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TRANSMUTATION OF CENTRAL FORCES AND BERTRAND'S THEOREM

Christian Carimalo

Abstract. The transmutation of central forces, or dual law, is a transformation linking potentials in power law relative to the distance, that is, those having a positive exponent to those having a negative exponent. A well known example is that of the Newtonian and Hookean potentials, which are also strongly linked by Bertrand's famous theorem. This article shows how the use of dual law provides a better understanding of this theorem, and a new way to complete its demonstration.

1. Introduction

Looking for possible trajectories of a classical and nonrelativistic particle moving in a spherically symmetric potential produced by a point source O, one finds that stable circular orbits and closed quasi-circular orbits can only be realized *at any distance r from the source* by attractive potentials of the form $V(r) = -K/r^{\alpha}$ with $K\alpha > 0$ and $\alpha = 2 - \beta^2$, where β is a rational number, [1]. In addition, Bertrand's theorem states that among such potentials, only the Newtonian potential $-K_N/r$ ($K_N > 0, \beta_N = 1$) and the Hookean potential $K_H r^2/2$ ($K_H > 0, \beta_H = 2$) satisfy these requirements, [2]. As is well known, these closed orbits are ellipses for the two potentials, having the source as a focus in the Newtonian case and as a center in the Hookean case. This curious concordance has already been noted and described by I. Newton in his Principia, [3]. In fact, this similarity between closed orbits given by two different laws of forces can be explained by what T. Needham has called the *transmutation of central forces*, [4]. Its generalization to other power laws and mathematical extensions of the idea have been carried out by various authors in the past, [5–8], or even recently, see [9]. Its description in current formulation can also be found in [10].

In this article, we investigate how this property manifests in the proof of Bertrand's theorem.

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2. Duality formulas

Let us briefly describe the context. The trajectories of a particle moving in a spherically symmetric potential $V(r)$ are planar. Let us consider a bounded trajectory (produced by an attractive potential), lying inside a circular crown defined by $r_m \leqslant r \leqslant r_M$, the values of r_m and r_M depending on the values of the energy E and the angular momentum $L > 0$ of the particle. The orthoradial velocity of the particle having mass m is given by $v_\theta = r\dot{\theta} = L/(mr)$ and is always positive if $L \neq 0$. Consequently, the particle inside the said crown turns indefinitely around the source. When $r = r_m$ or $r = r_M$, we have turning points where the radial velocity $v_r = \dot{r}$ of the particle cancels. Then,

$$E = \frac{m\dot{r}^2}{2} + \frac{L^2}{2mr^2} + V(r) = \frac{L^2}{2mr_m^2} + V(r_m) = \frac{L^2}{2mr_M^2} + V(r_M) \tag{2.1}$$

Using the two equations

$$\dot{\theta} = \frac{L}{mr^2}, \quad \dot{r} = \pm\sqrt{\frac{2}{m}\Big(E - \frac{L^2}{2mr^2} - V(r)\Big)},$$

the function $\theta(r)$, which is the inverse of the polar equation $r(\theta)$ of the trajectory, is obtained from the integral

$$\theta(r) = \sqrt{\frac{L^2}{2m}} \int_{r_m}^{r} \frac{ds}{s^2\sqrt{\big(E - \frac{L^2}{2ms^2} - V(s)\big)}},$$

the positivity of $\dot{\theta}$ being taken into account, and setting $\theta(r_m) = 0$. Let P_m and P_M be two successive turning points. The separation angle between the two axes OP_m and OP_M is thus given by

$$\Delta\theta = \sqrt{\frac{L^2}{2m}} \int_{r_m}^{r_M} \frac{dr}{r^2\sqrt{\big(E - \frac{L^2}{2mr^2} - V(r)\big)}} \tag{2.2}$$

It can be shown that if the trajectory is closed, these two axes are axes of symmetry, thanks to the spherical symmetry of the potential. It is known that a plane curve having two crossing axes of symmetry is closed if and only if their separation angle is commensurable with π. This is exactly the matter of Bertrand's theorem to find which potentials can give to Eq. (2.2) a value commensurable with π that additionally must not depend on the values of E and $L(\neq 0)$. The first selection is obtained by imposing that, *at any distance from the source*, the potential produces: 1) stable circular orbits and 2) closed quasi-circular orbits. As said previously, the only potentials satisfying these requirements are those of the form $-K/r^\alpha$ with $K\alpha > 0$ and $\alpha = 2 - \beta^2$ in order to satisfy point 1), and where β must be a rational number in order to satisfy point 2). We then obtain

$$\Delta\theta = \frac{\pi}{\beta} \tag{2.3}$$

It is thus sufficient to deal with the following two cases

i) case (a): $V_a(r) = -K_a/r^a$ with $K_a > 0$ and $a = 2 - \beta_a^2 > 0$,

ii) case (b): $V_b(r) = K_b r^b/2$ with $K_b > 0$ and $b = \beta_b^2 - 2 > 0$,

where, at this stage, β_a and β_b need not be rational, but must verify $0 < \beta_a < \sqrt{2}$ and $\beta_b > \sqrt{2}$.

First, let us give to Eq. (2.2) a more manageable form. In case (a), using (2.1) and making the substitution $r = r_m/x$, we obtain

$$\Delta\theta = \Delta\theta_a = \int_{x_a}^{1} \frac{dx}{\sqrt{A_a(x)}},$$

where

$$A_a(x) = 1 - x^2 - \chi_a(1 - x^a), \quad \text{and} \quad \chi_a = \frac{1 - x_a^2}{1 - x_a^a} > 1, \quad x_a = \frac{r_m}{r_M} < 1 \tag{2.4}$$

The equivalent form for case (b) is

$$\Delta\theta = \Delta\psi_b = \int_{y_b}^{1} \frac{dy}{\sqrt{B_b(y)}},$$

where

$$B_b(y) = 1 - y^2 - \eta_b(y^{-b} - 1), \quad \text{and} \quad \eta_b = \frac{1 - y_b^2}{y_b^{-b} - 1}, \quad y_b = \frac{r_m}{r_M} < 1 \tag{2.5}$$

It is easy to check that $\Delta\theta_a = \pi$, independent of x_a, for $a = 1$, and $\Delta\psi_b = \pi/2$, independent of y_b, for $b = 2$. Thus the Newtonian and the Hookean potentials satisfy requirements 1) and 2).

In Eq. (2.5), let us now make the substitution $x = y^{1+b/2}$. We obtain

$$dx = (1 + b/2)y^{b/2}dy, \quad y^{2+b} = x^2, \quad y^b = x^c \quad \text{with} \quad c = \frac{2b}{2+b} < 2,$$

whence

$$\Delta\psi_b = \frac{2}{2+b}\int_{x_c}^{1} \frac{dx}{\sqrt{A_c(x)}} \quad \text{with} \quad A_c(x) = 1 - x^2 - \chi_c(1 - x^c)$$

and

$$\chi_c = (1 - x_c^2)/(1 - x_c^c), \quad x_c = y_b^{1+b/2}, \quad \text{and} \quad \chi_c = 1 + \eta_b > 1$$

Thus, we can establish a correspondence between cases (a) and (b) by setting $c = a$, which leads to

$$\Delta\psi_b = \frac{2}{2+b}\Delta\theta_a = \frac{2-a}{2}\Delta\theta_a, \quad x_a = y_b^{1+b/2} \tag{2.6}$$

This duality between the two kinds of potentials may be summarized in terms of their parameters β_a and β_b by the simple relation

$$\beta_a\beta_b = 2 \tag{2.7}$$

from which it is clear that β_a and β_b are rational or not together.

3. Transmutation of central forces, dual law

The duality observed in the preceding section can be explained in the framewok of a transformation between potentials with negative exponents and potentials with positive exponents, called *transmutation of central forces* by Needham or *dual law* by Arnol'd. We give below a brief presentation of the formalism (see [10] for more details and references).

As trajectories provided by central forces are planar, we can as well represent them in the complex plane. Define $z = x + iy = |z|e^{i\psi}$, x and y being the Cartesian coordinates of the particle moving in the xOy plane, and ψ its polar angle relative to the axis Ox. By means of complex numbers and with appropriate units, the fundamental equation of motion reads

$$\ddot{z} = F(z), \quad \text{with} \quad F(z) = -\frac{z}{|z|}\frac{dV}{d|z|}(|z|) \tag{3.1}$$

where $V(|z|)$ is supposed to be given by

$$V(|z|) = \frac{|z|^\mu}{\mu} \quad \text{with} \quad \mu > 0, \quad \text{hence} \quad F(z) = -z|z|^{\mu-2}$$

To simplify, we have taken $m = 1$. The energy E takes the form

$$E = \frac{|\dot{z}|^2}{2} + V(|z|) = \text{constant} > 0$$

Consider the transformation

$$z \to Z = |Z|e^{i\theta} = z^\sigma, \quad t \to \tau \quad \text{with} \quad \frac{d\tau}{dt} = |z|^\gamma \tag{3.2}$$

where σ and γ are such that the Kepler's law of areas is unchanged, i.e with $\theta = \sigma\psi$,

$$|Z|^2\frac{d\theta}{d\tau} = \sigma|z|^{2\sigma-\gamma-2}|z|^2\frac{d\psi}{dt} = \text{constant}$$

Clearly, this is satisfied if and only if

$$\sigma = 1 + \frac{\gamma}{2}$$

The new fundamental equation is then

$$\begin{aligned}\frac{d^2Z}{d\tau^2} &= \frac{d}{d\tau}\Big(\frac{dz^\sigma}{d\tau}\Big) = \frac{1}{|z|^\gamma}\frac{d}{dt}\Big(\frac{1}{|z|^\gamma}\frac{dz^\sigma}{dt}\Big) = \frac{\sigma}{|z|^\gamma}\frac{d}{dt}\Big(\frac{z^{\gamma/2}\dot{z}}{|z|^\gamma}\Big) = \frac{\sigma}{|z|^\gamma}\frac{d}{dt}\Big(\frac{\dot{z}}{z^{\star\gamma/2}}\Big) \\ &= \frac{\sigma}{|z|^\gamma}\Big[\frac{-z|z|^{\mu-2}}{z^{\star\gamma/2}} - \frac{\gamma}{2}\frac{|\dot{z}|^2}{z^{\star 1+\gamma/2}}\Big] = -\frac{\sigma\gamma}{|z|^\gamma z^{\star 1+\gamma/2}}\Big[\frac{|z|^\mu}{\gamma} + \frac{|\dot{z}|^2}{2}\Big].\end{aligned} \tag{3.3}$$

In Eq. (3.3), the expression in square brackets is the energy E at the condition that γ is taken equal to μ. In that case we obtain

$$\frac{d^2Z}{d\tau^2} = -\frac{\sigma\gamma E z^\sigma}{|z|^{2\sigma+\gamma}} = -\sigma\gamma E Z|Z|^{-2-\gamma/\sigma}$$

that is, an equation similar to (3.1) with a force deriving from the *attractive* potential $V'(|Z|) = K|Z|^\nu/\nu$ where

$$\nu = -\gamma/\sigma = -\frac{2\mu}{2+\mu} < 0, \quad K = \sigma\gamma E > 0$$

Thus, the transformation (3.2) subjected to the above conditions allows us to convert the potential r^b with $b = \gamma = \mu > 0$ into the potential $-K_a/r^a$ with $a = -\nu = \frac{2b}{b+2} \leqslant 2$, $K_a = K/a > 0$. In this operation, any variation of the angle $\theta = \theta_a$ is correlated to that of the angle $\psi = \psi_b$ through the simple relation

$$\Delta\theta_a = \sigma\Delta\psi_b = \frac{b+2}{2}\Delta\psi_b = \frac{2}{2-a}\Delta\psi_b$$

in accordance with Eq. (2.6), which is thus simply due to the above described duality.

It is worth mentioning here that the transmutation (3.2) can also be viewed as a generalized canonical transformation, as shown in [11]. In this respect, let us note that the energy equation Eq. (2.1) can be rewritten in terms of the polar equations $x(\theta) = r_m/r(\theta)$ or $y(\psi) = r_m/r(\psi)$ in the form

$$\left(\frac{dx}{d\theta}\right)^2 = A_a(x) \quad \text{for case (a)}, \qquad \left(\frac{dy}{d\psi}\right)^2 = B_b(y) \quad \text{for case (b)} \tag{3.4}$$

As seen above, these expressions are linked by the transformation

$$x = y^{1+\frac{b}{2}}, \quad \theta = \left(1+\frac{b}{2}\right)\psi, \quad a = \frac{2b}{b+2}$$

4. Bertrand's theorem

Bertrand's theorem is achieved by requiring that for closed orbits the separation angle $\Delta\theta$ between symmetry axes must be an intrinsic property of the potential, independent of the parameters characterizing these orbits, namely, the energy E and the angular momentum L. Then, its value can be found giving to E some extreme value allowing an easy computation of the integrals in Eqs. (2.4) and (2.5). The way to do this is suggested by Eq. (2.1): since we require closed orbits at any distance from the source, we should consider the limit $r_M \to +\infty$, which amounts to $E \to 0$ for case (a) and $E \to +\infty$ for case (b). This is the final step of the original demonstration by Bertrand, proposed as an exercise by Arnol'd in [12]. Thanks to the dual law, it is sufficient to consider the limit $y_b = r_m/r_M \to 0$ in Eq. (2.5), which immediately leads to $B_b(y) \to 1 - y^2$ and

$$\lim_{y_b \to 0} \Delta\psi_b = \int_0^1 \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2} \tag{4.1}$$

for any $b > 0$. The conclusion of Bertrand's theorem is then obtained by combining this result with two important relations: first, Eq. (2.3), which leads to $\beta_b = 2$ (Hooke's potential), and subsequently the duality relation Eq. (2.7), which leads to

$\beta_a = 1$ (Newton's potential). If we consider instead the limit $x_a \to 0$ in Eq. (2.4), we have $\chi_a \to 1$, $A_a(x) \to x^a - x^2$ and

$$\lim_{x_a \to 0} \Delta\theta_a = \int_0^1 \frac{dx}{\sqrt{x^a - x^2}} = \frac{\pi}{2-a} = \frac{\pi}{\beta_a^2} \tag{4.2}$$

for any $a \in]0, 2[$. Then, $\beta_a = 1$ from Eq. (2.3) and $\beta_b = 2$ from Eq. (2.7). However, there is now a subtle difference. Indeed, taking the extreme limits E infinite or $E = 0$ without precaution can lead us into the domain of unbounded trajectories. Hopefully, there is no problem considering E infinite in the Hookean case, because all corresponding trajectories are closed (ellipses). But in the Newtonian case, it is known that taking $E = 0$ gives parabolas which are not closed orbits. Thus, having found a value $\Delta\theta$ for unbounded trajectories, we would have to prove additionally that it is the same for closed orbits, and that it is finally well-founded to identify this result with Eq. (2.3). This may be considered as a drawback of the method and one has to be cautious in interpreting Eq. (4.2).

In this regard, let us mention that S. A. Chin in [13] and later J. Galbraith and J. Williams in [14] claimed to have found a new "elementary" proof of Bertrand's theorem, by directly searching the solutions of Eqs. (3.4) when x_a or y_b equals zero. This method is viewed as an attempt to avoid handling integrals, but is in fact not very different from the previous one. However, it really has the above-mentioned drawback. Considering case (a) with $x_a \to 0$, it consists in writing the corresponding Eq. (3.4) in the simplified form

$$\left(\frac{dx}{d\theta}\right)^2 = A_a(x) \simeq x^a - x^2$$

which is in fact a differential version of Eq. (4.2). Then, using the substitution $u = x^{1-a/2}$ that is currently used to simplify the integral in the latter equation, we obtain

$$\left(\frac{du}{d\phi}\right)^2 = 1 - u^2, \quad \text{with} \quad \phi = \Omega\theta, \quad \Omega = 1 - \frac{a}{2}$$

The general solution of this new equation is $u = \mu\cos\phi + \nu\sin\phi$. Here, the constants μ and ν are fixed using the constraints $u(0) = 1$, $du/d\phi(0) = 0$. We find $u = \cos\phi$ and finally

$$r = \frac{r_m}{(\cos\phi)^{2/(2-a)}}$$

with the additional condition $\cos\phi \geqslant 0$. This is not the polar equation of a closed orbit and the angle gap $\Delta\theta = \pi/(2-a)$ is that between the position at $r = r_m$ and a position at infinity. It remains to show that it is also the value of the separation angle between the axes of symmetry of the closed orbits that the potential is supposed to provide.

This is indeed the case for the Newtonan potential and it is probably the main merit of this discussion to highlight this fact. It is the consequence of the symmetry generated by the celebrated Laplace-Runge-Lenz vector, [15–17], initially discovered for closed orbits (ellipses) but that also extends to unbounded trajectories (parabolas, hyperbolas).

Independently of all this, having a rigorous proof of (4.1) for closed orbits is advisable anyway. By means of suitable bounds of the integral in Eq. (2.5), it was done in [18] by V. Jovanović, who already pointed out the lack of mathematical details of previous demonstrations. However, even there, no mention is made about Eq (4.2), its subtlety and how it is related to Eq. (4.1). Moreover, in our opinion, all previous proofs do not sufficiently highlight the exceptionality of the Newtonian and Hookean potentials, nor how the limit in Eq. (4.1) is reached for any other exponent b. That is why we propose in the next section a new demonstration of this equation, based on a particular property of $B_b(y)$. The choice to deal with $B_b(y)$ instead of $A_a(x)$ comes from the difficulty of finding an upper bound for $\Delta\theta_a$, due to the limitation $a < 2$..

5. Another demonstration of Eq. (4.1)

As shown in the short Appendix by elementary methods, at fixed y and y_b, $B_b(y)$ is a strictly increasing function of the exponent b. It is thus bounded by its limits $B_0(y)$ and $B_\infty(y)$ for $b \to 0$ and for $b \to +\infty$:

$$B_0(y) < B_b(y) < B_\infty(y)$$

The upper limit corresponds to an infinite potential and the lower one to a potential written as a constant plus a logarithmic component that cannot produce closed orbits, apart circles ($(r/r_0)^b \simeq 1 + b\ln(r/r_0)$ for $b \ll 1$). Thus, these limits are related to unphysical situations that should be discarded. However, we are justified to consider them from a pure mathematical point of view. They lead to the following bounding of $\Delta\psi_b$:

$$\Delta\psi_\infty < \Delta\psi_b < \Delta\psi_0$$

with

$$\Delta\psi_0 = \int_{y_b}^{1} \frac{dy}{\sqrt{B_0(y)}}, \quad \Delta\psi_\infty = \int_{y_b}^{1} \frac{dy}{\sqrt{B_\infty(y)}} \tag{5.1}$$

It is easy to show that

$$B_\infty(y) = 1 - y^2 \quad \text{for} \quad y_b < y \leqslant 1, \quad B_\infty(y_b) = 0$$

and

$$B_0(y) = 1 - y^2 - (1 - y_b^2)\frac{\ln y}{\ln y_b}$$

Thus,

$$\Delta\psi_\infty = \int_{y_b}^{1} \frac{dy}{\sqrt{1 - y^2}} = \frac{\pi}{2}\Big[1 - \frac{2}{\pi}\sin^{-1} y_b\Big] \tag{5.2}$$

Following the same reasoning used in the Appendix, we get

$$-\frac{\ln y}{1-y} \leqslant -\frac{\ln y_b}{1-y_b} \quad \text{or} \quad \frac{\ln y}{\ln y_b} \leqslant \frac{1-y}{1-y_b},$$

and

$$B_0(y) \geqslant (1-y)(y-y_b),$$

whence

$$\Delta\psi_0 \leqslant \int_{y_b}^{1} \frac{dy}{\sqrt{(1-y)(y-y_b)}} = \pi$$

This rather natural bound ensures that all of the above integrals are convergent whatever the value of $y_b \in [0,1]$. Let us now investigate what happens to the integrand of $\Delta\psi_0$ in the vicinity of the lower bound y_b when the latter goes to zero. Taking $y = y_b(1+\epsilon)$ with $\epsilon \ll 1$, we get

$$B_0 \simeq \epsilon\Big[-2y_b^2 - \frac{1-y_b^2}{\ln y_b}\Big] \simeq \frac{y-y_b}{y_b|\ln y_b|}$$

Thus, for $y_b < \nu \ll 1$,

$$\int_{y_b}^{\nu} \frac{dy}{\sqrt{B_0}} \simeq \sqrt{y_b|\ln y_b|}\int_{y_b}^{\nu} \frac{dy}{\sqrt{y-y_b}} = 2\sqrt{y_b|\ln y_b|}\sqrt{\nu - y_b} \to 0$$

From this result we conclude that we can safely take $y_b = 0$ in $\Delta\psi_0$, getting $\lim_{y_b\to 0}\Delta\psi_0 = \pi/2$. Moreover, from Eq. (5.2) we have $\lim_{y_b\to 0}\Delta\psi_\infty = \pi/2$. Thus, as the upper and lower bounds of $\Delta\psi_b$ take the same value $\pi/2$ in the limit $y_b \to 0$, we can finally assert that Eq. (4.1) is verified for any $b \geqslant 0$.

Let us add a few comments. Clearly, $\Delta\psi_\infty$ and $\Delta\psi_0$ do depend on y_b: as y_b tends to zero, they both move to $\pi/2$ which is a maximum value for the former and a minimum value for the second. The bounding equation Eq. (5.1) shows that almost all $\Delta\psi_b$ also depend on y_b, as they move to the same limit $\pi/2$ as $y_b \to 0$. The only exception is for $b = 2$, for which $\Delta\psi_2 = \pi/2$, whatever the value of y_b. For potential with positive exponents, this latter case is thus the only one satisfying the condition of Bertrand's theorem, i.e. giving $\Delta\psi_b = \pi/\beta_b$ with β_b being a rational number independent of y_b (here $\beta_b = 2$). The case of potentials with negative exponents is subsequently solved applying Eqs. (2.6) and (2.7), which leads to $\beta_a = 1$ $(a = 1)$ and $\Delta\theta_1 = \pi$.

For other potentials, including non-homogeneous ones, finding rational values of $\Delta\psi/\pi$ is not excluded. But Bertrand's result implies that they can be obtained only with particular closed orbits. For homogeneous potentials, Eq. (2.3) may not even be valid for such orbits. Moreover, if for some b, a closed orbit exists with rational $\Delta\psi_b/\pi$, then from Eq. (2.6), there exists a corresponding trajectory given by the dual potential, which is also closed only if b and thus a are rational. Finally, note that $\Delta\psi_b(y_b)$, with given b, depends on y_b only. Hence, the fact that $\Delta\psi_b/\pi$ is rational or not generally concerns a continuous set of (bounded) trajectories having the same y_b, generated by varying E and L between their allowed limits.

6. Conclusion

In this article, several points regarding Bertrand's theorem have been clarified. We have shown how the amazing association found in this theorem between the Hookean and Newtonian potentials is fully understood in the framework of the transmutation of forces, also called dual law. The final step of the original demonstration of Bertrand's theorem has also been discussed. We have drawn attention to the fact that the usual method considering extreme values of energy can lead to non-closed trajectories, which are out of the scope of Bertrand's theorem, and thus, can lead to a misinterpretation of the result. We have proposed another proof of Eq. 4.1, and then Eq. (4.2), based on a simple property of $B_b(y)$ as a function of b, and the dual law, with the aim to highlight both the exceptionality property of the Hookean and Newtonian potentials to provide a value of $\Delta\psi$ or $\Delta\theta$ that is commensurable with π and independent of the orbit, and the way these values are only reached as limits for other homogeneous potentials. We have also emphasized the fact that the value π found in the Newtonian case applies as well to unbounded trajectories, which is due to the corresponding underlying dynamical symmetry.

We hope that these clarifications will be useful for a better understanding of Bertrand's theorem.

Appendix. $B_b(y)$ as a function of b

We have

$$\frac{\partial B_b}{\partial b} = \eta_b \frac{y^{-b} - 1}{b} [g(u_b) - g(u)],$$

where $u = y^b$, $u_b = y_b^b$, and

$$g(u) = -\frac{\ln u}{1 - u}$$

The function $g(u)$ is such that $g(0) = +\infty$, $g(1) = 1$ and

$$g'(u) = \frac{h(v)}{(1-u)^2}, \quad \text{where} \quad h(v) = 1 - v + \ln v \quad \text{with} \quad v = 1/u \geqslant 1$$

The function $h(v)$ is such that $h(1) = 0$, $h(+\infty) = -\infty$ and

$$h'(v) = -1 + 1/v \leqslant 0$$

Hence, $h(v)$ being a stricly decreasing function remains negative for $v > 1$ and correspondingly $g'(u) < 0$. Then, $g(u)$ is also a stricly decreasing function and remains less than $g(u_b)$ for $u_b < u \leqslant 1$. Consequently, at fixed y and y_b, $B_b(x)$ is a strictly increasing function of $b \in [0, +\infty[$.

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ТРАНСМУТАЦИЈА ЦЕНТРАЛНИХ СИЛА И БЕРТРАНДОВА ТЕОРЕМА

Резиме. Трансмутација централних сила, или дуални закон, је трансформација која повезује потенцијале различитог степена у односу на растојање, то јест, оне који имају позитиван експонент са онима који имају негативан експонент. Добро је познат пример Њутновог и Хуковог потенцијала, који су такође повезани Бертрандовом чувеном теоремом. Овај рад показује како употреба дуалног закона омогућава боље разумевање ове важне теореме.

Sorbonne Université
Campus Pierre et Marie Curie
Paris
France
christian.carimalo@sorbonne-universite.fr

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