_{\sigma 2}^{(n)}$:

$$\big(K_{\sigma 0}^{(n)}, K_{\sigma 1}^{(n)}, K_{\sigma 2}^{(n)}\big)^T = Q_\sigma^n \cdot (1,0,0)^T, \quad n = 0,1,2,\dots, \quad Q_\sigma = \begin{pmatrix} 0 & 0 & \Delta_\sigma \\ 1 & 0 & -J_\sigma \\ 0 & 1 & I_{\sigma 1} \end{pmatrix}$$

and write the series (4.11) in the form of a three-term relation

$$\boldsymbol{\varepsilon} = D_0\,\mathbf{I} + D_1\,\boldsymbol{\sigma} + D_2\,\boldsymbol{\sigma}^2 \tag{4.16}$$

where

$$D_0 = \sum_{n=0}^{\infty} B_n K_{\sigma 0}^{(n)}, \quad D_1 = \sum_{n=1}^{\infty} B_n K_{\sigma 1}^{(n)}, \quad D_2 = \sum_{n=2}^{\infty} B_n K_{\sigma 2}^{(n)}$$

Substituting the tensor function (4.9) into the inverse of it (4.16), after the transformations, we obtain a connection of the material functions C_0, C_1, C_2 and D_0, D_1, D_2. It follows from the solution of a linear inhomogeneous system of equations with respect to D_0, D_1 and D_2:

$$D_0 + C_0 D_1 + (C_0^2 + 2C_1C_2\Delta_\varepsilon + C_2^2 I_{\varepsilon 1}\Delta_\varepsilon) D_2 = 0 \tag{4.17}$$

$$C_1 D_1 + (2C_0C_1 - 2C_1C_2J_\varepsilon - C_2^2 I_{\varepsilon 1} J_\varepsilon + C_2^2\Delta_\varepsilon) D_2 = 1 \tag{4.18}$$

$$C_2 D_1 + (2C_0C_2 + C_1^2 + 2C_1C_2 I_{\varepsilon 1} + C_2^2 I_{\varepsilon 1}^2 - C_2^2 J_\varepsilon) D_2 = 0 \tag{4.19}$$

and substitutions in the solution of invariants $I_{\varepsilon n}$ by $I_{\sigma n}$, $n = 1,2,3$, according to (4.15). From the equations (4.18) and (4.19), D_1 and D_2 are found, after which D_0 is determined from the equation (4.17).

4.3. Tensor Taylor series. An important special case is the situation when all coefficients A_n, $n = 0,1,2,\dots$, in (4.1) are constant, i.e., do not depend on the invariants of the tensor $\boldsymbol{\varepsilon}$. Then they can be considered as coefficients of the Taylor series near zero $A_n = F^{(n)}(0)/n!$ of some scalar function $F(x)$, and the series itself (4.1) is interpreted as a tensor function $\boldsymbol{\sigma} = \mathbf{F}(\boldsymbol{\varepsilon})$ generated (by means of the set A_n) by the scalar function F.

Examples of such constructions are inverse to each other tensor (matrix) exponent and logarithm:

$$\boldsymbol{\sigma} = \beta\Big(\exp\frac{\boldsymbol{\varepsilon}}{\alpha} - \mathbf{I}\Big) \equiv \beta \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\frac{\boldsymbol{\varepsilon}}{\alpha}\Big)^n,$$

$$\boldsymbol{\varepsilon} = \alpha \ln\Big(\mathbf{I} + \frac{\boldsymbol{\sigma}}{\beta}\Big) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\Big(\frac{\boldsymbol{\sigma}}{\beta}\Big)^n$$

where α and β are typical quantities with physical dimensions that coincide with the dimensions of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, respectively.

For the function $\boldsymbol{\sigma} = \mathbf{F}(\boldsymbol{\varepsilon})$, which admits a three-term representation (4.9), (4.10), in matrix form we can write

$$\boldsymbol{\sigma} = (\mathbf{I}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^2) \cdot \sum_{n=0}^{\infty} A_n \big(K_{\varepsilon 0}^{(n)}, K_{\varepsilon 1}^{(n)}, K_{\varepsilon 2}^{(n)}\big)^T$$

$$= (\mathbf{I}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^2) \cdot \sum_{n=0}^{\infty} A_n Q_\varepsilon n \cdot (1,0,0)^T = (\mathbf{I}, \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^2) \cdot F(Q_\varepsilon) \cdot (1,0,0)^T$$

This means that the column $(C_0, C_1, C_2)^T$ coincides with the first column of the matrix $F(Q_\varepsilon)$.

4.4. Tensor linearity and nonlinearity. In the theory of constitutive relations, there are usually two equivalent definitions of tensor linearity of functions (4.9) and (4.16) (in the terminology of [16] quasilinearity). One of them is related to the identical vanishing of the coefficients C_2 in (4.9) and D_2 in (4.16), and the other is due to the fact that the angle between the deviators $\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} - I_{\varepsilon 1}\mathbf{I}/3$ and $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - I_{\sigma 1}\mathbf{I}/3$ is equal to zero [8], i.e.

$$\bar{\boldsymbol{\varepsilon}} : \bar{\boldsymbol{\sigma}} = \sqrt{\bar{\boldsymbol{\varepsilon}} : \bar{\boldsymbol{\varepsilon}}}\,\sqrt{\bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\sigma}}}$$

The conditions for separating the deviatory and spherical properties of isotropic tensor functions in the general case of nonlinearity and in the case of quasilinearity are covered in detail in [7].

Inverse to each other tensor functions (4.9) and (4.16) are quasilinear or non-quasilinear at the same time. The presence in series (4.1) and (4.11) of terms with tensor degrees greater than the first does not yet indicate the tensor nonlinearity of the corresponding functions. So, for example, if $A_2 = -I_{\varepsilon 1} A_3$, then the function $\boldsymbol{\sigma} = A_2 \boldsymbol{\varepsilon}^2 + A_3 \boldsymbol{\varepsilon}^3$ is nevertheless tensorically linear and equal to $A_3(\Delta_\varepsilon \mathbf{I} - J_\varepsilon \boldsymbol{\varepsilon})$.

Acknowledgments. The paper is supported by the Russian Scientific Fund (22-21-00077).

The author is grateful to the Organizing Committee of the 9th International Congress of the Serbian Society of Mechanics for the opportunity to make an invited report, based on the materials of which this article was written, as well as to Božidar Jovanović and Borislav Gajić for the invitation to submit it to the Journal.

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СВОЈСТВА ОПЕРАТОРСКИХ КОНСТИТУТИВНИХ РЕЛАЦИЈА У МЕХАНИЦИ ДЕФОРМАБИЛНОГ ЧВРСТОГ ТЕЛА

Резиме. Разматрају се конститутивне релације између напона и деформација у механици деформабилног чврстог тела, укључујући њихову операторску везу. Описана су нека важна и честа својства модула тангенте и савитљивости тангенте као тензора четвртог ранга. У зависности од тога, предлаже се могућа класификација непрекидних средина. Разликују се склерономске и реономске средине, хомогене и нехомогене средине (посебно композити), средине са меморијом, просторно нелокалне средине, материјали са тврдим или меким карактеристикама. За нелинеарно еластичне изотропне средине развијен је апарат тензорских нелинеарних изотропних функција једног аргумента. Посебна пажња је посвећена трочланом развијању тензора у тродимензионалном простору, реверзибилности тензорских функција, Тејлоровом тензорском реду, линеарности тензора (квазилинеарности) и нелинеарности.

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(Received 19.07.2023)
(Available online 18.10.2023)