THEORETICAL AND APPLIED MECHANICS
Volume 50 (2023) Issue 1, 13–24 DOI: https://doi.org/10.2298/TAM220811002L

ARBITRARY DECAY FOR A NONLINEAR EULER–BERNOULLI BEAM WITH NEUTRAL DELAY

Ibrahim Lakehal, Djamila Benterki, and Khaled Zennir

Abstract. In this paper, the free transverse vibration of a nonlinear Euler-Bernoulli beam under a neutral type delay is considered. In order to suppress the beam transverse vibrations, a boundary control based on the Lyapunov method is designed. The novelty of this paper is the ability to get a wide variety of energy decay rates under free vibration conditions.

1. Introduction

Due to the requirement for high-precision control of numerous mechanical systems, such as marine risers for oil and gas transportation, spacecraft with flexible attachments, or flexible robot arms, the boundary control of flexible systems has been an important topic of study in recent years [2–5, 9, 14]. The time delay is one of several elements that have a significant impact on the dynamic properties of systems. It became evident that its existence could not be fully neglected in many systems, and with the rapid growth of numerous engineering disciplines, including mechanical engineering, a more precise system analysis was necessary. Time delays in these systems can lead to poor performance and unstable dynamic systems [8, 10]. As a result, throughout the past few decades, the stability issue with time-delay systems has received a lot of attention. In [11], exponential stability result for a viscoelastic Timoshenko beam was established. The researchers in [12] used the LMI (linear matrix inequality) technique to investigate global exponential stability for neutral differential systems with time-varying or constant delay. The asymptotic stability of delay differential equations of neutral type has been extensively studied in [1, 13]. We consider in this paper the neutrally retarded nonlinear

2020 *Mathematics Subject Classification:* 34K30; 35R09; 35R10.
Key words and phrases: Euler-Bernoulli beam, neutral delay, arbitrary decay, boundary control.

Euler-Bernoulli beam for $x \in (0, L) \times [0, \infty), L > 0$

$$\rho A\left[u_t + \int_0^t \kappa(t-s)u_t(x,s)ds\right]_t + EIu_{xxxx} - P_0 u_{xx} - \frac{1}{2}EA(u_x^3)_x = 0, \tag{1.1}$$

under the boundary

$$\begin{cases} u_{xx}(0,t) = u_{xx}(L,t) = u(0,t) = 0, & \forall t \geqslant 0, \\ EIu_{xxx}(L,t) = P_0 u_x(L,t) + \frac{1}{2}EAu_x^3(L,t) + \alpha u_t(L,t), & \forall t \geqslant 0,\ \alpha > 0, \end{cases} \tag{1.2}$$

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in (0, L), \tag{1.3}$$

where EI is the beam's flexural rigidity, ρA is the beam's mass per unit length, and $u(x,t)$ represents transverse displacement at time t with respect to the spatial coordinate x, EA the axial stiffness, P_0 the tension force. In this paper we consider the transverse dynamics of a beam in bending vibration and we neglect the coupling between longitudinal and transversal displacements. Assuming that the change in length due to the axial force is small and negligible, we take only the elongation of the beam due to the curvature. We prove a general decay result for the problem (1.1)–(1.3). The rest of our paper is arranged as follows: In Section 2, we give some assumptions and lemmas necessary for our work. Section 3 discusses the arbitrary decay of the energy result.

2. Notation and main results

We introduce the following notation

$$(\kappa \circ u)(t) = \int_0^L \int_0^t \kappa(t-s)[u(x,t) - u(x,s)]^2 ds\, dx$$

For the kernel κ we assume:

(K1): The kernel κ is a nonnegative summable function $C^1(\mathbb{R}_+)$ satisfying $\kappa'(t) \leqslant 0$ for all $t \geqslant 0$.

(K2): $0 < \bar{k} = \int_0^{+\infty} \kappa(s)ds < 1$.

(K3): There exists a positive increasing function $g(t)$ such that $\mu(t) = \frac{g'(t)}{g(t)}$ is a decreasing function and

$$\int_0^{+\infty} \kappa(s)g(s)ds < +\infty, \qquad \int_0^{+\infty} |\kappa'(s)|g(s)ds < +\infty.$$

We denote for $t^* > 0$, $\kappa^* = \int_0^{t^*} \kappa(s)ds$,

$$\mathcal{V} = \{u \in H^2(0,L) \mid u(0) = 0\},$$
$$\mathcal{H} = \{u \in \sqcap \cap H^4(0,L) \mid u_{xx}(0) = u_{xx}(L) = 0\}$$

and $(.\,,.)$, $\|.\|$ the inner product and the norm of the space $L^2(0,L)$, respectively. The existence and uniqueness of a solution in $L^\infty([0,T), \mathcal{H})$ can be demonstrated by combining the results in [6, 7] and [3]. We define the (classical) energy of the problem (1.1)–(1.3) by

$$E(t) = \frac{1}{2}\Big[\rho A\|u_t\|^2 + EI\|u_{xx}\|^2 + P_0\|u_x\|^2 + \frac{EA}{4}\|u_x^2\|^2 + \rho A\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds\Big]. \tag{2.1}$$

We need the following auxiliary result:

Lemma 2.1. [11] *We have the following identity:*

$$\int_0^L u_t(t)\int_0^t \kappa(t-s)u_{tt}(s)ds\,dx = -\frac{1}{2}(\kappa'\circ u_t)(t) + \frac{1}{2}\frac{d}{dt}\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds + \frac{\kappa(t)}{2}\|u_t(t)\|^2 - \kappa(t)\int_0^L u_t(t)u_t(0)dx$$

for all $u_t \in C^1([0,\infty); L^2(0,L))$ and $\kappa \in C^1[0,\infty)$.

Proposition 2.1. *The modified energy $E(t)$ is non-increasing and uniformly bounded. More precisely, we have*

$$E'(t) = \frac{\rho A}{2}(\kappa'\circ u_t)(t) - \rho A\frac{\kappa(t)}{2}\|u_t(t)\|^2 - \alpha u_t^2(L,t) \leqslant 0, \quad t \geqslant 0.$$

Proof. Multiplying equation (1.1) by u_t and integrating the result over $(0, L)$ by parts and using the boundary conditions, we get

$$\frac{1}{2}\frac{d}{dt}\Big[\rho A\|u_t(t)\|^2 + EI\|u_{xx}(t)\|^2 + P_0\|u_x(t)\|^2 + \frac{EA}{4}\|u_x^2(t)\|^2\Big] + \rho A\kappa(t)\int_0^L u_t(t)u_t(0)dx + \rho A\int_0^L u_t\int_0^t \kappa(t-s)u_{tt}(s)ds\,dx + \big[EIu_{xxx}(L,t) - P_0u_x(L,t) - \tfrac{1}{2}EAu_x^3(L,t)\big]u_t(L,t).$$

Utilizing Lemma 2.1, we determine the relation in the proposition. □

Next, we introduce the functionals

$$\Psi_1(t) = \rho A\int_0^L u_t\int_0^t \kappa(t-s)u_t(s)ds\,dx,$$

$$\Psi_2(t) = \rho A\int_0^L u\Big(u_t + \int_0^t \kappa(t-s)u_t(s)ds\Big)dx,$$

$$\Psi_3(t) = \frac{P_0}{2}\int_0^t K_g(t-s)\|u_x(s)\|^2 ds,$$

$$\Psi_4(t) = \frac{\rho A}{2}\int_0^t (\tilde{K}_g(t-s) + K_g(t-s))\|u_t(s)\|^2 ds,$$

and

$$\Psi_5(t) = \frac{EI}{2}\int_0^L\int_0^t K_g(t-s)u_{xx}(s)^2 ds\,dx + \frac{EA}{2}\int_0^L\int_0^t K_g(t-s)u_x(s)^4 ds\,dx,$$

$$K_g(t) = g^{-1}(t)\int_t^{+\infty} |\kappa'(s)|g(s)ds,\;\; \tilde{K}_g(t) = g^{-1}(t)\int_t^{+\infty} \kappa(s)g(s)ds$$

and $g(t)$ is specified below. We define the second modified functional by

$$F(t) = E(t) + \sum_{i=1}^{5} \lambda_i \Psi_i(t), \quad t \geqslant 0, \tag{2.2}$$

for $\lambda_i > 0$, $i = 1, \dots, 5$ to be specified later. Our first study indicated that this functional is reasonable to consider.

Proposition 2.2. *There exist* $n_i > 0, i = 1, 2$ *such that*

$$\begin{aligned} n_1(E(t) + \Psi_3(t) + \Psi_4(t) + \Psi_5(t)) &\leqslant F(t) \\ &\leqslant n_2(E(t) + \Psi_3(t) + \Psi_4(t) + \Psi_5(t)), \quad t \geqslant 0. \end{aligned} \tag{2.3}$$

Proof. It is easy to see, from the above definitions, that

$$\begin{aligned} \Psi_1(t) &\leqslant \frac{\rho A}{2}\|u_t(t)\|^2 + \frac{\rho A}{2}\bar{k}\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds \\ &\leqslant q_1\bigg(\frac{\rho A}{2}\|u_t(t)\|^2 + \frac{\rho A}{2}\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds\bigg) \end{aligned}$$

where $q_1 = L\max(1, \bar{k})$.

$$\begin{aligned} \Psi_2(t) &\leqslant \frac{\rho A}{2}\|u_t(t)\|^2 + \frac{2\rho A L^2}{P_0}\frac{P_0}{2}\|u_x(t)\|^2 + \frac{\rho A}{2}\bar{k}\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds \\ &\leqslant q_2\bigg(\frac{\rho A}{2}\|u_t(t)\|^2 + \frac{P_0}{2}\|u_x(t)\|^2 + \frac{\rho A}{2}\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds\bigg). \end{aligned}$$

where $q_2 = L\max(1, \bar{k}, \frac{2\rho A L^2}{P_0})$. Taking into account these considerations, we have

$$\begin{aligned} F(t) \leqslant{}& (1 + \lambda_1 q_1 + \lambda_2 q_2)\frac{\rho A}{2}\|u_t(t)\|^2 + \frac{EI}{2}\|u_{xx}(t)\|^2 + \frac{P_0}{2} + (1 + \lambda_2 q_2)\|u_x(t)\|^2 \\ &+ \frac{EA}{8}\|u_x^2(t)\|^2 + (1 + \lambda_1 q_1 + \lambda_2 q_2)\frac{\rho A}{2}\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds \\ &+ \lambda_3\Psi_3(t) + \lambda_4\Psi_4(t) + \lambda_5\Psi_5(t) \end{aligned}$$

and

$$\begin{aligned} 2F(t) \geqslant{}& (1 - q_1\lambda_1 - \lambda_2 q_2)\rho A\|u_t(t)\|^2 + EI\|u_{xx}(t)\|^2 + \frac{EA}{4}\|u_x^2(t)\|^2 \\ &+ (1 - \lambda_2 q_2)P_0\|u_x(t)\|^2 + (1 - q_1\lambda_1 - \lambda_2 q_2)\rho A\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds \\ &+ 2\lambda_3\Psi_3(t) + 2\lambda_4\Psi_4(t) + 2\lambda_5\Psi_5(t),\; t \geqslant 0. \end{aligned}$$

Therefore,

$$\begin{aligned} n_1(E(t) + \lambda_3\Psi_3(t) + \lambda_4\Psi_4(t) + \lambda_5\Psi_5(t)) &\leqslant L(t) \\ &\leqslant n_2(E(t) + \lambda_3\Psi_3(t) + \lambda_4\Psi_4(t) + \lambda_5\Psi_5(t)) \end{aligned}$$

for some $n_i > 0$ and λ_i, $i = 1, 2$ such that $\lambda_1 < \frac{1-\lambda_2 q_2}{2q_1}$, $\lambda_2 < \frac{1-\lambda_1 q_1}{2q_2}$. □

3. Asymptotic behavior

In the following section, we state and prove our main result.

THEOREM 3.1. *Let us suppose that κ and g satisfy the hypotheses (K1)–(K3). Then, there exist positive constants C and σ such that*

$$E(t) \leqslant Cg(t)^{-\sigma}, \quad t \geqslant 0.$$

PROOF. Differentiating $\Psi_1(t)$, with respect to t and utilizing the first equation of (1.1)–(1.3), we obtain

$$\begin{aligned}\Psi_1(t) &= -\rho A \int_0^L u_t \int_0^t \kappa(t-s)u_t(s)ds\,dx\\ \Psi_1'(t) &= -\rho A \int_0^L u_t \Big(\int_0^t \kappa(t-s)u_t(s)ds\Big)_t dx\\ &\quad - \rho A \int_0^L u_{tt} \int_0^t \kappa(t-s)u_t(s)ds\,dx = I_1 + I_2.\end{aligned}$$

Clearly

$$\begin{aligned}I_1 &= -\rho A \int_0^L u_t(\kappa(0)u_t + \int_0^t \kappa'(t-s)u_t(s)ds)\\ I_1 &= -\rho A\kappa(0)\|u_t\|^2 - \rho A \int_0^L u_t \int_0^t \kappa'(t-s)u_t(s)ds\,dx\\ &\leqslant -\rho A\kappa(0)\|u_t\|^2 + \frac{\rho A}{4\delta_0}\|u_t\|^2 + \rho A\delta_0\kappa(0)\int_0^t |\kappa'(t-s)|\|u_t(s)\|^2 ds\\ &\leqslant \rho A\Big(\frac{1}{4\delta_0} - \kappa(0)\Big)\|u_t\|^2 + \rho A\delta_0\kappa(0)\int_0^t |\kappa'(t-s)|\|u_t(s)\|^2 ds, \quad \delta_0 > 0.\end{aligned}$$

The equation (1.1) allows us to write

$$\begin{aligned}I_2 &= \rho A \int_0^L \Big(\int_0^t \kappa(t-s)u_t(s)ds\Big)_t \Big(\int_0^t \kappa(t-s)u_t(s)ds\Big)dx\\ &\quad + \int_0^L \Big(EIu_{xxxx} - P_0u_{xx} - \frac{EA}{2}(u_x^3)_x\Big)\\ &\quad \times \Big(\kappa(0)u - \kappa(t)u_0 + \int_0^t \kappa'(t-s)u(s)ds\Big)dx = I_{21} + I_{22},\end{aligned}$$

and

$$\begin{aligned}I_{21} &= \rho A \int_0^L \Big(\kappa(0)u_t + \int_0^t \kappa'(t-s)u_t(s)ds\Big)\Big(\int_0^t \kappa(t-s)u_t(s)ds\Big)dx\\ &= \rho A\kappa(0)\int_0^L u_t \int_0^t \kappa(t-s)u_t(s)ds\,dx + \rho A \int_0^L \int_0^t \kappa'(t-s)u_t(s)ds \int_0^t \kappa(t-s)u_t(s)ds\,dx\\ &\leqslant \rho A\kappa(0)^2\delta_0\|u_t\|^2 + \frac{\rho A}{4\delta_0}\bar{k}\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds\end{aligned}$$

$$+\rho A\kappa(0)\delta_0\int_0^t|\kappa'(t-s)|\|u_t(s)\|^2ds+\frac{\rho A\bar{k}}{4\delta_0}\int_0^t\kappa(t-s)\|u_t(s)\|^2ds$$
$$\leqslant\rho A\kappa(0)^2\delta_0\|u_t\|^2+\frac{\rho A\bar{k}}{2\delta_0}\int_0^t\kappa(t-s)\|u_t(s)\|^2ds$$
$$+\rho A\kappa(0)\delta_0\int_0^t|\kappa'(t-s)|\|u_t(s)\|^2ds.$$

In addition, for $\delta_1,\ \delta_2>0$

$$\begin{aligned}I_{22}&=\int_0^L\Big(EIu_{xxx}-P_0u_x-\frac{EA}{2}u_x^3\Big)_x\Big(\kappa(0)u-\kappa(t)u_0+\int_0^t\kappa'(t-s)u(s)ds\Big)dx\\&=\Big(EIu_{xxx}(L,t)-P_0u_x(L,t)-\frac{EA}{2}u_x^3(L,t)\Big)\\&\quad\times\Big(\kappa(0)u(L,t)-\kappa(t)u(L,0)+\int_0^t\kappa'(t-s)u(L,s)ds\Big)\\&\quad-\int_0^L\Big(EIu_{xxx}-P_0u_x-\frac{EA}{2}u_x^3\Big)\Big(\kappa(0)u_x-\kappa(t)u_{x0}+\int_0^t\kappa'(t-s)u_x(s)ds\Big)dx.\end{aligned}$$

Using boundary control, we find

$$\begin{aligned}I_{22}&\leqslant\alpha u_t(L,t)\Big(\kappa(0)u(L,t)-\kappa(t)u(L,0)+\int_0^t\kappa'(t-s)u(L,s)ds\Big)\\&\quad EI(\kappa(0)+\kappa(t)\delta_1)\|u_{xx}\|^2+P_0(\kappa(0)+\kappa(t)\delta_2)\|u_x\|^2+\frac{EI\,\kappa(t)}{4\delta_1}\|u_{xx0}\|^2\\&\quad+\frac{EA}{2}\Big(\kappa(0)+\frac{\kappa(t)(1+\delta_3)}{2}\Big)\|u_x^2\|^2+\frac{P_0\kappa(t)}{4\delta_2}\|u_{x0}\|^2+\frac{EA}{16\delta_3}\kappa(t)\|u_{x0}^2\|^2\\&\quad-\int_0^L\Big(EIu_{xxx}-P_0u_x-\frac{EA}{2}u_x^3\Big)\Big(\int_0^t\kappa'(t-s)u_x(s)ds\Big)dx.\end{aligned}$$

Young inequality gives us

$$\begin{aligned}-\int_0^L\Big(EIu_{xxx}-P_0u_x-\frac{EA}{2}u_x^3\Big)\Big(\int_0^t\kappa'(t-s)u_x(s)ds\Big)dx&\leqslant EI\delta_4\|u_{xx}\|^2\\&+\frac{EI\,\kappa(0)}{4\delta_4}\int_0^t|\kappa'(t-s)|\|u_{xx}(s)\|^2ds+P_0\delta_5\|u_x\|^2\\&+\frac{P_0\kappa(0)}{4\delta_5}\int_0^t|\kappa'(t-s)|\|u_x(s)\|^2ds\\&+\frac{EA}{2}\int_0^L u_x^2\int_0^t\kappa'(t-s)u_x(s)u_xds\,dx,\quad\delta_4,\delta_5>0.\end{aligned}$$

For $\delta_6>0$, the estimation

$$\begin{aligned}&\int_0^L u_x^2\int_0^t\kappa'(t-s)u_x(s)u_xds\,dx\\&\quad\leqslant\Big(\int_0^L(u_x^2)^2dx\Big)^{1/2}\Big(\int_0^L\Big(\int_0^t|\kappa'(t-s)|u_x(s)u_xds\Big)^2dx\Big)^{1/2}\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{2}\|u_x^2\|^2 + \frac{1}{2}\int_0^L \int_0^t |\kappa'(t-s)| u_x^2(s) ds \int_0^t |\kappa'(t-s)| u_x^2 ds\, dx \\
&\leqslant \frac{1}{2}\|u_x^2\|^2 + \frac{1}{2}\Big(\frac{\kappa(0)\delta_6}{4}\int_0^t |\kappa'(t-s)| \|u_x^2(s)\|^2 ds + \frac{\kappa(0)}{\delta_6}\|u_x^2\|^2\Big) \\
&\leqslant \frac{1}{2}\Big(1 + \frac{\kappa(0)}{\delta_6}\Big)\|u_x^2\|^2 + \frac{\delta_6 \kappa(0)}{8}\int_0^t |\kappa'(t-s)| \|u_x^2(s)\|^2 ds.
\end{aligned}$$

Applying Young's inequality, we find

$$u_t(L,t)\int_0^t \kappa'(t-s) u(L,s) ds \leqslant \frac{1}{2b_0} u_t^2(L,t) + \frac{b_0}{2} L\kappa(0)\int_0^t |\kappa'(t-s)| \|u_x(s)\|^2 ds,$$

$$u_t(L,t)u(L,t) \leqslant \frac{1}{2b_0}u_t^2(L,t) + \frac{b_0 L}{2}\|u_x\|^2,$$

$$-u_t(L,t)u(L,0) \leqslant \frac{1}{2b_0}u_t^2(L,t) + \frac{b_0 L}{2}\|u_{x0}\|^2.$$

Hence,

$$\begin{aligned}
I_{22} \leqslant{}& \alpha \frac{1+\kappa(0)+\kappa(t)}{2b_0} u_t^2(L,t) + P_0\Big(\kappa(0) + \kappa(t)\delta_2 + \delta_5 + \frac{b_0 L \kappa(0)}{2P_0}\Big)\|u_x\|^2 \\
&+ EI(\kappa(0) + \kappa(t)\delta_1 + \delta_4)\|u_{xx}\|^2 + \frac{\kappa(0)}{2}\Big(\frac{P_0}{2\delta_5} + \alpha L b_0\Big)\int_0^t |\kappa'(t-s)| \|u_x(s)\|^2 ds \\
&+ \frac{EA}{2}\Big(\kappa(0)\Big(1 + \frac{1}{2\delta_6}\Big) + \frac{1+\kappa(t)(1+\delta_3)}{2}\Big)\|u_x^2\|^2 + \frac{EI\,\kappa(t)}{4\delta_1}\|u_{xx0}\|^2 \\
&+ \Big(\frac{P_0}{2\delta_2} + \alpha b_0 L\Big)\frac{\kappa(t)}{2}\|u_{x0}\|^2 + \frac{EA}{16\delta_3}\kappa(t)\|u_{x0}^2\|^2 \\
&+ \frac{EI\kappa(0)}{4\delta_4}\int_0^t |\kappa'(t-s)| \|u_{xx}(s)\|^2 ds + \frac{\delta_6 \kappa(0)}{16} EA \int_0^t |\kappa'(t-s)| \|u_x^2(s)\|^2 ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{(3.1)}\quad \Psi_1'(t) \leqslant{}& \alpha \frac{(1+\kappa(0)+\kappa(t))}{2b_0} u_t^2(L,t) + \rho A\Big(\frac{1}{4\delta_0} - \kappa(0) + \kappa(0)^2\delta_0\Big)\|u_t\|^2 \\
&+ 2\rho A \kappa(0)\delta_0 \int_0^t |\kappa'(t-s)| \|u_t(s)\|^2 ds + \frac{\rho A \bar{k}}{2\delta_0}\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds \\
&+ EI(\kappa(0) + \kappa(t)\delta_1 + \delta_4)\|u_{xx}\|^2 \\
&+ P_0\Big(\kappa(0) + \kappa(t)\delta_2 + \delta_5 + \frac{\alpha b_0 L \kappa(0)}{2P_0}\Big)\|u_x\|^2 \\
&+ \frac{EA}{2}\Big(\kappa(0)\Big(1 + \frac{1}{2\delta_6}\Big) + \frac{1+\kappa(t)(1+\delta_3)}{2}\Big)\|u_x^2\|^2 + \frac{EI}{4\delta_1}\kappa(t)\|u_{xx0}\|^2 \\
&+ \Big(\frac{P_0}{2\delta_2} + b_0 L\Big)\frac{\kappa(t)}{2}\|u_{x0}\|^2 + \frac{EA}{16\delta_3}\kappa(t)\|u_{x0}^2\|^2 \\
&+ \frac{EI\,\kappa(0)}{4\delta_4}\int_0^t |\kappa'(t-s)| \|u_{xx}(s)\|^2 ds
\end{aligned}$$

$$+\frac{\kappa(0)}{2}\Big(\frac{P_0}{2\delta_5}+\alpha L b_0\Big)\int_0^t |\kappa'(t-s)|\|u_x(s)\|^2 ds$$
$$+\frac{\delta_6\kappa(0)}{16}EA\int_0^t |\kappa'(t-s)|\|u_x^2(s)\|^2 ds.$$

In view of equation (1.1), The derivative of $\Psi_2(t)$ is given by

$$\begin{aligned}\Psi_2'(t) = \rho A\int_0^L u_t\Big(u_t+\int_0^t k(t-s)u_t(s)ds\Big)dx \\ +\rho A\int_0^L u\frac{\partial}{\partial t}\Big(u_t+\int_0^t k(t-s)u_t(s)ds\Big)dx \\ =\rho A\|u_t\|^2+\rho A\int_0^L u_t\int_0^t k(t-s)u_t(s)ds\,dx \\ +\int_0^L u\Big(-EIu_{xxxx}+\frac{EA}{2}(u_x^3)_x+P_0u_{xx}\Big)dx.\end{aligned}$$

Therefore, for $\delta_4>0$,

$$\Psi_2'(t)\leqslant \rho A(1+\frac{\delta_4}{2})\|u_t\|^2+\frac{\rho A\bar{k}}{2\delta_4}\int_0^t k(t-s)\|u_t(s)\|^2ds-EI\|u_{xx}\|^2$$
$$-P_0\|u_x\|^2-\frac{EA}{2}\|u_x^2\|^2+u(L,t)(-EIu_{xxx}(L,t)+P_0u_x(L,t)+\frac{EA}{2}u_x^3(L,t)).$$

It follows from the boundary conditions that

$$\begin{aligned}\Psi_2'(t)\leqslant \rho A(1+\frac{\delta_4}{2})\|u_t\|^2+\frac{\rho A\bar{k}}{2\delta_4}\int_0^t k(t-s)\|u_t(s)\|^2ds \\ -EI\|u_{xx}\|^2-P_0\Big(1-\frac{\alpha L b_0}{2P_0}\Big)\|u_x\|^2-\frac{EA}{2}\|u_x^2\|^2+\frac{\alpha}{2b_0}u_t^2(L,t).\end{aligned}\tag{3.2}$$

The derivative of $\Psi_3(t)$ satisfies

$$\begin{aligned}\Psi_3'(t) &= \frac{P_0}{2}K_g(0)\|u_x\|^2+\frac{P_0}{2}\int_0^t K_g'(t-s)\|u_x(s)\|^2ds \\ &\leqslant \frac{P_0}{2}K_g(0)\|u_x\|^2-\frac{P_0}{2}\mu\int_0^t K_g(t-s)\|u_x(s)\|^2ds \\ &-\frac{P_0}{2}\int_0^t |\kappa'(t-s)|\|u_x(s)\|^2ds, t\geqslant 0.\end{aligned}\tag{3.3}$$

Further, differentiating $\Psi_4(t)$ yields

$$\begin{aligned}\Psi_4'(t) &= \frac{\rho A}{2}(\tilde{K}_g(0)+K_g(0))\|u_t(t)\|^2 \\ &+\frac{\rho A}{2}\int_0^t (\tilde{K}_g'(t-s)+K_g'(t-s))\|u_t(s)\|^2ds \\ &\leqslant \frac{\rho A}{2}(\tilde{K}_g(0)+K_g(0))\|u_t(t)\|^2\end{aligned}\tag{3.4}$$

$$- \frac{\rho A}{2}\mu(t)\int_0^t (\tilde{K}_g(t-s)+K_g(t-s))\|u_t(s)\|^2 ds$$
$$- \frac{\rho A}{2}\int_0^t (\kappa(t-s)+|\kappa'(t-s)|)\|u_t(s)\|^2 ds, t \geqslant 0.$$

Direct computations give us

$$\Psi_5'(t) = \frac{EI}{2}K_g(0)\|u_{xx}\|^2 + \frac{EI}{2}\int_0^t K_g'(t-s)\|u_{xx}(s)\|^2 ds$$
$$+ \frac{EA}{2}K_g(0)\|u_x^2\|^2 + \frac{EA}{2}\int_0^t K_g'(t-s)\|u_x^2(s)\|^2 ds,$$

that is

$$\Psi_5'(t) \leqslant \frac{EI}{2}K_g(0)\|u_{xx}\|^2 + \frac{EA}{2}K_g(0)\|u_x^2\|^2 - \frac{EI}{2}\int_0^t |\kappa'(t-s)|\|u_{xx}(s)\|^2 ds$$
$$- \frac{EA}{2}\mu(t)\int_0^t K_g(t-s)\|u_x^2(s)\|^2 ds - \frac{EA}{2}\int_0^t |\kappa'(t-s)|\|u_x^2(s)\|^2 ds$$
$$- \frac{EI}{2}\mu(t)\int_0^t K_g(t-s)\|u_{xx}(s)\|^2 ds,\ t \geqslant 0. \tag{3.5}$$

Collecting the estimations (3.1)–(3.5), we find

$$F'(t) \leqslant \frac{\rho A}{2}(\kappa'\circ u_t)(t) + \frac{\rho A}{2}\Big\{ -\kappa(t) + 2\lambda_1\Big(\frac{1}{4\delta_0} - \kappa(0) + \kappa(0)^2\delta_0\Big)$$
$$+ 2\lambda_2(1+\delta_4) + \lambda_4(\tilde{K}_g(0)+K_g(0))\Big\}\|u_t\|^2$$
$$+ P_0\Big(\lambda_1\Big(\kappa(0)+\kappa(t)\delta_2+\delta_5+\frac{\alpha L b_0 \kappa(0)}{2P_0}\Big) - \lambda_2\Big(1-\frac{\alpha L b_0}{2P_0}\Big) + \frac{\lambda_3}{2}K_g(0)\Big)\|u_x\|^2$$
$$+ EI\Big(\lambda_1\Big(\kappa(0)+\kappa(t)\delta_1+\delta_4\Big) - \lambda_2 + \frac{\lambda_5}{2}K_g(0)\Big)\|u_{xx}\|^2$$
$$+ \frac{EA}{2}\Big(\lambda_1\Big(\frac{1+\kappa(t)(1+\delta_3)}{2} + \kappa(0)\Big(1+\frac{1}{\delta_6}\Big)\Big) - \lambda_2 + \lambda_5 K_g(0)\Big)\|u_x^2\|^2$$
$$+ \frac{\rho A}{2}\Big(\frac{\lambda_1\bar{\kappa}}{\delta_0} + \frac{\lambda_2\bar{\kappa}}{\delta_4} - \lambda_4\Big)\int_0^t \kappa(t-s)\|u_t(s)\|^2 ds$$
$$+ \rho A\Big(2\lambda_1\kappa(0)\delta_0 - \frac{\lambda_4}{2}\Big)\int_0^t |\kappa'(t-s)|\|u_t(s)\|^2 ds$$
$$+ \Big(-\lambda_3\frac{P_0}{2} + \lambda_1\frac{\kappa(0)}{2}\big(\frac{P_0}{2\delta_5}+\alpha b_0 L\big)\Big)\int_0^t |\kappa'(t-s)|\|u_x(s)\|^2 ds$$
$$+ \frac{EI}{2}\Big(\lambda_1\frac{\kappa(0)}{2\delta_4} - \lambda_5\Big)\int_0^t |\kappa'(t-s)|\|u_{xx}(s)\|^2 ds - \lambda_3\mu(t)\Psi_3(t) - \lambda_4\mu(t)\Psi_4(t)$$
$$+ \frac{EA}{2}\Big(\lambda_1\frac{\delta_6\kappa(0)}{8} - \lambda_5\Big)\int_0^t |\kappa'(t-s)|\|u_x^2(s)\|^2 ds + \lambda_1\kappa(t)\frac{EI}{4\delta_1}\|u_{xx0}\|^2$$
$$+ \lambda_1\kappa(t)\frac{EA}{16\delta_3}\|u_{x0}^2\|^2 + \lambda_1\frac{\kappa(t)}{2}\Big(\frac{P_0}{2\delta_0}+\alpha b_0 L\Big)\|u_{x0}\|^2$$
$$+ \alpha\Big(-1+\lambda_1\frac{1+\kappa(0)+\kappa(t)}{2b_0} + \frac{\lambda_2}{2b_0}\Big)u_t^2(L,t) - \lambda_5\mu(t)\Psi_5(t).$$

Choosing

$$b_0 = 2, \qquad \alpha = \frac{P_0}{2L}, \qquad \delta_0 = \frac{1}{\kappa(0)}, \qquad \delta_3 = \delta_1 = 1,$$
$$\delta_6 = \frac{2}{\kappa(0)}, \qquad \delta_4 = 2\kappa(0), \qquad \lambda_5 = \frac{\lambda_1}{4}, \qquad \lambda_3 = \kappa(0)\Big(\frac{1}{2\delta_5}+1\Big)\lambda_1.$$

Such that

$$\begin{aligned}
F'(t) \leqslant{}& \frac{\rho A}{2}\Big\{-\kappa(t)+\lambda_1\frac{\kappa(0)}{2}+2\lambda_2(1+2\kappa(0))+\lambda_4(\tilde{K}_g(0)+K_g(0))\Big\}\|u_t\|^2\\
&+\frac{\rho A}{2}(\kappa'\circ u_t)(t)+P_0\Big(\lambda_1\Big(\kappa(t)\delta_2+\delta_5+\frac{3\kappa(0)}{2}+\frac{\kappa(0)}{2}(\frac{1}{2\delta_5}+1)K_g(0)\Big)-\frac{\lambda_2}{2}\Big)\|u_x\|^2\\
&+EI\Big(\lambda_1(3\kappa(0)+\kappa(t)+\frac{K_g(0)}{8})-\lambda_2\Big)\|u_{xx}\|^2\\
&+\rho A\Big(2\lambda_1-\frac{\lambda_4}{2}\Big)\int_0^t|\kappa'(t-s)|\|u_t(s)\|^2ds\\
&+\frac{EA}{2}\Big(\lambda_1\Big(\frac{1+2\kappa(t)+\kappa(0)(2+\kappa(0))}{2}+\frac{K_g(0)}{4}\Big)-\lambda_2\Big)\|u_x^2\|^2-\lambda_3\mu(t)\Psi_3(t)\\
&+\frac{\rho A}{2}\Big(\lambda_1\bar{\kappa}\kappa(0)+\frac{\lambda_2\bar{\kappa}}{2\kappa(0)}-\lambda_4\Big)\int_0^t\kappa(t-s)\|u_t(s)\|^2ds-\lambda_4\mu(t)\Psi_4(t)-\lambda_5\mu(t)\Psi_5(t)\\
&+\kappa(t)\frac{\lambda_1}{2}\Big(\frac{EI}{2}\|u_{xx0}\|^2+\frac{EA}{8}\|u_{x0}^2\|^2+P_0\Big(\frac{\kappa(0)}{2}+1\Big)\|u_{x0}\|^2\Big)\\
&+\alpha\Big(-1+\lambda_1\frac{1+\kappa(0)+\kappa(t)}{4}+\frac{\lambda_2}{4}\Big)u_t^2(L,t).
\end{aligned}$$

We need λ_1 so small that

$$\begin{cases}
\lambda_1\frac{\kappa(0)}{2}+2\lambda_2(1+2\kappa(0))+\lambda_4(\tilde{K}_g(0)+K_g(0))<\kappa(t),\\
\lambda_1(3\kappa(0)+\kappa(t)+\frac{K_g(0)}{8})<\lambda_2,\\
\lambda_1\Big(\frac{1+2\kappa(t)+\kappa(0)(2+\kappa(0))}{2}+\frac{K_g(0)}{4}\Big)<\lambda_2,\\
\lambda_1\bar{\kappa}\kappa(0)+\frac{\lambda_2\bar{\kappa}}{2\kappa(0)}<\lambda_4,\\
\lambda_1<\frac{\lambda_4}{4},\\
\lambda_1\Big(\kappa(t)\delta_2+\delta_5+\frac{3\kappa(0)}{2}+\frac{\kappa(0)}{2}(\frac{1}{2\delta_5}+1)K_g(0)\Big)<\frac{\lambda_2}{2},\\
\lambda_1\frac{1+\kappa(0)+\kappa(t)}{4}+\frac{\lambda_2}{4}<1.
\end{cases}$$

As a result, for $t \geqslant t^*$

$$F'(t) \leqslant -C_1E(t)-\lambda_3\mu(t)\Psi_3(t)-\lambda_4\mu(t)\Psi_4(t)-\lambda_5\mu(t)\Psi_5(t)+C_2\kappa(t), \quad t\geqslant t^*,$$

where

$$C_2=\frac{\lambda_1}{2}\Big(\frac{EI}{2}\|u_{xx0}\|^2+\frac{EA}{8}\|u_{x0}^2\|^2+P_0\Big(\frac{\kappa(0)}{2}+1\Big)\|u_{x0}\|^2\Big).$$

As $\mu(t)$ is nonincreasing, we can write

$$F'(t) \leqslant -\frac{C_1}{\mu(0)}\mu(t)E(t)-\lambda_3\mu(t)\Psi_3(t)-\lambda_4\mu(t)\Psi_4(t)-\lambda_5\mu(t)\Psi_5(t)+C_2\kappa(t), \quad t\geqslant t^*.$$

Using the equivalence (2.3), we obtain

$$F'(t) \leqslant -C_3\mu(t)F(t)+C_2\kappa(t), \quad t\geqslant t^*, \tag{3.6}$$

where $C_i, i = 1, \dots, 3$ are positive constants. A simple integration of (3.6) over $[t^*, t]$ gives

$$F(t) \leqslant Ne^{-C_3 \int_{t^*}^{t} \mu(s)ds}, \quad t \geqslant t^*,$$

for some positive constant N. Then utilizing the inequality (2.3) of Proposition 2.2, we get

$$n_1(E(t) + \Psi_3(t) + \Psi_4(t) + \Psi_5(t)) \leqslant Ne^{-C_3 \int_{t^*}^{t} \mu(s)ds}, \quad t \geqslant t^*.$$

Due to the continuity of $E(t)$ over the interval $[0, t^*]$, we deduce

$$E(t) \leqslant \frac{C}{g(t)^{\sigma}}, \quad t \geqslant 0,$$

for some positive constants C and σ. □

References

1. R. P. Agarwal, S. R. Grace, *Asymptotic Stability of certain neutral differential equations*, Appl. Math. Letter **3** (2000), 9–15.
2. D. Benterki, N.-e. Tatar, *Stabilization of a nonlinear Euler–Bernoulli beam*, Arab. J. Math. **11** (2022), 479–496.
3. M. P. Fard, S. I. Sagatun, *Boundary control of a transversely vibrating beam via Lyapunov method*, Proceedings of 5th IFAC Conference on Manoeuvring and Control of Marine Craft. Aalborg, Denmark, pp. 263–268, 2000.
4. G. Chen, *Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain*, J. Math. Pure. Appl. **58** (1979), 249–273.
5. S. A. Compell, J. Bélair, T. Ocira, J. Milton, *Complex dynamics and multistability in a damped harmonic oscillator with delayed negative feedback*, Chaos **5**(4) (1995), 640–645.
6. R. Grimmer, R. Lenczewski, W. Schappacher, *Well-posedness of hyperbolic equations with delay in the boundary conditions*, in: P. Clement, S. Invernizzi, E. Mitidieri, I. Vrabie (eds.), *Semigroup Theory and Applications*, Lect. Notes Pure Appl. Math. **116**, Marcel Dekker, New York, NY, USA, 1989.
7. D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, London, 1996.
8. S. Lakshmanan, T. Senthilkumar, P. Balasubramaniam, *Improved results on robust stability of neutral systems with mixed time-varying delays and nonlinear perturbations*, Appl. Math. Model **35** (2011), 5355–5368.
9. O. Morgül, *On the boundary control of beam equation*, in: Proc. of the 15th IFAC World Congress on Automatic Control, 2002.
10. J. Sun, G. Liu, J. Chen, D. Rees, *Improved delay-range-dependent stability criteria for linear systems with time-varying delays*, Automatica **46** (2010), 466–470.
11. S. Kerral, N. E. Tatar, *Exponential stabilization of a neutrally delayed viscoelastic Timoshenko beam*, Turk. J. Math. **43** (2019), 595–611.
12. M. Liu, *Global exponential stability analysis for neutral delay-differential systems, an LMI approach*, Int. J. Syst. Sci. **37**(11) (2006), 773–783.
13. M. S. Ali, *On exponential stability of neutral differential system with nonlinear uncertainties*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012), 595–601.
14. O. Morgül, *An exponential stability result for the wave equation*, Autoniatica **38** (2002), 731–735.

ДИСИПАЦИЈА ЕНЕРГИЈЕ НЕЛИНЕАРНЕ ОЈЛЕР–БЕРНУЛИЈЕВЕ ГРЕДЕ СА НЕУТРАЛНИМ КАШЊЕЊЕМ

Резиме. У раду се разматрају слободне попречне осцилације нелинеарне Ојлер-Берноулијеве греде при кашњењу неутралног типа. Да би се сузбиле попречне осцилације штапа, пројектована је контрола граничних услова заснована на методи Љапунова. Новина овог рада је могућност добијања широког спектра брзине дисипације енергије у условима слободних осцилација.

Faculty of Mathematics and Computer Science
University Mohamed El Bachir El Ibrahimi
Bordj Bou Arredridj
Algeria
ibrahim.lakehal@univ-bba.dz

djamila.benterki@univ-bba.dz

Department of Mathematics
College of Sciences and Arts
Qassim University
Ar-Rass
Saudi Arabia

Laboratoire de Mathématiques Appliquées et de Modélisation
Université 8 Mai 1945 Guelma
Guelma
Algérie
k.zennir@qu.edu.sa

(Received 11.08.2022)
(Revised 15.01.2023)
(Available online 09.02.2023)