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INVERSE DYNAMICS IN RIGID BODY MECHANICS

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Abstract. Inverse Dynamics is used to calculate the forces and moments in the joints of multibody systems investigated in fields such as Biomechanics or Robotics. In a didactic spirit, this paper begins with an overview of the derivations of the kinematical and dynamical equations of rigid bodies from the point of view of modern Continuum Mechanics. Then, it introduces a matrix formulation for the solution of Inverse Dynamics problems and, finally, reports a simple two-dimensional example of application to a problem in Biomechanics.

1. Introduction

The dynamics of a rigid body is entirely determined by the translational and rotational equations of motion,

$$\dot{\boldsymbol{P}} = \boldsymbol{F}, \tag{1.1a}$$

$$\dot{\boldsymbol{L}}_G = \boldsymbol{M}_G, \tag{1.1b}$$

where $\boldsymbol{P}$ is the total linear momentum, $\boldsymbol{F}$ is the resultant of the external forces, $\boldsymbol{L}_G$ is the total angular momentum with respect to the centre of mass x_G and $\boldsymbol{M}_G$ is the resultant of the external moments with respect to the centre of mass x_G. Equations (1.1) constitute a system of 6 non-linear ordinary differential equations in 6 unknowns. The unknowns are the 6 generalised coordinates given, usually, by the 3 coordinates of the centre of mass x_G and 3 angular coordinates defining the orientation of the rigid body in space. For this reason, we say that a rigid body has 6 degrees of freedom. The resultant external generalised forces $\boldsymbol{F}$ and $\boldsymbol{M}_G$ (3 components of force plus 3 components of moment, for a total of 6 generalised forces) are assumed to be known. The equations of motion (1.1) can be integrated once appropriate initial conditions are given. This is the traditional approach to Dynamics, which we can call *Direct Dynamics*.

We speak of *Inverse Dynamics* [4] when the full kinematics of a rigid body is known experimentally (e.g., by means of a motion capture system, such as those used customarily in Biomechanics [24]), and thus the trajectory of the system in the configuration space is entirely known, but some of the generalised external

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force components are unknown. The number of unknown generalised forces must be equal to the number of degrees of freedom of the system, because this is also the number of available dynamical equations. In this way, we are guaranteed that the problem is solvable (equal number of equations and unknowns). Since the kinematical quantities featuring in the derivatives $\dot{\boldsymbol{P}}$ and $\dot{\boldsymbol{L}}_G$ are known, the equations of motion (1.1) become *algebraic equations* in the unknown generalised forces. So, from a computational point of view, Inverse Dynamics is enormously simpler than Direct Dynamics, in which we deal with a system of non-linear ordinary differential equations.

For a single rigid body in space (which has 6 degrees of freedom), the typical scenario of application of Inverse Dynamics is that in which the entire kinematics is known, the external force vector and external moment vector applied at one point ($3+3=6$ generalised forces) are unknown, and the remaining forces applied at other points are known. This is the situation attained in each segment of a multibody system with joints such as a robotic arm or a limb in the human body. The unknowns in these systems are the generalised forces at each joint. These generalised forces are *internal* for the system as a whole, but are *external* for each segment, when considered individually.

These multibody systems with joints are solved by means of Inverse Dynamics in an iterative manner. Let us use the anatomical terms *distal* and *proximal*, for the joints of a segment that are farthest and closest, respectively, to a suitable reference part of the system (e.g., the sagittal plane for the case of the human body, or the fixed "shoulder" joint of an industrial robotic arm). The distal end of the most distal segment may or may not be subjected to external forces, which would be measurable, but the six generalised forces at the proximal end, which is a joint, are unknown. One thus solves for the proximal joint forces of the most distal segment. For the subsequent segment, the distal forces are simply equal and opposite to the proximal forces calculated for the previous segment. Then, the procedure is repeated again to solve for the subsequent joint, and so on.

As an illustrative example, which we shall develop in a simplified two-dimensional setting in Section 6, imagine that we want to determine the forces and moments acting on the foot at the ankle joint. The foot is subjected to its own weight, to the reaction force of the ground, which can be measured by means of a force plate [**24**], and to the resultant force and moment at the ankle joint. If the kinematics is measured by means of a camera-based motion analysis system [**1**], we are in the case depicted above: a single body, in which the force vector and moment vector at one point (the ankle) are unknown, but for which all other forces and the whole kinematics are known.

In a didactic spirit, this paper aims at elucidating the method of Inverse Dynamics, devoting particular attention to a matrix formalism that facilitates numerical implementation. The general setting is that of modern Continuum Mechanics [**21**, **25**]. From a methodological point of view, and particularly for the case of kinematics, this approach may in some instances differ with that typically adopted in rigid body mechanics [**5**, **17**, **18**, **20**, **22**]. In Section 2, we briefly introduce the kinematics of a continuum body, and we show how general continuum kinematics is specialised to the case of a rigid body in Section 3. In Section 4, we describe rigid body dynamics. In Section 5, we present the method of Inverse Dynamics and the associated matrix formalism. In Section 6, we report a simple two-dimensional example of application to a problem in Biomechanics. Section 7 summarises the work.

2. Continuum kinematics

We describe a rigid body as a *particular case* of continuum body, and thus employ the customary setting of modern Continuum Mechanics [**21**, **25**]. The notation is that of some previous works [**2**, **14**].

The physical space is represented by the affine space $\mathcal{S} = \mathbb{E}^3$, obtained by employing $\mathbb{R}^3$ *both* as the point space and as the modelling vector space [**10**, **15**]. This means that, given two points p and x, their *difference* $p-x$ is a vector attached at the point x. The space of all vectors attached at x that are tangent to curves passing by x is called the *tangent space* $T_x\mathcal{S}$. The disjoint union of all tangent spaces in $\mathcal{S}$ is the *tangent bundle* $T\mathcal{S}$.

A body is identified with a convenient, but arbitrary, reference configuration $\mathcal{B}$, a subset of the physical space $\mathcal{S}$. The tangent space $T_X\mathcal{B}$ at a point X and the tangent bundle $T\mathcal{B}$ are defined analogously as for the case of $\mathcal{S}$.

In the treatment of kinematics, we shall follow the standard convention of denoting *material* points and quantities in the body $\mathcal{B}$ with uppercase symbols and *spatial* points and quantities in the physical space $\mathcal{S}$ with lowercase symbols, with some exceptions that we shall point out.

The physical space $\mathcal{S}$ and the body $\mathcal{B}$ are assumed to be endowed with the spatial metric tensor $\boldsymbol{g}$ and the material metric tensor $\boldsymbol{G}$, respectively. The metric tensors $\boldsymbol{g}$ and $\boldsymbol{G}$ define the spatial scalar product $\boldsymbol{y} \,.\, \boldsymbol{z} \equiv \langle \boldsymbol{y}, \boldsymbol{z} \rangle_{\boldsymbol{g}} = y^i\, g_{ij}\, z^j$ and the material scalar product $\boldsymbol{Y} \,.\, \boldsymbol{Z} \equiv \langle \boldsymbol{Y}, \boldsymbol{Z} \rangle_{\boldsymbol{G}} = Y^I\, G_{IJ}\, Z^J$, respectively. Also, they induce the Euclidean norms $\|\boldsymbol{y}\| = \sqrt{\langle \boldsymbol{y}, \boldsymbol{y} \rangle_{\boldsymbol{g}}} = \sqrt{\boldsymbol{y} \,.\, \boldsymbol{y}}$ and $\|\boldsymbol{Y}\| = \sqrt{\langle \boldsymbol{Y}, \boldsymbol{Y} \rangle_{\boldsymbol{G}}} = \sqrt{\boldsymbol{Y} \,.\, \boldsymbol{Y}}$. If orthogonal Cartesian coordinates are used in $\mathcal{S}$ and $\mathcal{B}$, then the bases in the tangent bundles $T\mathcal{S}$ and $T\mathcal{B}$ are orthonormal. In this case, the metric tensors are both represented by identity matrices, the distinction between vectors and covectors (and thus between contravariant and covariant indices) fades out and the scalar products reduce to $\boldsymbol{y} \,.\, \boldsymbol{z} \equiv \langle \boldsymbol{y}, \boldsymbol{z} \rangle_{\boldsymbol{g}} = y_i\, z_i$ and $\boldsymbol{Y} \,.\, \boldsymbol{Z} \equiv \langle \boldsymbol{Y}, \boldsymbol{Z} \rangle_{\boldsymbol{G}} = Y_I\, Z_I$. We shall assume Cartesian coordinates and orthonormal bases throughout and write all indices as subscripts.

Motion of a continuum body. A time-dependent point map

$$\phi \colon \mathcal{B} \times \mathcal{I} \to \mathcal{S} \colon (X, t) \mapsto x = \phi(X, t)$$

is called a *motion* of the continuum body $\mathcal{B}$ if it is twice differentiable in the time interval $\mathcal{I} \subset \mathbb{R}$ and, for every time $t \in \mathcal{I}$, the map

$$\phi(\,\cdot\,, t) \colon \mathcal{B} \to \mathcal{S} \colon X \mapsto x = \phi(X, t), \tag{2.1}$$

called the *configuration map* of the body $\mathcal{B}$ at time t, is an *embedding*, i.e., its codomain-restriction to the *current configuration* $\phi(\mathcal{B}, t)$ is a *diffeomorphism*. For

$\phi(\,\cdot\,,t)$ to be a diffeomorphism, it must be continuous, differentiable and with a continuous and differentiable inverse

$$\Phi(\,\cdot\,,t)\colon \phi(\mathcal{B},t) \to \mathcal{B}\colon x \mapsto X = \Phi(x,t).$$

At a given material point X, the curve

$$\phi(X,\,\cdot\,)\colon \mathcal{I} \to \mathcal{S}\colon t \mapsto x = \phi(X,t), \tag{2.2}$$

is the *trajectory* of the material point X in the physical space $\mathcal{S}$.

Deformation gradient and polar decomposition. The differentiability of the motion ϕ implies that its *directional derivative* at X in the direction of the material vector $\boldsymbol{M} \in T_X\mathcal{B}$ is *linear* in $\boldsymbol{M}$, i.e.,

$$(\partial_{\boldsymbol{M}}\phi)(X,t) = \lim_{h\to 0} \frac{\phi(X + h\,\boldsymbol{M},t) - \phi(X,t)}{h} = [(T\phi)(X,t)]\,\boldsymbol{M} \in T_x\mathcal{S},$$

where $x = \phi(X,t)$. The linear map

$$\boldsymbol{F}(X,t) \equiv (T\phi)(X,t)\colon T_X\mathcal{B} \to T_x\mathcal{S} \tag{2.3}$$

with components

$$F_{iK}(X,t) = \phi_{i,K}(X,t) \equiv \frac{\partial\phi_i}{\partial X_K}(X),$$

is the *tangent map* (or, simply, the *derivative*) of the point map $\phi(\,\cdot\,,t)$ and is traditionally called *deformation gradient*.[1] The vector

$$\boldsymbol{m} = \boldsymbol{F}(X,t)\,\boldsymbol{M} \in T_x\mathcal{S},$$

obtained by applying $\boldsymbol{F}(X,t)$ on the material vector $\boldsymbol{M}$, is called the *push-forward* of $\boldsymbol{M} \in T_X\mathcal{B}$. The determinant $J = \det \boldsymbol{F}$ is guaranteed to be strictly positive, and thus non-singular, by the fact that ϕ is an embedding and thus invertible. Since it maps between two *different* tangent spaces, the deformation gradient is said to be a *two-point* tensor. The deformation gradient is the primary measure of deformation, from which all other ones descend.

Cauchy's *Polar Decomposition Theorem* (for a concise and elegant proof, see [8]) ensures that, as any non-singular tensor, the deformation gradient admits the *right* and *left* multiplicative decompositions

$$\boldsymbol{F}(X,t) = \boldsymbol{R}(X,t)\,\boldsymbol{U}(X,t) \qquad \boldsymbol{F}(X,t) = \boldsymbol{V}(x,t)\,\boldsymbol{R}(X,t), \tag{2.4a}$$

$$F_{iK}(X,t) = R_{iJ}(X,t)\,U_{JK}(X,t) \qquad F_{iK}(X,t) = V_{ij}(x,t)\,R_{jK}(X,t). \tag{2.4b}$$

In the polar decompositions (2.4), $\boldsymbol{R}(X,t)$ is a two-point proper orthogonal tensor (i.e., with $\det \boldsymbol{R} = 1$, since $\det \boldsymbol{F} > 0$), i.e., $\boldsymbol{R}(X,t) \in \mathrm{Orth}^+(T_X\mathcal{B}, T_x\mathcal{S})$, called *rotation tensor*, $\boldsymbol{U}(X,t)\colon T_X\mathcal{B} \to T_X\mathcal{B}$ is a completely material symmetric and positive definite tensor, called *right stretch tensor*, and $\boldsymbol{V}(x,t)\colon T_x\mathcal{S} \to T_x\mathcal{S}$

[1] The deformation gradient is in fact *not* a gradient, in so far as ϕ is *not* a tensor field (specifically, not a vector field) but a point map. Thus, differentiation of ϕ *does not* need the introduction of a covariant derivative, and its components are simple partial derivatives and *do not* contain terms in the Christoffel symbols. This is explained in, e.g., [10, 21].

is a completely spatial[2] symmetric and positive definite tensor, called *left stretch tensor*. Thus, the theorem states that the deformation can be decomposed as a material deformation followed by a two-point rotation (right decomposition) or a two-point rotation followed by a spatial deformation (left decomposition).

We recall that the orthogonality of $\boldsymbol{R}$ means that, for any two material vectors $\boldsymbol{Y}$ and $\boldsymbol{Z}$, their scalar product is preserved, i.e., if $\boldsymbol{y} = \boldsymbol{R}\boldsymbol{Y}$ and $\boldsymbol{z} = \boldsymbol{R}\boldsymbol{Z}$, then

$$\begin{aligned} \boldsymbol{Y} \,.\, \boldsymbol{Z} = \langle \boldsymbol{Y}, \boldsymbol{Z} \rangle_{\boldsymbol{G}} = \langle \boldsymbol{R}\boldsymbol{Y}, \boldsymbol{R}\boldsymbol{Z} \rangle_{\boldsymbol{g}} &= (\boldsymbol{R}\boldsymbol{Y}) \,.\, (\boldsymbol{R}\boldsymbol{Z}) \\ &= \langle \boldsymbol{y}, \boldsymbol{z} \rangle_{\boldsymbol{g}} = \boldsymbol{y} \,.\, \boldsymbol{z}. \end{aligned} \tag{2.5}$$

Since it preserves the scalar product induced by the material and spatial metric tensors $\boldsymbol{G}$ and $\boldsymbol{g}$, an orthogonal tensor such as $\boldsymbol{R}$ is said to be a two-point *isometry* [13, 21], i.e., a metric-preserving two-point tensor. The definition (2.5) of isometry implies that its transpose $\boldsymbol{R}^T$ and inverse $\boldsymbol{R}^{-1}$ coincide:[3]

$$\boldsymbol{R}^{-1} \equiv \boldsymbol{R}^T. \tag{2.6}$$

Lagrangian and Eulerian velocity. With reference to Figure 1, let us consider the trajectory $\phi(X, \cdot\,)$ of the material point X, defined in Equation (2.2). By definition, the velocity of the material point X is given by the time derivative

$$\dot{\phi}(X,t) \equiv \partial_t \phi(X,t) = \lim_{h\to 0} \frac{\phi(X, t+h) - \phi(X,t)}{h}. \tag{2.7}$$

Indeed, as shown in Figure 1, the numerator of the incremental ratio in Equation (2.7) is the *displacement* $\phi(X, t+h) - \phi(X,t)$, which is a vector attached at the current placement $x = \phi(X,t)$ of the material point X. Moreover, when the time increment h tends to zero, the limit of the ratio of the displacement $\phi(X, t+h) - \phi(X,t)$ to h tends to be tangent to the trajectory of X at the current placement $x = \phi(X,t)$ of X, and is precisely the velocity.

The velocity defined in Equation (2.7) is the *Lagrangian velocity* of the material point X, and we must emphasise that it is a *spatial* vector, as it is attached at $x = \phi(X,t)$ (see Figure 1)[4]. The *Eulerian velocity* is defined as the spatial vector field such that

$$\boldsymbol{v}(x,t) \equiv \dot{\phi}(\Phi(x,t),t) = \dot{\phi}(X,t),$$

i.e., it obtained by expressing the Lagrangian velocity $\dot{\phi}$ as a function of the spatial point x.

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[2] The left stretch tensor $\boldsymbol{V}$ is the first exception to the uppercase/lowercase convention of Continuum Mechanics: it is virtually universally adopted, to avoid confusion with the velocity $\boldsymbol{v}$.

[3] Strictly speaking, here we are referring to what we call the *metric transpose* [12, 13] and denote $\boldsymbol{R}^t$ and to what Marsden and Hughes [21] call *dual* and denote $\boldsymbol{R}^*$. If $\boldsymbol{R}(X,t)$ maps the tangent spaces, i.e., $\boldsymbol{R}(X,t)\colon T_X\mathcal{B} \to T_x\mathcal{S}$, and has components $R^a{}_B$, the *algebraic transpose* [12, 13] $\boldsymbol{R}^T(x,t)$ maps the cotangent spaces (*dual spaces* of the tangent spaces), i.e., $\boldsymbol{R}^T(x,t)\colon T^\star_x\mathcal{S} \to T^\star_X\mathcal{B}$ and has components $(\boldsymbol{R}^T)_B{}^a$. In contrast, the metric transpose maps, like the inverse, the tangent spaces, i.e., $\boldsymbol{R}^t(x,t)\colon T_x\mathcal{S} \to T_X\mathcal{B}$, and has components $(\boldsymbol{R}^t)^A{}_b = G^{AB}\,(\boldsymbol{R}^T)_B{}^a\,g_{ab}$. With Cartesian coordinates and orthonormal bases, the distinction between the two types of transpose fades out.

[4] A detailed discussion on the distinction that we make between spatial and Eulerian on the one hand and of material and Lagrangian on the other hand can be found in [14].

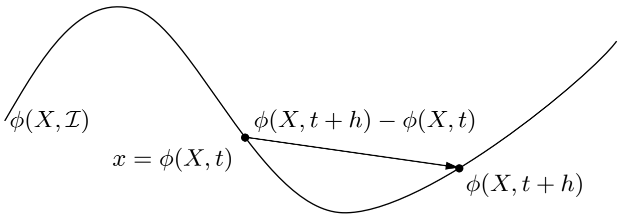

FIGURE 1. The trajectory $\phi(X,\mathcal{I})$ described by the motion of a material point X during the time interval $\mathcal{I}$; the point is placed at $x=\phi(X,t)$ at time t, and at $\phi(X,t+h)$ at time $t+h$. The vector $\phi(X,t+h)-\phi(X,t)$ is the displacement after the increment of time h. The Lagrangian velocity is given by the limit of the incremental ratio $[\phi(X,t+h)-\phi(X,t)]/h$ for $h\to 0$, and is, by definition of derivative, tangent to the trajectory at x.

Velocity gradient and sym-skew decomposition. The gradient of the velocity,[5]

$$\boldsymbol{l} = \operatorname{grad}\boldsymbol{v}, \qquad l_{ij} = v_{i,j},$$

is, unsurprisingly, called *velocity gradient* and it can be decomposed into its symmetric and skew-symmetric parts, as

$$\boldsymbol{l} = \boldsymbol{d} + \boldsymbol{w},$$

where the symmetric part is the *rate of deformation tensor* [7] or *stretching rate tensor* [25]

$$\boldsymbol{d} = \operatorname{sym}(\boldsymbol{l}) = \tfrac{1}{2}\,(\boldsymbol{l}+\boldsymbol{l}^T), \qquad d_{ij} = \tfrac{1}{2}\,(v_{i,j}+v_{j,i}),$$

and the skew-symmetric part is the is the *spin tensor* [7, 25]

$$\boldsymbol{w} = \operatorname{skew}(\boldsymbol{l}) = \tfrac{1}{2}\,(\boldsymbol{l}-\boldsymbol{l}^T), \qquad w_{ij} = \tfrac{1}{2}\,(v_{i,j}-v_{j,i}).$$

3. Rigid body kinematics

Rigid bodies do not actually exist in nature: any material body will deform if subjected to external forces, changes of temperature, etc. The *rigid body approximation* is a convenient tool to describe all those systems in which the deformations are negligible with respect to the overall motion and for the specific problem at study. For instance, in Biomechanics, the bones are customarily regarded as rigid when studying the overall motion of a subject [24], but must be regarded as deformable when studying, e.g., their remodelling [9]. When the deformations can be neglected, the motion of a body can be approximated as being *rigid*, i.e., distances and angles are *preserved* throughout the motion. Such a motion is the combination of a translation and a rotation, i.e., a *Euclidean motion*: its deformation gradient

[5] Rigorously, in the component expression, we should use a *covariant derivative* and write, distinguishing contravariant and covariant indices, $l^a{}_b = v^a{}_{|b}$. Since we assume Cartesian coordinates throughout, the Christoffel symbols vanish identically and the covariant derivative $v^a{}_{|b}$ reduces to the simple partial derivative $v^a{}_{,b}$.

$\boldsymbol{F}$ reduces to the rotation tensor $\boldsymbol{R}$, which is an *isometry*, i.e., is orthogonal. Since isometries preserve the metric, they preserve lengths (Euclidean norms) and angles. Conventionally, a body subjected to the *kinematical constraint* of rigid motion is called a *rigid body*.

Rigid body motion. There are two equivalent ways to introduce the kinematical constraint of rigid motion. The first is to impose that the distance between two arbitrary points be constant throughout the motion. The second is to impose that the symmetric part of the velocity gradient of the motion be identically zero and then integrate to obtain the motion. We choose to follow the former method and then briefly discuss the equivalence of the latter.

With reference to Figure 2, we arbitrarily choose a point $X_P \in \mathcal{B}$ and describe its motion by the curve

$$x_P \equiv \phi(X_P, \cdot\,)\colon \mathcal{I}\colon t \mapsto x_P(t) \equiv \phi(X_P, t). \tag{3.1}$$

If X is another material point in $\mathcal{B}$, the rigidity constraint imposes that the distance between X_P and X, i.e., the norm $\|\boldsymbol{\Xi}(X)\| = \|X - X_P\|$ of the vector

$$\boldsymbol{\Xi}(X) = X - X_P,$$

called *material local position vector* of X, be preserved throughout the motion. Thus, at any arbitrary time t, the distance between $x_P(t) = \phi(X_P, t)$ and $x = \phi(X, t)$, i.e., the norm $\|\boldsymbol{\xi}(x, t)\| = \|x - x_P(t)\|$ of the vector

$$\boldsymbol{\xi}(x, t) = x - x_P(t),$$

called *spatial local position vector* of $x = \phi(X, t)$, must equal the norm $\|\boldsymbol{\Xi}(X)\| = \|X - X_P\|$ of $\boldsymbol{\Xi}(X) = X - X_P$, i.e.,

$$\|\boldsymbol{\xi}(x, t)\|^2 = \|x - x_P(t)\|^2 = \|X - X_P\|^2 = \|\boldsymbol{\Xi}(X)\|^2,$$

or, equivalently,

$$[x - x_P(t)]\,.\,[x - x_P(t)] = [X - X_P]\,.\,[X - X_P].$$

By definition of isometry, this can only happen if, and only if, there exists a curve of two-point isometries,

$$\boldsymbol{R}\colon \mathcal{I} \to \mathrm{Orth}^+(T\mathcal{B}, T\mathcal{S})\colon t \mapsto \boldsymbol{R}(t) \in \mathrm{Orth}^+(T_X\mathcal{B}, T_x\mathcal{S}),$$

which are *independent* of the material point X and such that, for every $X \in \mathcal{B}$, $\boldsymbol{\xi}(x, t)$ is obtained by rotating $\boldsymbol{\Xi}(X)$ through $\boldsymbol{R}(t)$, i.e.,

$$\boldsymbol{\xi}(x, t) = \boldsymbol{R}(t)\,\boldsymbol{\Xi}(X), \tag{3.2}$$

or, explicitly,

$$x - x_P(t) = \boldsymbol{R}(t)\,[X - X_P].$$

By solving for x and recalling the general definition (2.1) of motion, we obtain the definition of rigid body motion as

$$x = \phi(X, t) = x_P(t) + \boldsymbol{R}(t)\,[X - X_P] = x_P(t) + \boldsymbol{R}(t)\,\boldsymbol{\Xi}(X). \tag{3.3}$$

Using the property (2.6) of isometries, we can invert (3.3) into

$$X = \Phi(x, t) = X_P + \boldsymbol{R}^T(t)\,[x - x_P(t)] = X_P + \boldsymbol{R}^T(t)\,\boldsymbol{\xi}(x, t). \tag{3.4}$$

The rigid motion (3.3) is said to be *Euclidean*, i.e., it is the superposition of the translation described by the curve x_P and the rotation described by the rotation $\boldsymbol{R}$. Therefore, the configuration space of the rigid body $\mathcal{B}$ is

$$\mathcal{E} = \mathcal{S} \times \mathrm{Orth}^+(T\mathcal{B}, T\mathcal{S}).$$

Rigid body deformation gradient. Since the motion is linear in the coordinates of the material point X, it is straightforward to see that the deformation gradient of the rigid motion (3.3) is *independent* of X and coincident with the rotation $\boldsymbol{R}(t)$, for every t, i.e.,

$$\boldsymbol{F}(X,t) \equiv (T\phi)(X,t) = \boldsymbol{R}(t).$$

Consequently, the right and left stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ reduce to the material and spatial identity tensors, respectively. The fact that $\boldsymbol{F}(X,t) = \boldsymbol{R}(t)$ also implies that that (3.2) is in fact a *push-forward*.

Rigid body velocity and velocity gradient. The Lagrangian velocity of the rigid motion (3.3) is

$$\begin{aligned} \dot{\phi}(X,t) &= \boldsymbol{v}_P(t) + \dot{\boldsymbol{R}}(t)\,[X - X_P] \\ &= \boldsymbol{v}_P(t) + \dot{\boldsymbol{R}}(t)\,\boldsymbol{\Xi}(X), \end{aligned} \tag{3.5}$$

where $\boldsymbol{v}_P = \dot{x}_p$ is the velocity of the arbitrarily chosen point X_P with current position x_P. The Eulerian velocity is obtained by substituting the inverse rigid motion (3.4) into the Lagrangian velocity (3.5):

$$\begin{aligned} \boldsymbol{v}(x,t) = \dot{\phi}(\Phi(x,t),t) &= \boldsymbol{v}_P(t) + \dot{\boldsymbol{R}}(t)\,\boldsymbol{R}^T(t)\,[x - x_P(t)] \\ &= \boldsymbol{v}_P(t) + \dot{\boldsymbol{R}}(t)\,\boldsymbol{R}^T(t)\,\boldsymbol{\xi}(x,t). \end{aligned}$$

The tensor $\dot{\boldsymbol{R}}\boldsymbol{R}^T$ is computed by taking the derivative of the (spatial) identity tensor[6] $\boldsymbol{I} = \dot{\boldsymbol{R}}\boldsymbol{R}^T$ Since the identity tensor $\boldsymbol{I}$ does *not* depend on time, its derivative is the zero tensor $\mathbf{0}$, i.e.,

$$\mathbf{0} = (\boldsymbol{R}\boldsymbol{R}^T)^{\cdot} = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T = \dot{\boldsymbol{R}}\boldsymbol{R}^T + (\dot{\boldsymbol{R}}\boldsymbol{R}^T)^T,$$

and thus

$$\dot{\boldsymbol{R}}\boldsymbol{R}^T = -(\dot{\boldsymbol{R}}\boldsymbol{R}^T)^T.$$

Since it equals the negative of its transpose, the tensor $\dot{\boldsymbol{R}}\boldsymbol{R}^T$ is a *skew-symmetric* tensor, called *rigid spin tensor* [6] or *angular velocity tensor*:[7]

$$\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T. \tag{3.6}$$

Therefore the Eulerian velocity field for a rigid body takes the final form

$$\boldsymbol{v}(x,t) = \boldsymbol{v}_P(t) + \boldsymbol{\Omega}(t)\,\boldsymbol{\xi}(x,t). \tag{3.7}$$

[6] Here is the second exception to the uppercase/lowercase convention of Continuum Mechanics. Since we do not need the material identity, we employ the symbol $\boldsymbol{I}$ for the spatial identity, in place of the more notationally consistent $\boldsymbol{i}$.

[7] Another deviation from the uppercase/lowercase convention: we use an uppercase symbol for the spatial tensor $\boldsymbol{\Omega}$ because we reserve the lowercase $\boldsymbol{\omega}$ to the angular velocity, which we shall introduce shortly.

Note that, since $\boldsymbol{v}_P(t)$ does not depend on x and the local position vector $\boldsymbol{\xi}(x,t) = x - x_G(t)$ is a linear function of x, the velocity gradient $\boldsymbol{l} = \operatorname{grad}\boldsymbol{v}$ equals the rigid spin tensor $\boldsymbol{\Omega}$. From this, and from the skew symmetry of the rigid spin tensor $\boldsymbol{\Omega}$, follows that the symmetric part of $\boldsymbol{l}$, the rate of deformation $\boldsymbol{d} = \frac{1}{2}(\boldsymbol{l}+\boldsymbol{l}^T)$, vanishes identically, and that the spin tensor $\boldsymbol{w}$ is identically equal to the rigid spin tensor $\boldsymbol{\Omega}$.[8] Therefore, as anticipated, the rigid body motion (3.3) could be obtained by integrating the differential equation $\boldsymbol{l}(x,t) = (\operatorname{grad}\boldsymbol{v})(x,t) \equiv \boldsymbol{\Omega}(t)$ first with respect to x, to obtain the velocity (3.7), and then, after a change of variables from x to X, with respect to t.

It is customary to express the Eulerian velocity field (3.7) for a rigid body in terms of the *axial vector* [7, 19] of the rigid spin tensor $\boldsymbol{\Omega}$, i.e., the vector $\boldsymbol{\omega}$ such that, for every vector $\boldsymbol{u}$,

$$\boldsymbol{\omega} \times \boldsymbol{u} = \boldsymbol{\Omega}\,\boldsymbol{u}. \tag{3.8}$$

The vector $\boldsymbol{\omega}$ is called *angular velocity* and its components are related to those of the rigid spin tensor $\boldsymbol{\Omega}$ via

$$\Omega_{ik} = \epsilon_{ijk}\,\omega_j, \qquad [\Omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{3.9}$$

where we made use of the Ricci/Levi-Civita symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for} \quad i,j,k \quad \text{even permutation of } 1,2,3, \\ -1 & \text{for} \quad i,j,k \quad \text{odd permutation of } 1,2,3, \\ 0 & \text{for} \quad i,j,k \quad \text{permutation with repetition of } 1,2,3. \end{cases}$$

Substitution into Equation (3.7) yields the customary form of the Eulerian velocity field of the rigid body, as

$$\boldsymbol{v}(x,t) = \boldsymbol{v}_P(t) + \boldsymbol{\omega}(t) \times \boldsymbol{\xi}(x,t). \tag{3.10}$$

The Eulerian rigid body velocity field, in either form (3.7) or form (3.10), is the fundamental kinematical relation for rigid bodies. We remark, however, that the form (3.7) featuring the rigid spin tensor $\boldsymbol{\Omega}$ is the most fundamental: the angular velocity $\boldsymbol{\omega}$ is only used to yield the more customary form (3.10), but all rigid body kinematics can be studied using (3.7) alone.

Material, fixed and moving reference frames. To establish our reference frames, we refer again to Figure 2.

We set a system of orthogonal Cartesian coordinates $\{X_K\}_{K=1}^3$ in the body $\mathcal{B}$, with origin at the point X_P, which will thus have null coordinates. The orthonormal basis associated with this system of coordinates is denoted $\{\boldsymbol{E}_K\}_{K=1}^3$. These are called *material frame* and *material basis*.

Similarly, in the physical space $\mathcal{S}$, we define a system of orthogonal Cartesian coordinates $\{x_i\}_{i=1}^3$ with origin in x_O and associated orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^3$,

[8] For a general motion, the spin tensor $\boldsymbol{w}$ and the rigid spin tensor $\boldsymbol{\Omega}$ *both* depend on the point x and *do not* coincide: the spin tensor $\boldsymbol{w}$ can be written in terms of $\boldsymbol{\Omega}$ and the right stretch tensor $\boldsymbol{U}$ [7, 25].

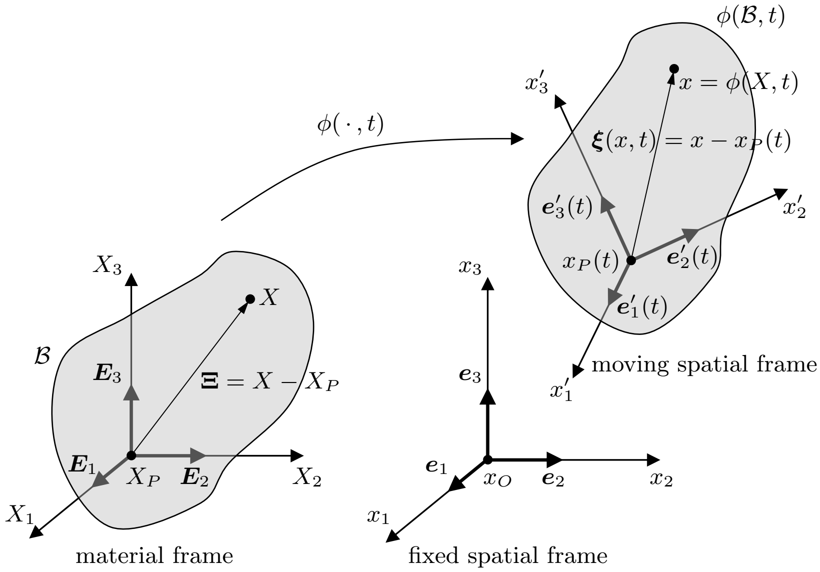

FIGURE 2. A rigid body $\mathcal{B}$, whose points are labelled by the material positions X in the material frame with origin X_P and coordinates X_1, X_2, X_3, mapped by the motion ϕ into its current configuration $\phi(\mathcal{B}, t) \subset \mathcal{S}$ at time t. The current positions $x = \phi(X, t)$ are in the fixed spatial frame with origin x_O and coordinates x_1, x_2, x_3. The moving frame with origin $x_P(t) = \phi(X_P, t)$ and coordinates x'_1, x'_2, x'_3 is attached to the body and is obtained as the push-forward of the material frame.

and we call these the *fixed spatial frame* and *fixed spatial basis*. The fixed spatial basis $\{\boldsymbol{e}_i\}_{i=1}^3$ is related to the material basis $\{\boldsymbol{E}_K\}_{K=1}^3$ by the *shifter* [11, 21], the two-point, time-independent orthogonal tensor $\mathbf{1} \in \mathrm{Orth}^+(T\mathcal{B}, T\mathcal{S})$ such that

$$\boldsymbol{e}_1 = \mathbf{1}\,\boldsymbol{E}_1, \quad \boldsymbol{e}_2 = \mathbf{1}\,\boldsymbol{E}_2, \quad \boldsymbol{e}_3 = \mathbf{1}\,\boldsymbol{E}_3. \tag{3.11}$$

By *convecting* the material coordinates $\{X_K\}_{K=1}^3$ through the motion ϕ, we obtain the *convected spatial coordinates*

$$x'_1 = X_1 \circ \phi, \quad x'_2 = X_2 \circ \phi, \quad x'_3 = X_3 \circ \phi.$$

Together with the origin $x_P(t) = \phi(X_P, t)$ defined in Equation 3.1, these coordinates constitute the *moving spatial frame*, which is rigidly attached to the body throughout its motion. The *moving spatial basis* $\{\boldsymbol{e}'_k(t)\}_{k=1}^3$ is obtained by pushing the material basis $\{\boldsymbol{E}_K\}_{K=1}^3$ forward through the motion ϕ, whose deformation gradient is the rotation $\boldsymbol{R}$, i.e.,

$$\boldsymbol{e}'_1(t) = \boldsymbol{R}(t)\,\boldsymbol{E}_1, \quad \boldsymbol{e}'_2(t) = \boldsymbol{R}(t)\,\boldsymbol{E}_2, \quad \boldsymbol{e}'_3(t) = \boldsymbol{R}(t)\,\boldsymbol{E}_3. \tag{3.12}$$

Since $\{\boldsymbol{e}'_k(t)\}_{k=1}^3$ is a spatial basis, it can also be obtained by a transformation of the fixed spatial basis $\{\boldsymbol{e}_i\}_{i=1}^3$. By inverting the relations in (3.11) and substituting them into the relations in (3.12), we can eliminate the material basis vectors and thus directly relate the fixed and moving bases via

$$\boldsymbol{e}'_k(t) = \boldsymbol{Q}(t)\,\boldsymbol{e}_k, \tag{3.13}$$

where the time-dependent, fully spatial tensor $\boldsymbol{Q}(t) \in \mathrm{Orth}^+(T\mathcal{S})$ is defined as[9]

$$\boldsymbol{Q}(t) = \boldsymbol{R}(t)\,\mathbf{1}^T, \quad Q_{ik}(t) = R_{iK}(t)\mathbf{1}_{kK}. \tag{3.14}$$

If the material coordinates $\{X_K\}_{K=1}^3$ and the fixed spatial coordinates $\{x_i\}_{i=1}^3$ are *collinear*, then the corresponding bases $\{\boldsymbol{E}_K\}_{K=1}^3$ and $\{\boldsymbol{e}_i\}_{i=1}^3$ are parallel, and the shifter $\mathbf{1}$ has components $\mathbf{1}_{iK} = \delta_{iK}$, i.e., its representing matrix is an identity. Under this hypothesis, the spatial rotation tensor $\boldsymbol{Q}(t)$ and the two-point rotation tensor $\boldsymbol{R}(t)$ have the *same* representing matrix. Indeed, we have $Q_{ik}(t) = R_{iK}(t)\,\delta_{kK}$.

Poisson's theorem. If we take the time derivative of the change of basis (3.13) from the fixed basis $\{\boldsymbol{e}_i\}_{i=1}^3$ to the moving basis $\{\boldsymbol{e}'_k(t)\}_{k=1}^3$, i.e.,

$$\dot{\boldsymbol{e}}'_k(t) = \dot{\boldsymbol{Q}}(t)\,\boldsymbol{e}_k,$$

and we eliminate the fixed basis vectors $\boldsymbol{e}_k$ by using the inverse $\boldsymbol{e}_k = \boldsymbol{Q}^T(t)\,\boldsymbol{e}'_k(t)$ of the change of basis (3.13), we obtain

$$\dot{\boldsymbol{e}}'_k(t) = \dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}^T(t)\,\boldsymbol{e}'_k(t).$$

This derivative contains again the rigid spin tensor of Equation (3.6). Indeed, by using the relations in (3.14), we have, by the orthogonality of the shifter $\mathbf{1}$,

$$\dot{\boldsymbol{Q}}(t)\,\boldsymbol{Q}^T(t) = \dot{\boldsymbol{R}}(t)\,\mathbf{1}^T\,\mathbf{1}\,\boldsymbol{R}^T(t) = \dot{\boldsymbol{R}}(t)\,\boldsymbol{R}^T(t) = \boldsymbol{\Omega}(t).$$

Therefore, we obtain, using also the definition (3.8) of angular velocity,

$$\dot{\boldsymbol{e}}'_k(t) = \boldsymbol{\Omega}(t)\,\boldsymbol{e}'_k(t) \equiv \boldsymbol{\omega}(t) \times \boldsymbol{e}'_k(t). \tag{3.15}$$

Equation (3.15) constitutes the enunciate of *Poisson's Theorem*. The same result could be obtained by taking the time derivative of the relation (3.12) between material and spatial moving bases, directly in terms of the two-point rotation tensor $\boldsymbol{R}$. However, the classical proof is in terms of the fully spatial rotation tensor $\boldsymbol{Q}$.

Poisson's Theorem allows for calculating the time derivatives of vectors and tensors that move rigidly with the body. Indeed, if $\boldsymbol{u}(t)$ is one such vector, it can be expressed in the fixed and moving bases as

$$\boldsymbol{u}(t) = u_i(t)\,\boldsymbol{e}_i = u'_k\,\boldsymbol{e}'_k(t),$$

where the components u'_k in the moving basis do *not* depend on time, since $\boldsymbol{u}(t)$ and the basis vectors $\boldsymbol{e}'_k(t)$ move rigidly together. Poisson's Theorem (3.15) implies

$$\dot{\boldsymbol{u}}(t) = u'_k\,\dot{\boldsymbol{e}}'_k(t) = u'_k\,\boldsymbol{\Omega}(t)\,\boldsymbol{e}'_k(t) = \boldsymbol{\Omega}(t)\,\boldsymbol{u}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{u}(t). \tag{3.16}$$

[9] In the customary convention of Continuum Mechanics, tensor $\boldsymbol{Q}$ should be denoted by the lowercase symbol $\boldsymbol{q}$.

This is the case of the local position vector $\boldsymbol{\xi}$, Equation (3.16), which reads

$$\dot{\boldsymbol{\xi}}(x,t) = \boldsymbol{\Omega}(t)\,\boldsymbol{\xi}(x,t) = \boldsymbol{\omega}(t) \times \boldsymbol{\xi}(x,t),$$

as we have seen in the expressions (3.7) and (3.10) of the velocity field. For the case of a second-order tensor $\boldsymbol{A}(t)$ such that[10]

$$\boldsymbol{A}(t) = A_{ij}(t)\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j = A'_{kl}\,\boldsymbol{e}'_k(t) \otimes \boldsymbol{e}'_l(t),$$

where, again, the components A'_{kl} in the moving basis do *not* depend on time, use of Poisson's Theorem (3.15) in its form in terms of the rigid spin tensor $\boldsymbol{\Omega}$ yields (we omit the argument t for the sake of a lighter notation)

$$\begin{aligned}
\dot{\boldsymbol{A}} &= A'_{kl}\left[(\boldsymbol{\Omega}\,\boldsymbol{e}'_k) \otimes \boldsymbol{e}'_l + \boldsymbol{e}'_k \otimes (\boldsymbol{\Omega}\,\boldsymbol{e}'_l)\right] \\
&= A'_{kl}\left[(\boldsymbol{\Omega}\,\boldsymbol{e}'_k) \otimes \boldsymbol{e}'_l + \boldsymbol{e}'_k \otimes (\boldsymbol{e}'_l\,\boldsymbol{\Omega}^T)\right] \\
&= A'_{kl}\left[(\boldsymbol{\Omega}\,\boldsymbol{e}'_k) \otimes \boldsymbol{e}'_l - \boldsymbol{e}'_k \otimes (\boldsymbol{e}'_l\,\boldsymbol{\Omega})\right] \\
&= A'_{kl}\left[\boldsymbol{\Omega}\,(\boldsymbol{e}'_k \otimes \boldsymbol{e}'_l) - (\boldsymbol{e}'_k \otimes \boldsymbol{e}'_l)\,\boldsymbol{\Omega}\right] \\
&= \boldsymbol{\Omega}\,\boldsymbol{A} - \boldsymbol{A}\,\boldsymbol{\Omega},
\end{aligned}$$

where we used the definition of transpose and the definition of skew-symmetric tensor. Finally, we have

$$\dot{\boldsymbol{A}}(t) = \boldsymbol{\Omega}(t)\,\boldsymbol{A}(t) - \boldsymbol{A}(t)\,\boldsymbol{\Omega}(t). \tag{3.17}$$

4. Rigid body dynamics

In order to derive the explicit expressions of the equations of motion (1.1), we need to first introduce the total mass m of the body, the centre of mass x_G and then derive the expressions of the total linear momentum $\boldsymbol{P}$ and the total angular momentum $\boldsymbol{L}_G$ with respect to the centre of mass x_G. From this point on, *all* quantities are spatial and, following the customary notation in rigid body mechanics, we shall use uppercase symbols for the total quantities $\boldsymbol{P}$ and $\boldsymbol{L}_G$ as well as for the external forces and moments.

Mass density. The *mass density* is the strictly positive scalar field

$$\varrho(\,\cdot\,,t)\colon \phi(\mathcal{B},t) \to \mathbb{R}^+ \colon x \mapsto \varrho(x,t)$$

such that its integral over the current configuration $\phi(\mathcal{B},t)$ of the body is the total mass

$$m = \int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\mathrm{d}v, \tag{4.1}$$

which is a *constant* during the motion, i.e., $\dot{m} = 0$.

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[10] For this spatial tensor, we have the last deviation from the uppercase/lowercase convention in this section.

Centre of mass. The *centre of mass* x_G is defined as the point with local position vector

$$m\,\boldsymbol{\xi}_G(t) = m\,[x_G(t) - x_P(t)] = \int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\boldsymbol{\xi}(x,t)\,\mathrm{d}v. \tag{4.2}$$

If the origin x_P of the moving frame coincides with the centre of mass x_G, then the local position vector of the centre of mass vanishes identically:

$$m\,\boldsymbol{\xi}_G(t) = m\,[x_G(t) - x_G(t)] = \int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\boldsymbol{\xi}(x,t)\,\mathrm{d}v \equiv \mathbf{0}. \tag{4.3}$$

With this choice, the kinematical equations (3.7) and (3.10) become

$$\boldsymbol{v}(x,t) = \boldsymbol{v}_G(t) + \boldsymbol{\Omega}(t)\,\boldsymbol{\xi}(x,t),$$

and

$$\boldsymbol{v}(x,t) = \boldsymbol{v}_G(t) + \boldsymbol{\omega}(t) \times \boldsymbol{\xi}(x,t). \tag{4.4}$$

Linear momentum. The total linear momentum $\boldsymbol{P}$ is given by the integral of the linear momentum density $\boldsymbol{p} = \varrho\,\boldsymbol{v}$, i.e.,

$$\boldsymbol{P}(t) = \int_{\phi(\mathcal{B},t)} \boldsymbol{p}(x,t)\,\mathrm{d}v = \int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\boldsymbol{v}(x,t)\,\mathrm{d}v.$$

Using the kinematical equation (4.4) and the linearity of the integral operator, and bringing out of the integral sign the quantities $\boldsymbol{v}_G$ and $\boldsymbol{\omega}$, which do not depend on the point x, we have

$$\begin{aligned}\boldsymbol{P}(t) &= \int_{\phi(\mathcal{B},t)} \varrho(x,t)\big(\boldsymbol{v}_G(t) + \boldsymbol{\omega}(t) \times \boldsymbol{\xi}(x,t)\big)\,\mathrm{d}v \\ &= \left[\int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\mathrm{d}v\right]\boldsymbol{v}_P(t) + \boldsymbol{\omega}(t) \times \int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\boldsymbol{\xi}(x,t)\,\mathrm{d}v.\end{aligned}$$

The first integral is the mass m (Equation (4.1)) and the second integral is the mass times the local position vector (4.2) of the centre of mass x_G, which vanishes identically (Equation (4.3)) since the origin of the moving frame is x_G itself. Therefore, the total linear momentum $\boldsymbol{P}$ reduces to the remarkably simple expression

$$\boldsymbol{P}(t) = m\,\boldsymbol{v}_G(t), \tag{4.5}$$

expressing that, as far as linear motion is concerned, a rigid body behaves like a single particle with mass equal to the total mass m and velocity equal to the velocity $\boldsymbol{v}_G$ of the centre of mass.

Angular momentum. The total angular momentum $\boldsymbol{L}_G$ with respect to the centre of mass x_G is the integral of the angular momentum density $\boldsymbol{\ell}_G(x,t) = \boldsymbol{\xi}(x,t) \times \big[\varrho(x,t)\,\boldsymbol{v}(x,t)\big]$, i.e.,

$$\boldsymbol{L}_O(t) = \int_{\phi(\mathcal{B},t)} \boldsymbol{\ell}_G(x,t)\,\mathrm{d}v = \int_{\phi(\mathcal{B},t)} \boldsymbol{\xi}(x,t) \times \big[\varrho(x,t)\,\boldsymbol{v}(x,t)\big]\,\mathrm{d}v.$$

Using again the kinematical equation (4.4) and the linearity of the integral operator, we get

$$\boldsymbol{L}_G(t) = \int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\boldsymbol{\xi}(x,t) \times \boldsymbol{v}_G(t)\,\mathrm{d}v + \int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\boldsymbol{\xi}(x,t) \times \big[\boldsymbol{\omega}(t) \times \boldsymbol{\xi}(x,t)\big]\,\mathrm{d}v. \tag{4.6}$$

In the second integral, we use Lagrange's identity $\boldsymbol{q} \times (\boldsymbol{r} \times \boldsymbol{s}) = (\boldsymbol{q} \,.\, \boldsymbol{s})\boldsymbol{r} - (\boldsymbol{q} \,.\, \boldsymbol{r})\boldsymbol{s}$ and write the double cross product as

$$\boldsymbol{\xi}(x,t) \times (\boldsymbol{\omega}(t) \times \boldsymbol{\xi}(x,t)) = \big[\boldsymbol{\xi}(x,t) \,.\, \boldsymbol{\xi}(x,t)\big]\,\boldsymbol{\omega}(t) - \big[\boldsymbol{\xi}(x,t) \,.\, \boldsymbol{\omega}(t)\big]\,\boldsymbol{\xi}(x,t),$$

which, by using the definitions of identity tensor, i.e., $\boldsymbol{\omega}(t) = \boldsymbol{I}\,\boldsymbol{\omega}(t)$, and of tensor product, i.e., $\big[\boldsymbol{\xi}(x,t) \,.\, \boldsymbol{\omega}(t)\big]\,\boldsymbol{\xi}(x,t) = \big[\boldsymbol{\xi}(x,t) \otimes \boldsymbol{\xi}(x,t)\big]\boldsymbol{\omega}(t)$, becomes

$$\begin{aligned} \boldsymbol{\xi}(x,t) \times (\boldsymbol{\omega}(t) \times \boldsymbol{\xi}(x,t)) &= \big[\boldsymbol{\xi}(x,t) \,.\, \boldsymbol{\xi}(x,t)\big]\,\boldsymbol{I}\,\boldsymbol{\omega}(t) - \big[\boldsymbol{\xi}(x,t) \otimes \boldsymbol{\xi}(x,t)\big]\boldsymbol{\omega}(t) \\ &= \big[\big(\boldsymbol{\xi}(x,t) \,.\, \boldsymbol{\xi}(x,t)\big)\,\boldsymbol{I} - \boldsymbol{\xi}(x,t) \otimes \boldsymbol{\xi}(x,t)\big]\,\boldsymbol{\omega}(t). \end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.6) and, again, bringing $\boldsymbol{v}_G$ and $\boldsymbol{\omega}$ out of the integral sign since they do not depend on x, we obtain

$$\begin{aligned} \boldsymbol{L}_G(t) &= \left[\int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\boldsymbol{\xi}(x,t)\,\mathrm{d}v\right] \times \boldsymbol{v}_G(t) \\ &\quad + \left[\int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\big[\big(\boldsymbol{\xi}(x,t) \,.\, \boldsymbol{\xi}(x,t)\big)\,\boldsymbol{I} - \boldsymbol{\xi}(x,t) \otimes \boldsymbol{\xi}(x,t)\big]\,\mathrm{d}v\right]\boldsymbol{\omega}(t). \end{aligned} \tag{4.8}$$

The first integral is $m\,\boldsymbol{\xi}_G$ and vanishes identically since the origin of the moving frame is in the centre of mass x_G (Equation (4.3)). The second integral is the *tensor of inertia* with respect to the centre of mass x_G [20],

$$\boldsymbol{J}_G(t) = \int_{\phi(\mathcal{B},t)} \varrho(x,t)\,\big[\big(\boldsymbol{\xi}(x,t) \,.\, \boldsymbol{\xi}(x,t)\big)\,\boldsymbol{I} - \boldsymbol{\xi}(x,t) \otimes \boldsymbol{\xi}(x,t)\big]\,\mathrm{d}v. \tag{4.9}$$

Using the expression (4.9) of the tensor of inertia, the angular momentum $\boldsymbol{L}_G$ with respect to the centre of mass x_G takes the final form

$$\boldsymbol{L}_G = \boldsymbol{J}_G\,\boldsymbol{\omega}. \tag{4.10}$$

Tensor of inertia. Note that, in contrast with the mass m, the tensor of inertia $\boldsymbol{J}_G$ *does* depend on time. This can be seen by looking at the representations of $\boldsymbol{J}_G$ in the fixed basis $\{\boldsymbol{e}_i\}_{i=1}^3$, which is time-independent, and in the moving basis $\{\boldsymbol{e}'_k(t)\}_{k=1}^3$, which is time-dependent:

$$\boldsymbol{J}_G(t) = J_{Gij}(t)\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j = J'_{Gkl}\,\boldsymbol{e}'_k(t) \otimes \boldsymbol{e}'_l(t).$$

The moving components J'_{Gkl} are *intrinsic* and time-independent and thus are preferred to the time-dependent fixed components J_{Gij}. We shall always assume that the intrinsic and time-independent components J'_{Gkl} in the moving basis are known from the outset, and that the time-dependent components J_{Gij} in the fixed basis are obtained by means of the transformation

$$J_{Gij}(t) = Q_{ik}(t)\,J'_{Gkl}\,Q_{jl}(t), \quad [J_G](t) = [Q](t)\,[J_G]'\,[Q]^T(t).$$

Kinetic energy. The most elegant way to introduce the tensor of inertia is in fact via the definition of *kinetic energy*, as done, e.g., by Landau and Lifshitz [20]. The kinetic energy $\mathcal{K}$ is the integral of the kinetic energy density $k = \frac{1}{2}\,\varrho\,\boldsymbol{v}\,.\,\boldsymbol{v}$, i.e.,

$$\mathcal{K}(t) = \int_{\phi(\mathcal{B},t)} k(x,t)\,\mathrm{d}v = \int_{\phi(\mathcal{B},t)} \frac{1}{2}\,\varrho(x,t)\,\boldsymbol{v}(x,t)\,.\,\boldsymbol{v}(x,t)\,\mathrm{d}v.$$

Substitution of the rigid body velocity field (3.10) and passages similar to those that brought the angular momentum $\boldsymbol{L}_G$ from the form (4.6) to the form (4.8) yield

$$\mathcal{K}(t) = \tfrac{1}{2}\,m\,\boldsymbol{v}_G(t)\,.\,\boldsymbol{v}_G(t) + \tfrac{1}{2}\,\boldsymbol{\omega}(t)\,.\,[\boldsymbol{J}_G(t)\,\boldsymbol{\omega}(t)],$$

which shows that the kinetic energy of a rigid body $\mathcal{B}$ is given by the sum of two terms. The first is the kinetic energy associated with the translational motion and equals that of a single particle whose mass is equal to that of $\mathcal{B}$ and whose motion is that of the centre of mass x_G. The second is an *intrinsic* kinetic energy term, associated with the rotational motion. In the second term, we note how the tensor of inertia $\boldsymbol{J}_G$ behaves as a *metric tensor*, by virtue of its symmetry and positive definiteness. Finally, we note that, by inserting an identity tensor in the first term, we can write the kinetic energy as the sum of two *quadratic forms*, i.e.,

$$\mathcal{K}(t) = \tfrac{1}{2}\,\boldsymbol{v}_G(t)\,.\,[m\,\boldsymbol{I}\,\boldsymbol{v}_G(t)] + \tfrac{1}{2}\,\boldsymbol{\omega}(t)\,.\,[\boldsymbol{J}_G(t)\,\boldsymbol{\omega}(t)]$$

where the spherical tensor $m\,\boldsymbol{I}$, called *mass tensor*, is to be considered a metric tensor as well, being symmetric and positive definite.

Equations of Motion. The derivatives of the linear momentum (4.5) and angular momentum (4.10) are given by

$$\dot{\boldsymbol{P}} = m\,\boldsymbol{a}_G, \tag{4.11a}$$

$$\dot{\boldsymbol{L}}_G = \boldsymbol{J}_G\,\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega}, \tag{4.11b}$$

where $\boldsymbol{a}_G = \dot{\boldsymbol{v}}_G$ is the acceleration of the centre of mass x_G and $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ is the angular acceleration. The derivative (4.11a) of the linear momentum is straightforward, but the derivative (4.11b) of the angular momentum deserves some attention. The term $\boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega}$ arises from the derivative of the tensor of inertia $\boldsymbol{J}_G$, which is evaluated using the application (3.17) of Poisson's theorem to second-order tensors. Using this result, along with the definition (3.8) of angular velocity $\boldsymbol{\omega}$ in terms of the rigid spin tensor $\boldsymbol{\Omega}$, we have

$$\begin{aligned}\dot{\boldsymbol{L}}_G &= \boldsymbol{J}_G\,\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{J}}_G\,\boldsymbol{\omega} = \boldsymbol{J}_G\,\boldsymbol{\alpha} + [\boldsymbol{\Omega}\,\boldsymbol{J}_G - \boldsymbol{J}_G\,\boldsymbol{\Omega}]\,\boldsymbol{\omega}\\ &= \boldsymbol{J}_G\,\boldsymbol{\alpha} + \boldsymbol{\Omega}\,\boldsymbol{J}_G\,\boldsymbol{\omega} - \boldsymbol{J}_G\,\boldsymbol{\Omega}\,\boldsymbol{\omega} = \boldsymbol{J}_G\,\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega} - \boldsymbol{J}_G\,[\boldsymbol{\omega} \times \boldsymbol{\omega}],\end{aligned}$$

which, for the vanishing of the cross product $\boldsymbol{\omega} \times \boldsymbol{\omega}$, yields (4.11b).
Using the derivatives (4.11), the equations of motion (1.1) take the explicit form

$$m\,\boldsymbol{a}_G = \boldsymbol{F}, \tag{4.12a}$$

$$\boldsymbol{J}_G\,\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega} = \boldsymbol{M}_G. \tag{4.12b}$$

In the rotational equation (4.12b), it is important to study the conditions under which the term $\boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega}$ vanishes. Since the tensor of inertia $\boldsymbol{J}_G$ is positive definite, $\boldsymbol{L}_G = \boldsymbol{J}_G\,\boldsymbol{\omega}$ is always non-zero, excluding the trivial case of null angular velocity. Therefore, the cross product $\boldsymbol{\omega}\times\boldsymbol{J}_G\,\boldsymbol{\omega}$ vanishes if, and only if, the angular momentum $\boldsymbol{L}_G = \boldsymbol{J}_G\,\boldsymbol{\omega}$ is *parallel* to the angular velocity $\boldsymbol{\omega}$. Parallelism means that

$$\boldsymbol{L}_G = \boldsymbol{J}_G\,\boldsymbol{\omega} = \lambda\,\boldsymbol{\omega},$$

i.e., that the angular velocity $\boldsymbol{\omega}$ is an *eigenvector* of the tensor of inertia $\boldsymbol{J}_G$. Therefore, when the condition $\boldsymbol{\omega}\times\boldsymbol{J}_G\,\boldsymbol{\omega} = \boldsymbol{0}$ is met, we say that the body is rotating about a *principal direction of inertia* and the rotational equation of motion (4.12b) reduces to $\boldsymbol{J}_G\,\boldsymbol{\alpha} = \boldsymbol{M}_G$. When the angular velocity $\boldsymbol{\omega}$ is *not* an eigenvector of the tensor of inertia $\boldsymbol{J}_G$ (i.e., the body is *not* rotating about a principal axis of inertia), the term $\boldsymbol{\omega}\times\boldsymbol{J}_G\,\boldsymbol{\omega}$ in (4.12b) tends to *deviate* the direction of the axis of rotation (i.e., the direction of $\boldsymbol{\omega}$). This has of course tremendous importance in Engineering applications, such as the dynamics of rotors.

5. Inverse dynamics

Let the rigid body $\mathcal{B}$ be subjected to a discrete number n of concentrated external forces $\boldsymbol{F}^{(i)}$ and concentrated external moments $\boldsymbol{M}^{(i)}$, applied at the n points with positions $x^{(i)}$ with local position vectors $\boldsymbol{\xi}^{(i)}$ with respect to the centre of mass x_G (see Figure 3). In this case, the resultant force $\boldsymbol{F}$ (not to be confused with the deformation gradient (2.3)) and resultant moment $\boldsymbol{M}_G$, evaluated with respect to the centre of mass x_G, are

$$\boldsymbol{F} = \sum_{i=1}^{n}\boldsymbol{F}^{(i)},$$
$$\boldsymbol{M}_G = \sum_{i=1}^{n}\boldsymbol{M}^{(i)} + \sum_{i=1}^{n}\boldsymbol{\xi}^{(i)}\times\boldsymbol{F}^{(i)}.$$

Under this hypothesis, the equations of motion (4.12) read

$$m\,\boldsymbol{a}_G = \sum_{i=1}^{n}\boldsymbol{F}^{(i)},$$
$$\boldsymbol{J}_G\,\boldsymbol{\alpha} + \boldsymbol{\omega}\times\boldsymbol{J}_G\,\boldsymbol{\omega} = \sum_{i=1}^{n}\boldsymbol{M}^{(i)} + \sum_{i=1}^{n}\boldsymbol{\xi}^{(i)}\times\boldsymbol{F}^{(i)}.$$

In the setting of Inverse Dynamics [4], we have that:

(1) The kinematical variables x_G, $\boldsymbol{v}_G$, $\boldsymbol{a}_G$, $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, along with all the positions $x^{(i)}$ and corresponding position vectors $\boldsymbol{\xi}^{(i)}$, are all *known* functions of time, determined experimentally;
(2) The forces $\boldsymbol{F}^{(i)}$ and moments $\boldsymbol{M}^{(i)}$ at the application points $x^{(i)}$ are known for all $i \in \{1, ..., n-1\}$;
(3) Only the force $\boldsymbol{F}^{(n)}$ and moment $\boldsymbol{M}^{(n)}$ at $x^{(n)}$ are unknown.

If we isolate the terms containing the force and moment at $x^{(n)}$ and group the terms containing the forces and moments at $x^{(1)}$,..., $x^{(n-1)}$, we can write the *inverse dynamics equations* as

$$m\,\boldsymbol{a}_G = \boldsymbol{F}^{(n)} + \boldsymbol{F}_0, \tag{5.1a}$$

$$\boldsymbol{J}_G\,\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega} = \boldsymbol{\xi}^{(n)} \times \boldsymbol{F}^{(n)} + \boldsymbol{M}^{(n)} + \boldsymbol{M}_0, \tag{5.1b}$$

where we defined the resultants of the known forces and moments as

$$\boldsymbol{F}_0 = \sum_{i=1}^{n-1} \boldsymbol{F}^{(i)},$$

$$\boldsymbol{M}_0 = \sum_{i=1}^{n-1} \boldsymbol{\xi}^{(i)} \times \boldsymbol{F}^{(i)} + \sum_{i=1}^{n-1} \boldsymbol{M}^{(i)}.$$

The two vectorial equations (5.1) constitute a system of 6 *algebraic* equations in the 6 unknowns (*generalised forces*) $F_1^{(n)}$, $F_2^{(n)}$, $F_3^{(n)}$, $M_1^{(n)}$, $M_2^{(n)}$, $M_3^{(n)}$.

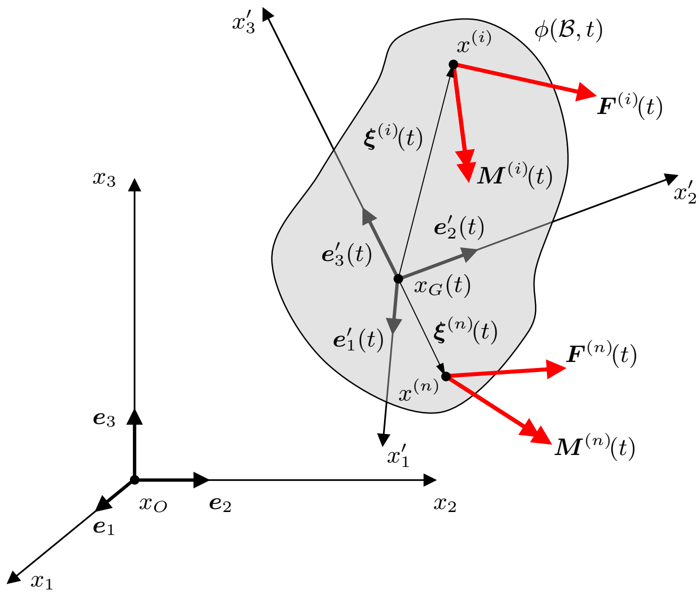

FIGURE 3. A rigid body subjected to external forces and moments at n points with positions $x^{(i)}$. The forces and moments at all points except $x^{(n)}$ and all kinematical variables are assumed to be known. Inverse Dynamics allows for calculating the unknown forces and moments at the point with position $x^{(n)}$.

In order to solve the system of algebraic equations constituted by Equations (5.1), we write them in a 6×6 matrix formalism as

$$m \underset{3\times1}{\{a_G\}} = \underset{3\times1}{\{F^{(n)}\}} + \underset{3\times1}{\{F_0\}}, \tag{5.2a}$$

$$\underset{3\times3}{[J_G]}\,\underset{3\times1}{\{\alpha\}} + \underset{3\times3}{[\Omega]}\,\underset{3\times3}{[J_G]}\,\underset{3\times1}{\{\omega\}} = \underset{3\times3}{[B]}\,\underset{3\times1}{\{F^{(n)}\}} + \underset{3\times1}{\{M^{(n)}\}} + \underset{3\times1}{\{M_0\}}. \tag{5.2b}$$

The cross product $\boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega}$ has been expressed in terms of the rigid spin tensor $\boldsymbol{\Omega}$ via (3.8), i.e.,

$$\boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega} = \boldsymbol{\Omega}\,\boldsymbol{J}_G\,\boldsymbol{\omega},$$

and then, with the matrix representation (3.9) of $\boldsymbol{\Omega}$, written in matrix form as

$$\underset{3\times1}{\{\boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega}\}} = \underset{3\times3}{[\Omega]}\,\underset{3\times1}{\{\boldsymbol{J}_G\,\boldsymbol{\omega}\}} = \underset{3\times3}{[\Omega]}\,\underset{3\times3}{[J_G]}\,\underset{3\times1}{\{\omega\}}.$$

The cross product $\boldsymbol{\xi}^{(n)} \times \boldsymbol{F}^{(n)}$ has been represented as

$$\underset{3\times1}{\{\boldsymbol{\xi}^{(n)} \times \boldsymbol{F}^{(n)}\}} = \underset{3\times3}{[B]}\,\underset{3\times1}{\{F^{(n)}\}},$$

where the matrix $[B]$ of the moment arms associated with the local position vector $\boldsymbol{\xi}^{(n)}$ is defined by

$$B_{ik} = \epsilon_{ijk}\,\xi_j^{(n)}, \quad [B] = \begin{bmatrix} 0 & -\xi_3^{(n)} & \xi_2^{(n)} \\ \xi_3^{(n)} & 0 & -\xi_1^{(n)} \\ -\xi_2^{(n)} & \xi_1^{(n)} & 0 \end{bmatrix},$$

omitting the superscript (n) in the symbols B_{ik} and $[B]$ for the sake of a lighter notation. Note that, while $[\Omega]$ is the matrix representation of the true skew-symmetric tensor $\boldsymbol{\Omega}$, the matrix $[B]$ does *not* represent any true tensor. Indeed, the transformation $[B] = [Q]\,[B]'\,[Q]^T$ is equivalent to the transformation $\{\xi^{(i)}\} = [Q]\,\{\xi^{(i)}\}'$ of the components of the *true* vector $\boldsymbol{\xi}^{(i)}$ *only* when $[Q]$ is a *proper orthogonal* matrix.[11]

Now, in (5.2), we use $\{a_G\} = \{\dot{v}_G\}$ and $\{\alpha\} = \{\dot{\omega}\}$, and insert an identity matrix $[I]$ between the mass m and the acceleration $\{a_G\} = \{\dot{v}_G\}$ of the centre of mass, and obtain

$$m\,\underset{3\times3}{[I]}\,\underset{3\times1}{\{\dot{v}_G\}} = \underset{3\times1}{\{F^{(n)}\}} + \underset{3\times1}{\{F_0\}}, \tag{5.3a}$$

$$\underset{3\times3}{[J_G]}\,\underset{3\times1}{\{\dot{\omega}\}} + \underset{3\times3}{[\Omega]}\,\underset{3\times3}{[J_G]}\,\underset{3\times1}{\{\omega\}} = \underset{3\times3}{[B]}\,\underset{3\times1}{\{F^{(n)}\}} + \underset{3\times1}{\{M^{(n)}\}} + \underset{3\times1}{\{M_0\}}. \tag{5.3b}$$

We are now ready to assemble Equations (5.3) into the single 6×6 equation

$$\begin{bmatrix} m\,\underset{3\times3}{[I]} & \underset{3\times3}{[0]} \\ \underset{3\times3}{[0]} & \underset{3\times3}{[J_G]} \end{bmatrix} \begin{bmatrix} \underset{3\times1}{\{\dot{v}_G\}} \\ \underset{3\times1}{\{\dot{\omega}\}} \end{bmatrix} + \begin{bmatrix} \underset{3\times3}{[0]} & \underset{3\times3}{[0]} \\ \underset{3\times3}{[0]} & \underset{3\times3}{[\Omega]} \end{bmatrix} \begin{bmatrix} m\,\underset{r\times3}{[I]} & \underset{3\times3}{[0]} \\ \underset{3\times3}{[0]} & \underset{3\times3}{[J_G]} \end{bmatrix} \begin{bmatrix} \underset{3\times1}{\{v_G\}} \\ \underset{3\times1}{\{\omega\}} \end{bmatrix}$$
$$= \begin{bmatrix} \underset{3\times3}{[I]} & \underset{3\times3}{[0]} \\ \underset{3\times3}{[B]} & \underset{3\times3}{[I]} \end{bmatrix} \begin{bmatrix} \underset{3\times1}{\{F^{(n)}\}} \\ \underset{3\times1}{\{M^{(n)}\}} \end{bmatrix} + \begin{bmatrix} \underset{3\times1}{\{F_0\}} \\ \underset{3\times1}{\{M_0\}} \end{bmatrix}, \tag{5.4}$$

[11] See the discussion on the transformation of two-forms, i.e., second-order skew-symmetric tensors, in [16].

having denoted with $[0]$ the 3×3 zero matrix. In compact form, Equation (5.4) reads

$$\underset{6\times6}{[\mathcal{M}]}\,\underset{6\times1}{\{\dot{u}\}} + \underset{6\times6}{[W]}\,\underset{6\times6}{[\mathcal{M}]}\,\underset{6\times1}{\{u\}} = \underset{6\times6}{[A]}\,\underset{6\times1}{\{\mathcal{F}\}} + \underset{6\times1}{\{\mathcal{F}_0\}}. \tag{5.5}$$

The 6×1 vector of the generalised velocities,

$$\underset{6\times1}{\{u\}} = \begin{bmatrix} \underset{3\times1}{\{v_G\}} \\ \underset{3\times1}{\{\omega\}} \end{bmatrix},$$

is a block vector, with the 1-block consisting of the components of the velocity $\boldsymbol{v}_G$ of the centre of mass and with the 2-block consisting of the components of the angular velocity $\boldsymbol{\omega}$. The 6×6 matrix of the generalised masses,

$$\underset{6\times6}{[\mathcal{M}]} = \begin{bmatrix} m \underset{3\times3}{[I]} & \underset{3\times3}{[0]} \\ \underset{3\times3}{[0]} & \underset{3\times3}{[J_G]} \end{bmatrix},$$

is a symmetric block-diagonal matrix, with the 11-block being an identity matrix $[I]$ multiplied by the mass m of the body and the 22-block being the symmetric matrix $[J_G]$ of the tensor of inertia $\boldsymbol{J}_G$. We emphasise again that the matrix $[J_G]$ *does* depend on time, as it is written in the fixed reference frame. The 6×6 generalised spin matrix,

$$\underset{6\times6}{[W]} = \begin{bmatrix} \underset{3\times3}{[0]} & \underset{3\times3}{[0]} \\ \underset{3\times3}{[0]} & \underset{3\times3}{[\Omega]} \end{bmatrix},$$

is a skew-symmetric block-diagonal matrix, with all blocks being null except for the 22-block, which is the skew-symmetric matrix $[\Omega]$ of the rigid spin tensor $\boldsymbol{\Omega}$. The generalised spin matrix $[W]$ "kills" all terms in $[\mathcal{M}]\{u\}$ except those corresponding to the term $[\Omega]\,[J_G]\,\{\omega\}$ in the rotational equation (5.3b) (term $\boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega}$ in (5.1b)). The 6×6 matrix of the generalised moment arms associated with $\boldsymbol{\xi}^{(n)}$ is the *block-lower-triangular* matrix

$$\underset{6\times6}{[A]} = \begin{bmatrix} \underset{3\times3}{[I]} & \underset{3\times3}{[0]} \\ \underset{3\times3}{[B]} & \underset{3\times3}{[I]} \end{bmatrix}.$$

The two identity blocks on the main diagonal simply "recopy" the three components of the force $\{F^{(n)}\}$ into the translational equation (5.3a) and the three components of the moment $\{M^{(n)}\}$ into the rotational equation (5.3b), respectively. The zero in the 12-block reflects the fact that *no* component of the moment $\{M^{(n)}\}$ can enter the translational equation (5.3a). The matrix $[B]$ in the 21-block provides the moment arms of the components of the unknown force $\{F^{(n)}\}$ in the rotational equation (5.3b). The 6×1 block vectors of the unknown generalised forces at point $x^{(n)}$ and of the resultants of the known generalised forces are defined as

$$\underset{6\times1}{\{\mathcal{F}\}} = \begin{bmatrix} \underset{3\times1}{\{F^{(n)}\}} \\ \underset{3\times1}{\{M^{(n)}\}} \end{bmatrix}, \quad \underset{6\times1}{\{\mathcal{F}_0\}} = \begin{bmatrix} \underset{3\times1}{\{F_0\}} \\ \underset{3\times1}{\{M_0\}} \end{bmatrix}.$$

Note that, similarly as for the case of $[B]$ (and, consequentially, of $[A]$), we omit the superscript (n) in the symbol $\{\mathcal{F}\}$.

The matrix form (5.5) of the inverse dynamics equations (5.1) is solved for the vector of the unknown generalised forces $\{\mathcal{F}\}$ by inversion of the matrix $[A]$ of the generalised moment arms, i.e.,

$$\underset{6\times 1}{\{\mathcal{F}\}} = \underset{6\times 6}{[A]}^{-1}\big(\underset{6\times 6}{[\mathcal{M}]}\,\underset{6\times 1}{\{\dot{u}\}} + \underset{6\times 6}{[W]}\,\underset{6\times 6}{[\mathcal{M}]}\,\underset{6\times 1}{\{u\}} - \underset{6\times 1}{\{\mathcal{F}_0\}}\big).$$

The inverse of $[A]$ is, perhaps surprisingly, simply obtained by changing the sign of the 21-block $[B]$, i.e.,

$$\underset{6\times 6}{[A]}^{-1} = \begin{bmatrix} \underset{3\times 3}{[I]} & \underset{3\times 3}{[0]} \\ -\underset{3\times 3}{[B]} & \underset{3\times 3}{[I]} \end{bmatrix}.$$

This is easy to verify by exploiting the block structure of $[A]$:

$$\begin{bmatrix} \underset{3\times 3}{[I]} & \underset{3\times 3}{[0]} \\ \underset{3\times 3}{[B]} & \underset{3\times 3}{[I]} \end{bmatrix} \begin{bmatrix} \underset{3\times 3}{[I]} & \underset{3\times 3}{[0]} \\ -\underset{3\times 3}{[B]} & \underset{3\times 3}{[I]} \end{bmatrix} = \begin{bmatrix} \underset{3\times 3}{[I]} & \underset{3\times 3}{[0]} \\ \underset{3\times 3}{[0]} & \underset{3\times 3}{[I]} \end{bmatrix}.$$

6. One-body example: inverse dynamics of the foot

Here, we present the two-dimensional analysis of a one-body problem in Inverse Dynamics, similar to that in the paper by Andrews [4]. We shall see how the matrix formulation described in Section (5) becomes particularly simple.

A runner's foot, schematised by a two-dimensional body contained in the plane spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, as shown in Figure 4, lands on a force plate with the ball of the foot. The resultant reaction force $\boldsymbol{F}_P$ recorded by a force plate is applied at the *centre of pressure* x_P [24]. The mass m, the moment of inertia $J_G \equiv J_{G33}$ with respect to the centre of mass x_G and the geometry of the foot (norms $\|\boldsymbol{\xi}_T\| = \ell$ and $\|\boldsymbol{\xi}_P\| = b$ of the local position vectors $\boldsymbol{\xi}_T$ and $\boldsymbol{\xi}_P$ of the talar joint (ankle) x_T and the centre of pressure x_P with respect to the centre of mass x_G, angle β between $\boldsymbol{\xi}_T$ and $\boldsymbol{\xi}_P$) are known. The kinematics (coordinates x_{G1} and x_{G2} of the centre of mass x_G as a function of the time t and rotation angle θ of $\boldsymbol{\xi}_T$ with respect to the horizontal direction $\boldsymbol{e}_1$ as a function of time t) is known from motion analysis data. Our goal is to find the resultant force $\boldsymbol{F}_T$ and the resultant moment $\boldsymbol{M}_T$ at the talar joint x_T.

Since the problem is in two dimensions and the body is contained in the $\boldsymbol{e}_1$-$\boldsymbol{e}_2$ plane, we have

$$\begin{aligned} \{\xi_T\}^T &= [\ell\,\cos\theta \;\; \ell\,\sin\theta \;\; 0] & \{\xi_P\}^T &= [b\,\cos(\theta+\beta) \;\; b\,\sin(\theta+\beta) \;\; 0] \\ \{\omega\}^T &= [0 \;\; 0 \;\; \omega] = [0 \;\; 0 \;\; \dot{\theta}] & \{\alpha\}^T &= [0 \;\; 0 \;\; \dot{\omega}] = [0 \;\; 0 \;\; \ddot{\theta}] \\ \{F_P\}^T &= [F_{P1} \;\; F_{P2} \;\; 0] & \{m\,g\}^T &= [0 \;\; -m\,g \;\; 0] \\ \{F_T\}^T &= [F_{T1} \;\; F_{T2} \;\; 0] & \{M_T\}^T &= [0 \;\; 0 \;\; M_T]. \end{aligned}$$

Note that, for the angles θ and β of Figure (4), both components of $\boldsymbol{\xi}_P$ are negative.

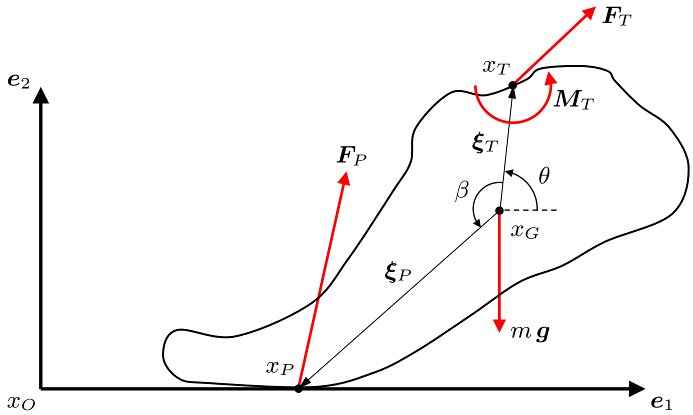

FIGURE 4. Schematised foot landing on a force plate, which measures the reaction $\boldsymbol{F}_P$ at the centre of pressure x_P; the goal of the Inverse Dynamics problem in this case is to calculate the resultant force and moment at the talar joint x_T. The senses of the unknown force $\boldsymbol{F}_T$ and moment $\boldsymbol{M}_T$ are *tentative*.

In the vectorial equation of equilibrium to translation,

$$\boldsymbol{F} = m\,\boldsymbol{g} + \boldsymbol{F}_P + \boldsymbol{F}_T = m\,\boldsymbol{a}_G,$$

only the first and second scalar equations are non-trivial:

$$F_{P1} + F_{T1} = m\,a_{G1} = m\,\dot{v}_{G1}, \tag{6.1a}$$

$$-m\,g + F_{P2} + F_{T2} = m\,a_{G2} = m\,\dot{v}_{G2}. \tag{6.1b}$$

The vectorial equation of equilibrium to rotation with respect to the centre of mass reads

$$\boldsymbol{M}_G = \cancel{\boldsymbol{\xi}_G \times (m\,\boldsymbol{g})} + \boldsymbol{\xi}_P \times \boldsymbol{F}_P + \boldsymbol{\xi}_T \times \boldsymbol{F}_T + \boldsymbol{M}_T = \boldsymbol{J}_G\,\boldsymbol{\alpha} + \cancel{\boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega}},$$

where $\boldsymbol{\xi}_G \times (m\,\boldsymbol{g}) = \boldsymbol{0}$ since $\boldsymbol{\xi}_G = \boldsymbol{0}$, and $\boldsymbol{\omega} \times \boldsymbol{J}_G\,\boldsymbol{\omega} = \boldsymbol{0}$ since we are in two dimensions (in a two-dimensional system, the direction orthogonal to the plane is a principal direction of inertia and thus the angular velocity, which is in the directional orthogonal to the plane, is an eigenvector of the tensor of inertia). Again because we are in two dimensions, the only non-trivial scalar equation is the third, which reads

$$b\,\cos(\theta+\beta)\,F_{P2} - b\,\sin(\theta+\beta)\,F_{P1} + \ell\,\cos\theta\,F_{T2} - \ell\,\sin\theta\,F_{T1} + M_T = J_G\,\dot{\omega}. \tag{6.2}$$

The three scalar equations (6.1) and (6.2) read, in explicit matrix form,

$$\left[\begin{array}{cc:c} m & 0 & 0 \\ 0 & m & 0 \\ \hdashline 0 & 0 & J_G \end{array}\right] \left[\begin{array}{c} \dot{v}_{G1} \\ \dot{v}_{G2} \\ \hdashline \dot{\omega} \end{array}\right] = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline -\ell\,\sin\theta & \ell\,\cos\theta & 1 \end{array}\right] \left[\begin{array}{c} F_{T1} \\ F_{T2} \\ \hdashline M_T \end{array}\right]$$

$$+ \left[\begin{array}{c} F_{R1} \\ F_{R2} - m\,g \\ \hdashline b\,\cos(\theta+\beta)\,F_{P2} - b\,\sin(\theta+\beta)\,F_{P1} \end{array}\right],$$

and, in compact form,

$$\underset{3\times 3}{[\mathcal{M}]}\,\underset{3\times 1}{\{\dot{u}\}} = \underset{3\times 3}{[A]}\,\underset{3\times 1}{\{\mathcal{F}\}} + \underset{3\times 1}{\{\mathcal{F}_0\}},$$

where $[\mathcal{M}]$ is the matrix of the generalised masses, $\{u\}$ is the vector of the generalised velocities, $[A]$ is the matrix of the generalised moment arms associated with $\boldsymbol{\xi}_T$, $\{\mathcal{F}\}$ is the vector of the unknown generalised joint forces at the talar joint x_T, and $\{\mathcal{F}_0\}$ is the vector of the known generalised external forces. The solution is found, as in the general case, by inversion of the matrix $[A]$, i.e.,

$$\underset{3\times 1}{\{\mathcal{F}\}} = \underset{3\times 3}{[A]^{-1}}\left(\underset{3\times 3}{[\mathcal{M}]}\,\underset{3\times 1}{\{\dot{u}\}} - \underset{3\times 1}{\{\mathcal{F}_0\}}\right),$$

where the inverse $[A]^{-1}$ is found similarly to the general three-dimensional case. Indeed, this 3×3 matrix $[A]$ too can be considered as a block matrix, i.e.,

$$\underset{3\times 3}{[A]} = \begin{bmatrix} \underset{2\times 2}{[I]} & \underset{2\times 1}{\{0\}} \\ \underset{1\times 2}{\{B\}} & \underset{1\times 1}{1} \end{bmatrix} = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline -\ell\,\sin\theta & \ell\,\cos\theta & 1 \end{array}\right],$$

and we can again verify that its inverse is

$$\underset{3\times 3}{[A]^{-1}} = \begin{bmatrix} \underset{2\times 2}{[I]} & \underset{2\times 1}{\{0\}} \\ -\underset{1\times 2}{\{B\}} & \underset{1\times 1}{1} \end{bmatrix} = \left[\begin{array}{cc:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline \ell\,\sin\theta & -\ell\,\cos\theta & 1 \end{array}\right].$$

7. Summary and outlook

We presented a fully three-dimensional matrix framework for the solution of problems in Inverse Dynamics, along with an example in the simplified two-dimensional setting. In a didactic spirit, the full kinematics and dynamics of rigid bodies have been derived, with an approach based on the methods of modern Continuum Mechanics, of which rigid body mechanics is, in fact, a particular case. The hope is that this work can be of aid in the classroom, for the delivery of rigid body mechanics in general, and of Inverse Dynamics in particular.

The continuum mechanical approach could be of help in future developments. One example could be the generalisation to pseudo-rigid bodies, i.e., bodies undergoing affine deformations (with stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ independent of X and x, respectively). Another example could be the study of the effect of small deformations, which could be seen as deviations from rigid body motion and represented as an infinitesimal displacement field $\boldsymbol{u}$ superposed to the rigid motion (3.3), i.e., a perturbed motion $\tilde{\phi}(X,t) = \phi(X,t) + \boldsymbol{u}(x,t)$ (see, e.g., [3, 15]). Finally, in Biomechanics, a body segment is normally regarded as being rigid but, while this approximation is well suited for the bones, it is not quite so for the soft tissues surrounding the bone, which can deform and vibrate considerably [23]. A continuum mechanical approach could help in devising strategies to optimally "filter" the effect of these “wobbling masses”, to extract rigid body kinematical data that is as faithful as possible to the movement of the bones.

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References

1. Y. I. Abdel-Aziz, H. M. Karara, *Direct linear transformation into object space coordinates in close-range photogrammetry*, Proceedings of the Symposium of Close-Range Photogrammetry, 1971, 1–18.
2. M. F. Alhasadi, M. Epstein, S. Federico, *Eshelby force and power for uniform bodies*, Acta Mech. **230** (2019), 1663–1684.
3. M. F. Alhasadi, Q. Sun, S. Federico, *Theory of uniformity applied to elastic dielectric materials and piezoelectricity*, Eur. J. Mech., A, Solids **91** (2022), 104391.
4. J. G. Andrews, *Biomechanical analysis of human motion*, Kinesiology **4** (1974), 32–42.
5. A. V. Arnold, *Mathematical Methods of Classical Mechanics*, Nauka, Moscow, USSR, 1974, English translation: Springer, Berlin, Germany, 1989.
6. C. Bellini, S. Federico, *Green–Naghdi rate of the Kirchhoff stress and deformation rate: the elasticity tensor*, Z. Angew. Math. Phys. **66** (2015), 1143–1163.
7. J. Bonet, R. D. Wood, *Nonlinear Continuum Mechanics for Finite Element Analysis*, 2^{nd} ed., Cambridge University Press, Cambridge, UK, 2008.
8. P. Chadwick, *Continuum Mechanics, Concise Theory and Problems*, George Allen & Unwin Ltd., London, UK, 1976.
9. S. C. Cowin, D. H. Hegedus, *Bone remodeling I: Theory of adaptive elasticity*, J. Elasticity **6**(3) (1976), 313–326.
10. M. Epstein, *The Geometrical Language of Continuum Mechanics*, Cambridge University Press, Cambridge, UK, 2010.
11. A. C. Eringen, *Mechanics of Continua*, Robert E. Krieger Publishing Company, Huntington, NY, USA, 1980.
12. S. Federico, *Porous materials with statistically oriented reinforcing fibres*, CISM Courses Lect. **559** (2015), 49–120
13. S. Federico, *Some remarks on metric and deformation*, Math. Mech. Solids **20** (2015), 522–539.
14. S. Federico, *The Truesdell rate in continuum mechanics*, Z. Angew. Math. Phys. **73** (2022), 109.
15. S. Federico, M. F. Alhasadi, A. Grillo, *Eshelby's inclusion theory in the light of Noether's theorem*, Math. Mech. Complex Syst. **7** (2019), 247–285.
16. S. Federico, A. Grillo, R. Segev, *Material description of fluxes in terms of differential forms*, Contin. Mech. Thermodyn. **28** (2016), 379–390.
17. Y. N. Fedorov, V. V. Kozlov, *Various aspects of n-dimensional rigid body dynamics*, Transl., Ser. 2, Am. Math. Soc. **168** (1995), 141–172.
18. L. C. García-Naranjo, *Hamiltonisation, measure preservation and first integrals of the multidimensional rubber Routh sphere*, Theoretical and Applied Mechanics **46** (2019), 65–88.
19. M. E. Gurtin, E. Fried, L. Anand, *The Mechanics and Thermodynamics of Continua*, Cambridge University Press, Cambridge, UK, 2009.
20. L. D. Landau, E. M. Lifshitz, *Mechanics*, Pergamon Press, Oxford, UK, 1960.
21. J. E. Marsden, T. J. R. Hughes, *Mathematical Foundations of Elasticity*, Prentice-Hall, Englewood Cliff, NJ, USA, 1983.
22. L. Meirovitch, *Methods of Analytical Dynamics*, McGraw-Hill, New York, USA, 1970.

23. B. M. Nigg, *Impact forces in running*, Current Opinion in Orthopaedics **8** (1997), 43–47.
24. B. M. Nigg, W. Herzog, *Biomechanics of the Musculo-Skeletal System*, 3$^{\text{rd}}$ ed., Wiley, Chichester, UK, 2007.
25. C. Tuesdell, W. Noll, *The non-linear field theories of mechanics*, S. Flügge, ed., *Encyclopedia of Physics*, vol. III, Springer-Verlag, Berlin, Germany, 1965.

ИНВЕРЗНА ДИНАМИКА У МЕХАНИЦИ КРУГОГ ТЕЛА

Резиме. Разматрамо проблем инверзне динамике, израчунавање сила и момената сила у зглобовима система више крутих тела, који се јавља у областима као што су биомеханика или роботика. У педагошком духу, рад почиње приказом извођења кинематичких и динамичких једначина крутог тела са становишта савремене механике континуума. Затим уводимо матричну формулацију за решење проблема инверзне динамике и, на крају, приказујемо једноставан дводимензионални пример примене проблема у биомеханици.

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