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THE THEORY OF GENERALIZED MICROPOLAR THERMOELASTIC DIFFUSION WITH DOUBLE POROSITY

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ABSTRACT. The main purpose of the paper is to derive the constitutive relations and field equations for anisotropic micropolar thermoelastic medium with mass diffusion and double porosity. In addition to this, the fundamental solution of system of equations in case of steady oscillations is also constructed.

1. Introduction

The classic theory of elasticity is not capable to represent the microstructure of various types of materials such as polycrystalline materials, materials with fibrous etc. The micropolar theory of elasticity takes into account the micro-structural motion of such types of materials. In this theory, the motion of solids are described by two vectors namely, displacement and microrotation. Eringen [1,2] and Nowacki [3–5] included thermal effects in the theory to become micropolar theory of thermoelasticity. Boschi and Iesan [6] extended a generalized theory of micropolar thermoelasticity.

The transfer of mass of a substance from the high concentration regions to low concentration regions is called diffusion. Nowacki [7–10], Sherief et al. [11], Aouadi [12] and Kansal and Kumar [13] established the different theories of thermoelastic diffusion to describe the coupled mechanical behaviour among temperature, concentration, and strain fields in elastic solids. Aouadi [14] derived the theory of generalized micropolar thermoelastic diffusion based on the theory of Lord–Shulman with one relaxation time.

Biot [15] presented the first model for single porosity deformable solid by using the classical Darcy's law. The double porosity model represents a double porous structure, one is macro porosity which is connected to pores and other is micro porosity which is connected to fissures. The theory for deformable materials with double porosity was developed by Aifantis and co-workers [16–18]. Khalili and Selvadurai [19,20] and Gelet et al. [21] established the basic governing homogeneous

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equations in the linear theory of thermoelasticity for solids with double porosity. After this, lots of works has been done in this field. Iesan and Quintanilla [22] derived a non-linear theory of thermoelastic solids with double porosity structure based upon Nunziato–Cowin theory of materials with voids. This theory was not based upon Darcy's law. Kansal [23] established a linear generalized theory of thermoelastic diffusion with double porosity.

Fundamental solutions are necessary to construct for solving boundary value problems of elasticity and thermoelasticity by potential method. The reason for constructing fundamental solutions is that an integral representation of solution of a boundary value problem by fundamental solution is easily solved by numerical methods rather than a differential equation with specified boundary and initial conditions. Various authors [24–29] constructed fundamental solutions in different theories of elasticity and thermoelasticity with double porosity.

The present article is an extension of the previous works [23,24]. In this paper, the constitutive relations, field equations for anisotropic generalized micropolar thermoelasticity with mass diffusion and double porosity based upon Lord–Shulman model are derived. Finally, in terms of elementary functions, the fundamental solution of basic governing equations in case of steady oscillations is constructed.

2. Basic Equations

Following [1–5], the equations of motions of a linear micropolar elasticity are

$$\sigma_{ji,j} + \rho F_i = \rho \ddot{u}_i, \tag{2.1}$$

$$\varepsilon_{ijk}\sigma_{jk} + \mu_{ji,j} + \rho G_i = \rho J_{ij}\ddot{\varphi}_j, \tag{2.2}$$

where σ_{ji} are the components of stress tensor, μ_{ji} are the components of moment of couple stress tensor, ρ is the density, u_i are the components of the displacement vector $\boldsymbol{u}$, F_i are the components of the external forces per unit mass, G_i are the components of the external applied couple per unit mass, ε_{ijk} is the alternating tensor, J_{ij} are the components of microinertia tensor, φ_j are the components of microrotation vector $\boldsymbol{\varphi}$.

The law of conservation of energy for an arbitrary material volume V bounded by a surface A at time t can be written as

$$\begin{aligned}\int_V \rho[\dot{u}_i\ddot{u}_i + J_{ij}\dot{\varphi}_i\ddot{\varphi}_j + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + \dot{U}]dV \\ = \int_V \rho[F_i\dot{u}_i + G_i\dot{\varphi}_i + \tilde{g}\dot{\nu}_1 + \tilde{l}\dot{\nu}_2]dV \\ + \int_A [f_i\dot{u}_i + \mu_i\dot{\varphi}_i + \Omega_i n_i\dot{\nu}_1 + \chi_i n_i\dot{\nu}_2 - q_i n_i]dA,\end{aligned} \tag{2.3}$$

where μ_i are the components of couple stress vector $\boldsymbol{\mu}$ occurring on the surface A, U is the internal energy per unit mass, q_i are the components of heat flux vector $\boldsymbol{q}$, f_i are the components of surface traction vector $\boldsymbol{f}$ occurring on the surface A, ν_1 and ν_2 are the volume fraction fields corresponding to pores and fissures respectively, k_1 and k_2 are coefficients of equilibrated inertia, $\tilde{g}$ and $\tilde{l}$ are, respectively, extrinsic equilibrated body forces per unit mass associated to macro pores and fissures, Ω_i, χ_i are the components of equilibrated stress vectors corresponding to ν_1, ν_2 measured per unit area of surface A respectively, n_i are the components of outward unit normal vector $\boldsymbol{n}$ to the surface A.

The components f_i and μ_i are, respectively, connected to stress and couple stress vectors by the relations

$$f_i = \sigma_{ji} n_j, \tag{2.4}$$

$$\mu_i = \mu_{ji} n_j. \tag{2.5}$$

Utilizing equations (2.4) and (2.5) in equation (2.3) and applying divergence theorem, we acquire

$$\begin{aligned}(2.6)\quad &\int_V \rho[\dot{u}_i\ddot{u}_i + J_{ij}\dot{\varphi}_i\ddot{\varphi}_j + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + \dot{U}]dV \\ &= \int_V \rho[F_i\dot{u}_i + G_i\dot{\varphi}_i + \tilde{g}\dot{\nu}_1 + \tilde{l}\dot{\nu}_2]dV \\ &+ \int_V [\sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \mu_{ji,j}\dot{\varphi}_i + \mu_{ji}\dot{\varphi}_{i,j} + \Omega_{i,i}\dot{\nu}_1 + \Omega_i\dot{\nu}_{1,i} + \chi_{i,i}\dot{\nu}_2 + \chi_i\dot{\nu}_{2,i} - q_{i,i}]dV.\end{aligned}$$

Since equation (2.6) is valid for every part of the body, therefore local form of conservation of energy is obtained as

$$\begin{aligned}(2.7)\quad &\rho[\dot{u}_i\ddot{u}_i + J_{ij}\dot{\varphi}_i\ddot{\varphi}_j + k_1\dot{\nu}_1\ddot{\nu}_1 + k_2\dot{\nu}_2\ddot{\nu}_2 + \dot{U}] = \rho[F_i\dot{u}_i + G_i\dot{\varphi}_i + \tilde{g}\dot{\nu}_1 + \tilde{l}\dot{\nu}_2] \\ &+ \sigma_{ji,j}\dot{u}_i + \sigma_{ji}\dot{u}_{i,j} + \mu_{ji,j}\dot{\varphi}_i + \mu_{ji}\dot{\varphi}_{i,j} + \Omega_{i,i}\dot{\nu}_1 + \Omega_i\dot{\nu}_{1,i} + \chi_{i,i}\dot{\nu}_2 + \chi_i\dot{\nu}_{2,i} - q_{i,i}.\end{aligned}$$

Equation (2.7) with the assistance of equations (2.1) and (2.2) yields a simplified form of conservation of energy

$$\rho\dot{U} = \sigma_{ji}\dot{\varepsilon}_{ji} + \mu_{ji}\dot{\varphi}_{i,j} + \Omega_i\dot{\nu}_{1,i} + \chi_i\dot{\nu}_{2,i} - q_{i,i} - \xi\dot{\nu}_1 - \zeta\dot{\nu}_2, \tag{2.8}$$

where ε_{ji}, ξ and ζ satisfy the relations

$$\varepsilon_{ji} = u_{i,j} - \varepsilon_{kji}\varphi_k, \quad \Omega_{i,i} + \xi + \rho\tilde{g} = \rho k_1\ddot{\nu}_1, \quad \chi_{i,i} + \zeta + \rho\tilde{l} = \rho k_2\ddot{\nu}_2. \tag{2.9}$$

Following Nowacki [30], the balance of entropy can be composed as

$$\begin{aligned}(2.10)\quad &\int_V \rho\dot{S}dV + \int_A \Big(\frac{q_i}{T}\Big)n_i dA - \int_A \Big(\frac{P\eta_i}{T}\Big)n_i dA \\ &= \int_V \Big[-\frac{q_i}{T^2}T_{,i} - \frac{P_{,i}}{T}\eta_i + \frac{P}{T^2}\eta_i T_{,i}\Big]dV.\end{aligned}$$

where S, P, are entropy and chemical potential per unit mass respectively, η_i are the components of mass diffusion flux vector $\boldsymbol{\eta}$, T is absolute temperature.

The equation (2.10) can be illustrated in the local form

$$\rho\dot{S} + \Big(\frac{q_i}{T}\Big)_{,i} - \Big(\frac{P\eta_i}{T}\Big)_{,i} = -\frac{q_i}{T^2}T_{,i} - \frac{P_{,i}}{T}\eta_i + \frac{P}{T^2}\eta_i T_{,i}. \tag{2.11}$$

The right hand side of above equation is the entropy source

$$\Re = -\frac{q_i}{T^2}T_{,i} - \frac{P_{,i}}{T}\eta_i + \frac{P}{T^2}\eta_i T_{,i} \geqslant 0.$$

On the basis of above equation, equation (2.11) can be represented in the form of an inequality called Clausius–Duhem inequality

$$\rho\dot{S}+\frac{q_{i,i}}{T}-\frac{q_i}{T^2}T_{,i}-\frac{P}{T}\eta_{i,i}-\frac{P_{,i}}{T}\eta_i+\frac{P}{T^2}\eta_iT_{,i}\geqslant 0. \tag{2.12}$$

The equation of conservation of mass is

$$\eta_{j,j}=-\dot{C}, \tag{2.13}$$

where C is the concentration of the diffusion material in the elastic body.

Equation (2.12) with the help of equations (2.8) and (2.13) becomes

$$\begin{aligned}\rho T\dot{S}-\rho\dot{U}+\sigma_{ji}\dot{\varepsilon}_{ji}+\mu_{ji}\dot{\varphi}_{i,j}+\Omega_i\dot{\nu}_{1,i}+\chi_i\dot{\nu}_{2,i}-\xi\dot{\nu}_1-\zeta\dot{\nu}_2\\ -\frac{q_i}{T}T_{,i}+P\dot{C}-P_{,i}\eta_i+\frac{P}{T}\eta_iT_{,i}\geqslant 0.\end{aligned} \tag{2.14}$$

Helmholtz free energy function Γ is stated as

$$\Gamma=U-TS. \tag{2.15}$$

Applying equation (2.15) in the equation (2.14), we get

$$\begin{aligned}-\rho[\dot{\Gamma}+\dot{T}S]+\sigma_{ji}\dot{\varepsilon}_{ji}+\mu_{ji}\dot{\varphi}_{i,j}+\Omega_i\dot{\nu}_{1,i}+\chi_i\dot{\nu}_{2,i}-\xi\dot{\nu}_1-\zeta\dot{\nu}_2\\ -\frac{q_i}{T}T_{,i}+P\dot{C}-P_{,i}\eta_i+\frac{P}{T}\eta_iT_{,i}\geqslant 0.\end{aligned} \tag{2.16}$$

The function Γ can be expressed in terms of independent variables ε_{ji}, $\varphi_{i,j}$, ν_1, $\nu_{1,i}$, ν_2, $\nu_{2,i}$, T, $T_{,i}$, C and $C_{,i}$. Therefore, we have

$$\begin{aligned}\dot{\Gamma}=\frac{\partial\Gamma}{\partial\varepsilon_{ji}}\dot{\varepsilon}_{ji}+\frac{\partial\Gamma}{\partial\varphi_{i,j}}\dot{\varphi}_{i,j}+\frac{\partial\Gamma}{\partial\nu_1}\dot{\nu}_1+\frac{\partial\Gamma}{\partial\nu_{1,i}}\dot{\nu}_{1,i}+\frac{\partial\Gamma}{\partial\nu_2}\dot{\nu}_2\\ +\frac{\partial\Gamma}{\partial\nu_{2,i}}\dot{\nu}_{2,i}+\frac{\partial\Gamma}{\partial T}\dot{T}+\frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i}+\frac{\partial\Gamma}{\partial C}\dot{C}+\frac{\partial\Gamma}{\partial C_{,i}}\dot{C}_{,i}.\end{aligned} \tag{2.17}$$

Equation (2.16) with the help of equation (2.17) becomes

$$\begin{aligned}\Big[\sigma_{ji}-\rho\frac{\partial\Gamma}{\partial\varepsilon_{ji}}\Big]\dot{\varepsilon}_{ji}+\Big[\mu_{ji}-\rho\frac{\partial\Gamma}{\partial\varphi_{i,j}}\Big]\dot{\varphi}_{i,j}+\Big[\Omega_i-\rho\frac{\partial\Gamma}{\partial\nu_{1,i}}\Big]\dot{\nu}_{1,i}\\ +\Big[\chi_i-\rho\frac{\partial\Gamma}{\partial\nu_{2,i}}\Big]\dot{\nu}_{2,i}-\Big[\xi+\rho\frac{\partial\Gamma}{\partial\nu_1}\Big]\dot{\nu}_1-\Big[\zeta+\rho\frac{\partial\Gamma}{\partial\nu_2}\Big]\dot{\nu}_2-\rho\Big[S+\frac{\partial\Gamma}{\partial T}\Big]\dot{T}\\ +\Big[P-\rho\frac{\partial\Gamma}{\partial C}\Big]\dot{C}-\rho\frac{\partial\Gamma}{\partial T_{,i}}\dot{T}_{,i}-\rho\frac{\partial\Gamma}{\partial C_{,i}}\dot{C}_{,i}-\frac{q_i}{T}T_{,i}-P_{,i}\eta_i+\frac{P}{T}\eta_iT_{,i}\geqslant 0.\end{aligned} \tag{2.18}$$

The inequality should be convinced for all rates $\dot{\varepsilon}_{ji}$, $\dot{\varphi}_{i,j}$, $\dot{\nu}_1$, $\dot{\nu}_{1,i}$, $\dot{\nu}_2$, $\dot{\nu}_{2,i}$, $\dot{T}$, $\dot{T}_{,i}$, $\dot{C}$ and $\dot{C}_{,i}$. Hence the coefficients of above variables must vanish, that is

$$\sigma_{ji}=\rho\frac{\partial\Gamma}{\partial\varepsilon_{ji}}, \tag{2.19}$$

$$\mu_{ji}=\rho\frac{\partial\Gamma}{\partial\varphi_{i,j}}, \tag{2.20}$$

$$\Omega_i = \rho \frac{\partial \Gamma}{\partial \nu_{1,i}}, \tag{2.21}$$

$$\chi_i = \rho \frac{\partial \Gamma}{\partial \nu_{2,i}}, \tag{2.22}$$

$$\xi = -\rho \frac{\partial \Gamma}{\partial \nu_1}, \tag{2.23}$$

$$\zeta = -\rho \frac{\partial \Gamma}{\partial \nu_2}, \tag{2.24}$$

$$S = -\frac{\partial \Gamma}{\partial T}, \tag{2.25}$$

$$P = \rho \frac{\partial \Gamma}{\partial C}, \tag{2.26}$$

$$\frac{\partial \Gamma}{\partial T_{,i}} = \frac{\partial \Gamma}{\partial C_{,i}} = 0, \tag{2.27}$$

$$-\frac{q_i}{T} T_{,i} - P_{,i}\eta_i + \frac{P}{T}\eta_i T_{,i} \geqslant 0. \tag{2.28}$$

Let us introduce the notations

$$\phi = \nu_1 - (\nu_1)_0, \quad \psi = \nu_2 - (\nu_2)_0, \quad \theta = T - T_0, \tag{2.29}$$

where T_0 is the reference temperature of the body chosen such that $|\frac{\theta}{T_0}| \ll 1$, $(\nu_1)_0$ and $(\nu_2)_0$ are the volume fraction fields in reference configuration.

In the linear theory, the independent variables are ε_{ji}, $\varphi_{i,j}$, $\phi, \phi_{,i}$, ψ, $\psi_{,i}$, θ and C. It is assumed that the undeformed body is free from stresses and has zero intrinsic equilibrated body forces and entropy. If the body has a centre of symmetry, then we have

$$\begin{aligned} 2\rho\Gamma = {} & c_{jikl}\varepsilon_{ji}\varepsilon_{kl} + f_{jikl}\varphi_{j,i}\varphi_{k,l} + d^*\phi^2 + f\psi^2 - \frac{\rho C_e \theta^2}{T_0} + bC^2 \\ & + q_{ij}\phi_{,i}\phi_{,j} + f_{ij}\psi_{,i}\psi_{,j} + 2\alpha_{ij}\phi_{,i}\psi_{,j} + 2p_{jikl}\varepsilon_{ji}\varphi_{k,l} + 2p_{ji}\varepsilon_{ji}\phi + 2\gamma_{ji}\varepsilon_{ji}\psi \\ & - 2a_{ji}\varepsilon_{ji}\theta - 2b_{ji}\varepsilon_{ji}C + 2V_{ji}\varphi_{i,j}\phi + 2T_{ji}\varphi_{i,j}\psi - 2A_{ji}\varphi_{i,j}\theta - 2B_{ji}\varphi_{i,j}C \\ & + 2\alpha_1\phi\psi - 2\gamma_1\phi\theta - 2v\phi C - 2\gamma_2\psi\theta - 2m\psi C - 2a\theta C. \end{aligned} \tag{2.30}$$

Since φ_j is an axial vector, therefore on the reflection of spatial axes, tenth, fifteenth, sixteenth, seventeenth and eighteenth terms change sign while other terms don't. For the function Γ to be invariant, $p_{jikl} = V_{ji} = T_{ji} = A_{ji} = B_{ji} = 0$. Using above equation in the equations (2.19)–(2.26), the following constitutive equations are obtained:

$$\sigma_{ji} = c_{jikl}\varepsilon_{kl} + p_{ji}\phi + \gamma_{ji}\psi - a_{ji}\theta - b_{ji}C, \tag{2.31}$$

$$\mu_{ji} = f_{ijkl}\varphi_{k,l}, \tag{2.32}$$

$$\Omega_i = q_{ij}\phi_{,j} + \alpha_{ij}\psi_{,j}, \tag{2.33}$$

$$\chi_i = \alpha_{ij}\phi_{,j} + f_{ij}\psi_{,j}, \tag{2.34}$$

$$\xi = -p_{ji}\varepsilon_{ji} - d^*\phi - \alpha_1\psi + \gamma_1\theta + vC, \tag{2.35}$$

$$\zeta = -\gamma_{ji}\varepsilon_{ji} - \alpha_1\phi - f\psi + \gamma_2\theta + mC, \tag{2.36}$$

$$\rho S = a_{ji}\varepsilon_{ji} + \gamma_1\phi + \gamma_2\psi + \frac{\rho C_e\theta}{T_0} + aC, \tag{2.37}$$

$$P = -b_{ji}\varepsilon_{ji} - v\phi - m\psi - a\theta + bC. \tag{2.38}$$

Equations (2.1), (2.2), $(2.9)_2$ and $(2.9)_3$ with the aid of equations (2.31)–(2.36) become

$$c_{jikl}\varepsilon_{kl,j} + p_{ji}\phi_{,j} + \gamma_{ji}\psi_{,j} - a_{ji}\theta_{,j} - b_{ji}C_{,j} + \rho F_i = \rho\ddot{u}_i, \tag{2.39}$$

$$\varepsilon_{ijk}(c_{jkpl}\varepsilon_{pl} + p_{jk}\phi + \gamma_{jk}\psi - a_{jk}\theta - b_{jk}C) + f_{ijkl}\varphi_{k,lj} + \rho G_i = \rho J_{ij}\ddot{\varphi}_j, \tag{2.40}$$

$$-p_{ji}\varepsilon_{ji} + q_{ij}\phi_{,ij} - d^*\phi + \alpha_{ij}\psi_{,ij} - \alpha_1\psi + \gamma_1\theta + vC + \rho\tilde{g} = \rho k_1\ddot{\phi}, \tag{2.41}$$

$$-\gamma_{ji}\varepsilon_{ji} + \alpha_{ij}\phi_{,ij} - \alpha_1\phi + f_{ij}\psi_{,ij} - f\psi + \gamma_2\theta + mC + \rho\tilde{l} = \rho k_2\ddot{\psi}. \tag{2.42}$$

The linearized form of equation (2.12) is

$$\rho T_0\dot{S} = -q_{i,i}. \tag{2.43}$$

Using equation (2.37) in equation (2.43), we get

$$a_{ji}T_0\dot{\varepsilon}_{ji} + \gamma_1 T_0\dot{\phi} + \gamma_2 T_0\dot{\psi} + \rho C_e\dot{\theta} + aT_0\dot{C} = -q_{i,i}. \tag{2.44}$$

Generalized Fourier's law of heat conduction equation is

$$q_i + \tau_0\dot{q}_i = -K_{ij}\theta_{,j}, \tag{2.45}$$

where K_{ij} are coefficients of thermal conductivity tensor, τ_0 is the thermal relaxation time which will ensure that the heat conduction equation will predict finite speeds of heat propagation.

Equation (2.45) with the help of equation (2.44) becomes

$$\left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)[T_0(a_{ji}\varepsilon_{ji} + \gamma_1\phi + \gamma_2\psi + aC) + \rho C_e\theta] = K_{ij}\theta_{,ij}. \tag{2.46}$$

Similar to equation (2.45), generalized Fick's law of mass diffusion is

$$\eta_i + \tau^0\dot{\eta}_i = -d_{ij}P_{,j}, \tag{2.47}$$

where d_{ij} are coefficients of diffusion tensor, τ^0 is the diffusion relaxation time which ensures that the equation satisfied by the concentration will also predict finite speeds of propagation of matter from one medium to the other.

Using equations (2.13) and (2.38) in equation (2.47), we get

$$-d_{ij}[b_{kl}\varepsilon_{kl,ij} + v\phi_{,ij} + m\psi_{,ij} + a\theta_{,ij} - bC_{,ij}] = \left(\frac{\partial}{\partial t} + \tau^0\frac{\partial^2}{\partial t^2}\right)C. \tag{2.48}$$

If we take

$$
\begin{gathered}
c_{jikl} = \lambda\delta_{ji}\delta_{kl} + (\mu + K^*)\delta_{jk}\delta_{il} + \mu\delta_{jl}\delta_{ik}, \quad f_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \gamma\delta_{ik}\delta_{jl} + \beta\delta_{il}\delta_{jk},\\
a_{ji} = \xi_1\delta_{ji}, \quad b_{ji} = \xi_2\delta_{ji}, \quad p_{ji} = p_1\delta_{ji}, \quad \gamma_{ji} = p_2\delta_{ji}, \quad q_{ij} = t_1\delta_{ij},\\
\alpha_{ij} = r_1\delta_{ij}, \quad f_{ij} = t_2\delta_{ij}, \quad K_{ij} = K\delta_{ij}, \quad d_{ij} = D\delta_{ij}, \quad J_{ij} = J\delta_{ij},
\end{gathered}
$$

in the equations (2.39)–(2.42), (2.46) and (2.48), the governing equations for homogeneous isotropic generalized micropolar thermoelastic diffusion with double porosity in absence of body forces and couples are obtained as

$$
\begin{gathered}
(\mu + K^*)\Delta\mathbf{u} + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + K^*\,\mathrm{curl}\,\boldsymbol{\varphi} + p_1\,\mathrm{grad}\ \phi\\
+ p_2\,\mathrm{grad}\ \psi - \xi_1\,\mathrm{grad}\ \theta - \xi_2\,\mathrm{grad}\ C = \rho\ddot{\mathbf{u}},\\
(\gamma\Delta - 2K^*)\boldsymbol{\varphi} + (\alpha + \beta)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi} + K^*\,\mathrm{curl}\ \mathbf{u} = \rho J\ddot{\boldsymbol{\varphi}},\\
-p_1\,\mathrm{div}\,\mathbf{u} + (t_1\Delta - d^*)\phi + (r_1\Delta - \alpha_1)\psi + \gamma_1\theta + vC = \rho k_1\ddot{\phi},\\
-p_2\ \mathrm{div}\ \mathbf{u} + (r_1\Delta - \alpha_1)\phi + (t_2\Delta - f)\psi + \gamma_2\theta + mC = \rho k_2\ddot{\psi},\\
\Big(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\Big)[T_0(\xi_1\,\mathrm{div}\mathbf{u} + \gamma_1\phi + \gamma_2\psi + aC) + \rho C_e\theta] = K\Delta\theta,\\
D\Delta[\xi_2\,\mathrm{div}\,\mathbf{u} + v\phi + m\psi + a\theta - bC] + \Big(\frac{\partial}{\partial t} + \tau^0\frac{\partial^2}{\partial t^2}\Big)C = 0,
\end{gathered}
\tag{2.49}
$$

where Δ is Laplacian operator.

In the upcoming sections, the chemical potential has been used as a state variable rather than concentration. In isotropic medium, the equation (2.38) becomes

$$
P = -\xi_2\,\mathrm{div}\,\mathbf{u} - v\phi - m\psi - a\theta + bC. \tag{2.50}
$$

The system of equations (2.49) with the aid of equation (2.50) can be rewritten as

$$
\begin{gathered}
(\mu + K^*)\Delta\mathbf{u} + (\lambda' + \mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + K^*\,\mathrm{curl}\,\boldsymbol{\varphi} + g_1\,\mathrm{grad}\,\phi\\
+ g_2\,\mathrm{grad}\,\psi - l_1\,\mathrm{grad}\,\theta - l_2\,\mathrm{grad}\,P = \rho\ddot{\mathbf{u}},\\
(\gamma\Delta - 2K^*)\boldsymbol{\varphi} + (\alpha + \beta)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi} + K^*\,\mathrm{curl}\ \mathbf{u} = \rho J\ddot{\boldsymbol{\varphi}},\\
-g_1\,\mathrm{div}\,\mathbf{u} + (t_1\Delta - d_1)\phi + (r_1\Delta - \varepsilon_{11})\psi + \xi_{11}\theta + wP = \rho k_1\ddot{\phi},\\
-g_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \varepsilon_{11})\phi + (t_2\Delta - d_2)\psi + \xi_{22}\theta + \nu P = \rho k_2\ddot{\psi},\\
-\Big(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\Big)T_0[l_1\,\mathrm{div}\,\mathbf{u} + \xi_{11}\phi + \xi_{22}\psi + \eta\theta + sP] + K\Delta\theta = 0,\\
-\Big(\frac{\partial}{\partial t} + \tau^0\frac{\partial^2}{\partial t^2}\Big)[l_2\,\mathrm{div}\,\mathbf{u} + w\phi + \nu\psi + s\theta + nP] + D\Delta P = 0,
\end{gathered}
\tag{2.51}
$$

where

$$
\begin{gathered}
n = \frac{1}{b}, \quad l_2 = n\xi_2, \quad g_1 = p_1 - vl_2, \quad l_1 = \xi_1 + al_2, \quad g_2 = p_2 - ml_2,\\
\lambda' = \lambda - l_2\xi_2, \quad s = an, \quad w = vn, \quad \nu = mn, \quad d_1 = d^* - vw, \quad \varepsilon_{11} = \alpha_1 - v\nu,\\
\xi_{11} = \gamma_1 + vs, \quad d_2 = f - m\nu, \quad \xi_{22} = \gamma_2 + ms, \quad \eta = \frac{\rho C_e}{T_0} + as.
\end{gathered}
$$

3. Steady Oscillations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space E^3,

$$|\mathbf{x}| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \quad \mathbf{D_x} = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big).$$

The displacement vector, microrotation vector, volume fraction fields, temperature change and chemical potential functions are assumed as:

$$\Big[\mathbf{u}(\mathbf{x},t), \boldsymbol{\varphi}(\mathbf{x},t), \phi(\mathbf{x},t), \psi(\mathbf{x},t), \theta(\mathbf{x},t), P(\mathbf{x},t)\Big] = \mathrm{Re}\Big[(\mathbf{u}^*, \boldsymbol{\varphi}^*, \phi^*, \psi^*, \theta^*, P^*)e^{-\iota\omega t}\Big], \tag{3.1}$$

where ω is oscillation frequency.

Using equation (3.1) in the system of equations (2.51) and omitting asterisk (*) for simplicity, the system of equations of steady oscillations are obtained as

$$\begin{aligned}
&[(\mu + K^*)\Delta + (\lambda' + \mu)\,\mathrm{grad}\,\mathrm{div} + \rho\omega^2]\mathbf{u} + K^*\,\mathrm{curl}\,\boldsymbol{\varphi} + g_1\,\mathrm{grad}\;\phi \\
&\qquad + g_2\,\mathrm{grad}\;\psi - l_1\,\mathrm{grad}\;\theta - l_2\,\mathrm{grad}\;P = \mathbf{0}, \\
&[\gamma\Delta - 2K^* + (\alpha+\beta)\,\mathrm{grad}\,\mathrm{div} + \rho J\omega^2]\boldsymbol{\varphi} + K^*\,\mathrm{curl}\;\mathbf{u} = \mathbf{0}, \\
&-g_1\,\mathrm{div}\,\mathbf{u} + \Big[t_1\Delta - d_1 + \rho k_1\omega^2\Big]\phi + (r_1\Delta - \varepsilon_{11})\psi + \xi_{11}\theta + wP = 0, \\
&-g_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \varepsilon_{11})\phi + [t_2\Delta - d_2 + \rho k_2\omega^2]\psi + \xi_{22}\theta + \nu P = 0, \\
&\tau_1 T_0[l_1\,\mathrm{div}\mathbf{u} + \xi_{11}\phi + \xi_{22}\psi] + [K\Delta + \tau_1\eta T_0]\theta + \tau_1 s T_0 P = 0, \\
&\tau^1[l_2\,\mathrm{div}\;\mathbf{u} + w\phi + \nu\psi + s\theta] + [D\Delta + \tau^1 n]P = 0,
\end{aligned} \tag{3.2}$$

where

$$\tau_1 = \iota\omega(1 - \iota\omega\tau_0),\; \tau^1 = \iota\omega(1 - \iota\omega\tau^0).$$

We introduce the second order matrix differential operators with constant coefficients

$$\mathbf{F}(\mathbf{D_x}) = \big(F_{gh}(\mathbf{D_x})\big)_{10\times 10}$$

where

$$F_{pq}(\mathbf{D_x}) = [\tilde{\mu}\Delta + \rho\omega^2]\delta_{pq} + (\lambda' + \mu)\frac{\partial^2}{\partial x_p \partial x_q},$$

$$F_{p;q+3}(\mathbf{D_x}) = F_{p+3;q}(\mathbf{D_x}) = K^*\sum_{i=1}^{3}\varepsilon_{piq}\frac{\partial}{\partial x_i},$$

$$F_{p7}(\mathbf{D_x}) = -F_{7p}(\mathbf{D_x}) = g_1\frac{\partial}{\partial x_p},$$

$$F_{p8}(\mathbf{D_x}) = -F_{8p}(\mathbf{D_x}) = g_2\frac{\partial}{\partial x_p},$$

$$F_{p9}(\mathbf{D_x}) = -l_1\frac{\partial}{\partial x_p},$$

$$F_{p;10}(\mathbf{D_x}) = -l_2 \frac{\partial}{\partial x_p},$$
$$F_{p+3;q+3}(\mathbf{D_x}) = (\gamma\Delta + \tilde{K})\delta_{pq} + (\alpha+\beta)\frac{\partial^2}{\partial x_p \partial x_q},$$
$$F_{p+3;j}(\mathbf{D_x}) = F_{j;p+3}(\mathbf{D_x}) = 0,$$
$$F_{77}(\mathbf{D_x}) = t_1\Delta + d_3,$$
$$F_{78}(\mathbf{D_x}) = F_{87}(\mathbf{D_x}) = r_1\Delta - \varepsilon_{11},$$
$$F_{79}(\mathbf{D_x}) = \xi_{11},$$
$$F_{7;10}(\mathbf{D_x}) = w,$$
$$F_{88}(\mathbf{D_x}) = t_2\Delta + d_4,$$
$$F_{89}(\mathbf{D_x}) = \xi_{22},$$
$$F_{8;10}(\mathbf{D_x}) = \nu,$$
$$F_{9q}(\mathbf{D_x}) = \tau_1 l_1 T_0 \frac{\partial}{\partial x_q},$$
$$F_{97}(\mathbf{D_x}) = \tau_1 \xi_{11} T_0,$$
$$F_{98}(\mathbf{D_x}) = \tau_1 \xi_{22} T_0,$$
$$F_{99}(\mathbf{D_x}) = K\Delta + \tau_1 \eta T_0,$$
$$F_{9;10}(\mathbf{D_x}) = \tau_1 s T_0,$$
$$F_{10;q}(\mathbf{D_x}) = \tau^1 l_2 \frac{\partial}{\partial x_q},$$
$$F_{10;7}(\mathbf{D_x}) = \tau^1 w,$$
$$F_{10;8}(\mathbf{D_x}) = \tau^1 \nu,$$
$$F_{10;9}(\mathbf{D_x}) = \tau^1 s,$$
$$F_{10;10}(\mathbf{D_x}) = D\Delta + \tau^1 n,$$
$$\tilde{\mu} = \mu + K^*,$$
$$\tilde{K} = -2K^* + \rho J \omega^2,$$
$$d_3 = -d_1 + \rho k_1 \omega^2,$$
$$d_4 = -d_2 + \rho k_2 \omega^2 \quad p,q = 1,2,3 \ j = 7,8,9,10,$$

and

$$\tilde{\mathbf{F}}(\mathbf{D_x}) = \big(\tilde{F}_{gh}(\mathbf{D_x})\big)_{10\times 10},$$

where

$$\tilde{F}_{pq}(\mathbf{D_x}) = \tilde{\mu}\Delta\delta_{pq} + (\lambda' + \mu)\frac{\partial^2}{\partial x_p \partial x_q},$$
$$\tilde{F}_{p+3;q+3}(\mathbf{D_x}) = \gamma\Delta\delta_{pq} + (\alpha+\beta)\frac{\partial^2}{\partial x_p \partial x_q},$$
$$\tilde{F}_{77}(\mathbf{D_x}) = t_1\Delta,$$
$$\tilde{F}_{78}(\mathbf{D_x}) = \tilde{F}_{87}(\mathbf{D_x}) = r_1\Delta,$$
$$\tilde{F}_{88}(\mathbf{D_x}) = t_2\Delta,$$
$$\tilde{F}_{99}(\mathbf{D_x}) = K\Delta,$$

$$\tilde{F}_{10;10}(\mathbf{D_x}) = D\Delta,$$
$$\tilde{F}_{p;q+3}(\mathbf{D_x}) = \tilde{F}_{p+3;q}(\mathbf{D_x}) = \tilde{F}_{le}(\mathbf{D_x}) = \tilde{F}_{el}(\mathbf{D_x}) = 0,$$
$$\tilde{F}_{ij}(\mathbf{D_x}) = \tilde{F}_{ji}(\mathbf{D_x}) = \tilde{F}_{9;10}(\mathbf{D_x}) = \tilde{F}_{10;9}(\mathbf{D_x}) = 0$$
$$p, q = 1, 2, 3 \quad e = 1, \ldots, 6 \quad i = 7, 8 \quad j = 9, 10 \quad l = 7, \ldots, 10.$$

The system of equations (3.2) can be represented as

$$\mathbf{F}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \phi, \psi, \theta, P)$ is a ten-component vector function on E^3. The matrix $\tilde{\mathbf{F}}(\mathbf{D_x})$ is called the principal part of operator $\mathbf{F}(\mathbf{D_x})$.

Definition 3.1. The operator $\mathbf{F}(\mathbf{D_x})$ is said to be elliptic if $|\tilde{\mathbf{F}}(\boldsymbol{\kappa})| \neq 0$, where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3)$.

Since $|\tilde{\mathbf{F}}(\boldsymbol{\kappa})| = \tilde{\mu}^2\gamma^2\tilde{\lambda}\tilde{\alpha}KD\sigma|\boldsymbol{\kappa}|^{20}$, $\tilde{\lambda} = \lambda' + 2\mu + K^*$, $\tilde{\alpha} = \alpha + \beta + \gamma$, $\sigma = t_1t_2 - r_1^2$. Therefore operator $\mathbf{F}(\mathbf{D_x})$ is an elliptic differential operator iff

$$\tilde{\mu}\gamma\tilde{\lambda}\tilde{\alpha}KD\sigma \neq 0. \tag{3.3}$$

Definition 3.2. The fundamental solution of the system of equations (3.2) (the fundamental matrix of operator $\mathbf{F}$) is the matrix $\mathbf{G}(\mathbf{x}) = \big(G_{gh}(\mathbf{x})\big)_{10\times 10}$ satisfying condition

$$\mathbf{F}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}), \tag{3.4}$$

where $\delta(\mathbf{x})$ is the Dirac delta, $\mathbf{I} = \big(\delta_{gh}\big)_{10\times 10}$ is the unit matrix and $\mathbf{x} \in \mathrm{E}^3$.

4. Construction of $\mathbf{G}(\mathbf{x})$ in terms of Elementary Functions

Let us consider the system of non-homogeneous equations

$$\begin{gathered}
[\tilde{\mu}\Delta + (\lambda' + \mu)\,\mathrm{grad\,div} + \rho\omega^2]\mathbf{u} + K^*\,\mathrm{curl}\,\boldsymbol{\varphi} - g_1\,\mathrm{grad}\,\phi \\
\qquad - g_2\,\mathrm{grad}\,\psi + \tau_1 l_1 T_0\,\mathrm{grad}\,\theta + \tau^1 l_2\,\mathrm{grad}\,P = \mathbf{H}, \\
K^*\,\mathrm{curl}\,\mathbf{u} + [\gamma\Delta + \tilde{K} + (\alpha+\beta)\,\mathrm{grad\,div}]\boldsymbol{\varphi} = \mathbf{V}, \\
g_1\,\mathrm{div}\,\mathbf{u} + (t_1\Delta + d_3)\phi + (r_1\Delta - \varepsilon_{11})\psi + \tau_1\xi_{11}T_0\theta + \tau^1 wP = L, \\
g_2\,\mathrm{div}\,\mathbf{u} + (r_1\Delta - \varepsilon_{11})\phi + (t_2\Delta + d_4)\psi + \tau_1\xi_{22}T_0\theta + \tau^1\nu P = M, \\
-l_1\,\mathrm{div}\mathbf{u} + \xi_{11}\phi + \xi_{22}\psi + (K\Delta + \tau_1\eta T_0)\theta + \tau^1 sP = Z, \\
-l_2\,\mathrm{div}\,\mathbf{u} + w\phi + \nu\psi + \tau_1 sT_0\theta + (D\Delta + \tau^1 n)P = X,
\end{gathered} \tag{4.1}$$

where $\mathbf{H}, \mathbf{V}$ are three-component vector functions on E^3; L, M, Z and X are scalar functions on E^3.

The system of equations (4.1) may also be written in the form

$$\mathbf{F}^{tr}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{Q}(\mathbf{x}), \tag{4.2}$$

where $\mathbf{F}^{tr}$ is the transpose of matrix $\mathbf{F}$, $\mathbf{Q} = (\mathbf{H}, \mathbf{V}, L, M, Z, X)$ and $\mathbf{x} \in \mathrm{E}^3$.

Applying operator div to the equations $(4.1)_1$ and $(4.1)_2$, we obtain

$$[\tilde{\lambda}\Delta + \rho\omega^2]\,\mathrm{div}\,\mathbf{u} - g_1\Delta\phi - g_2\Delta\psi + \tau_1 l_1 T_0\Delta\theta + \tau^1 l_2\Delta P = \mathrm{div}\,\mathbf{H}, \tag{4.3}$$

$$[\tilde{\alpha}\Delta + \tilde{K}]\operatorname{div}\boldsymbol{\varphi} = \operatorname{div}\mathbf{V}. \tag{4.4}$$

The equations $(4.1)_3$–$(4.1)_6$ and (4.3) may be expressed in the form

$$\mathbf{N}(\Delta)\mathbf{S} = \tilde{\mathbf{Q}}, \tag{4.5}$$

where $\mathbf{S} = (\operatorname{div}\mathbf{u}, \phi, \psi, \theta, P)$, $\tilde{\mathbf{Q}} = (w_1, \dots, w_5) = (\operatorname{div}\mathbf{H}, L, M, Z, X)$ and

$$\mathbf{N}(\Delta) = \big(N_{gh}(\Delta)\big)_{5\times 5} = \begin{pmatrix} \tilde{\lambda}\Delta + \rho\omega^2 & -g_1\Delta & -g_2\Delta & \tau_1 l_1 T_0\Delta & \tau^1 l_2\Delta \\ g_1 & t_1\Delta + d_3 & r_1\Delta - \varepsilon_{11} & \tau_1\xi_{11}T_0 & \tau^1 w \\ g_2 & r_1\Delta - \varepsilon_{11} & t_2\Delta + d_4 & \tau_1\xi_{22}T_0 & \tau^1\nu \\ -l_1 & \xi_{11} & \xi_{22} & K\Delta + \tau_1\eta T_0 & \tau^1 s \\ -l_2 & w & \nu & \tau_1 s T_0 & D\Delta + \tau^1 n \end{pmatrix}_{5\times 5} \tag{4.6}$$

The equations $(4.1)_3$–$(4.1)_6$ and (4.3) may also be written as

$$\Gamma_1(\Delta)\mathbf{S} = \mathbf{\Psi}, \tag{4.7}$$

where

$$\mathbf{\Psi} = (\Psi_1, \dots, \Psi_5), \quad \Psi_p = \frac{1}{M^*}\sum_{i=1}^{5} N^*_{ip} w_i,$$

$$\Gamma_1(\Delta) = \frac{1}{M^*}\,|\mathbf{N}(\Delta)|, \quad M^* = \tilde{\lambda} K D\sigma, \qquad p = 1, \dots, 5 \tag{4.8}$$

and N^*_{ip} is the cofactor of the element N_{ip} of the matrix $\mathbf{N}$.

From equations (4.6) and (4.8), we see that

$$\Gamma_1(\Delta) = \prod_{i=1}^{5}(\Delta + \lambda_i^2)$$

where λ_i^2, $i = 1, \dots, 5$ are the roots of the equation $\Gamma_1(-\kappa) = 0$ (with respect to κ).

Applying operators $\gamma\Delta + \tilde{K}$ and $K^*\operatorname{curl}$ to the equations $(4.1)_1$ and $(4.1)_2$, respectively, we obtain

$$\begin{aligned}(\gamma\Delta + \tilde{K})[\tilde{\mu}\Delta + (\lambda' + \mu)\operatorname{grad}\operatorname{div} + \rho\omega^2]\mathbf{u} + (\gamma\Delta + \tilde{K})K^*\operatorname{curl}\boldsymbol{\varphi} \\ = (\gamma\Delta + \tilde{K})[\mathbf{H} + g_1\operatorname{grad}\phi + g_2\operatorname{grad}\psi - \tau_1 l_1 T_0\operatorname{grad}\theta - \tau^1 l_2\operatorname{grad}P]\end{aligned} \tag{4.9}$$

and

$$(\gamma\Delta + \tilde{K})K^*\operatorname{curl}\boldsymbol{\varphi} = -K^{*^2}\operatorname{curl}\operatorname{curl}\mathbf{u} + K^*\operatorname{curl}\mathbf{V}. \tag{4.10}$$

Now

$$\operatorname{curl}\operatorname{curl}\mathbf{u} = \operatorname{grad}\operatorname{div}\mathbf{u} - \Delta\mathbf{u}. \tag{4.11}$$

Using equations (4.10) and (4.11) in equation (4.9), we obtain

$$\begin{aligned}&\{[(\gamma\Delta + \tilde{K})\tilde{\mu} + K^{*^2}]\Delta + (\gamma\Delta + \tilde{K})\rho\omega^2\}\mathbf{u} \\ &= -[(\gamma\Delta + \tilde{K})(\lambda' + \mu) - K^{*^2}]\operatorname{grad}\operatorname{div}\mathbf{u} + (\gamma\Delta + \tilde{K}) \\ &\times [\mathbf{H} + g_1\operatorname{grad}\phi + g_2\operatorname{grad}\psi - \tau_1 l_1 T_0\operatorname{grad}\theta - \tau^1 l_2\operatorname{grad}P] - K^*\operatorname{curl}\mathbf{V}.\end{aligned} \tag{4.12}$$

Applying operator $\Gamma_1(\Delta)$ to the equation (4.12) and using equation (4.7), we get

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathbf{u} = \mathbf{\Psi}', \tag{4.13}$$

where

$$\Gamma_2(\Delta) = \frac{1}{N^*}\begin{vmatrix} \tilde{\mu}\Delta + \rho\omega^2 & K^*\Delta \\ -K^* & \gamma\Delta + \tilde{K} \end{vmatrix}, \quad N^* = \gamma\tilde{\mu}$$

and

$$\begin{aligned} \mathbf{\Psi}' = \frac{1}{N^*}\{-[(\gamma\Delta + \tilde{K})(\lambda' + \mu) - {K^*}^2]\,\mathrm{grad}\ \Psi_1 + (\gamma\Delta + \tilde{K}) \\ \times [\Gamma_1(\Delta)\mathbf{H} + g_1\,\mathrm{grad}\ \Psi_2 + g_2\ \mathrm{grad}\,\Psi_3 - \tau_1 l_1 T_0\,\mathrm{grad}\ \Psi_4 \\ - \tau^1 l_2\,\mathrm{grad}\ \Psi_5] - K^*\Gamma_1(\Delta)\,\mathrm{curl}\,\mathbf{V}\}. \end{aligned} \tag{4.14}$$

It can be seen that

$$\Gamma_2(\Delta) = (\Delta + \lambda_6^2)(\Delta + \lambda_7^2),$$

where λ_6^2, λ_7^2 are the roots of the equation $\Gamma_2(-\kappa) = 0$ (with respect to κ).

From equation (4.4), it follows that

$$(\Delta + \lambda_8^2)\,\mathrm{div}\,\boldsymbol{\varphi} = \frac{1}{\tilde{\alpha}}\,\mathrm{div}\,\mathbf{V} = \Psi_6, \quad \lambda_8^2 = \frac{\tilde{K}}{\tilde{\alpha}}. \tag{4.15}$$

Applying operators K^* curl and $\tilde{\mu}\Delta + \rho\omega^2$ to the equations $(4.1)_1$ and $(4.1)_2$, respectively, we acquire

$$[\tilde{\mu}\Delta + \rho\omega^2]K^*\,\mathrm{curl}\,\mathbf{u} = K^*\,\mathrm{curl}\,\mathbf{H} - {K^*}^2\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{\varphi}, \tag{4.16}$$

and

$$\begin{aligned} (\tilde{\mu}\Delta + \rho\omega^2)(\gamma\Delta + \tilde{K})\boldsymbol{\varphi} + (\alpha + \beta)(\tilde{\mu}\Delta + \rho\omega^2)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi} \\ + K^*(\tilde{\mu}\Delta + \rho\omega^2)\,\mathrm{curl}\,\mathbf{u} = (\tilde{\mu}\Delta + \rho\omega^2)\mathbf{V}. \end{aligned} \tag{4.17}$$

Now

$$\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{\varphi} = \mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi} - \Delta\boldsymbol{\varphi}. \tag{4.18}$$

Using equations (4.16) and (4.18) in equation (4.17), we obtain

$$\begin{aligned} &\{[(\gamma\Delta + \tilde{K})\tilde{\mu} + {K^*}^2]\Delta + (\gamma\Delta + \tilde{K})\rho\omega^2\}\boldsymbol{\varphi} \\ &= -[(\alpha + \beta)(\tilde{\mu}\Delta + \rho\omega^2) - {K^*}^2]\ \mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi} + (\tilde{\mu}\Delta + \rho\omega^2)\mathbf{V} - K^*\,\mathrm{curl}\,\mathbf{H}. \end{aligned} \tag{4.19}$$

Applying operator $\Delta + \lambda_8^2$ to the equation (4.19) and using equation (4.15), we get

$$\Gamma_2(\Delta)(\Delta + \lambda_8^2)\boldsymbol{\varphi} = \mathbf{\Psi}'', \tag{4.20}$$

where

$$\begin{aligned} \mathbf{\Psi}'' = \frac{1}{N^*}\{-[(\alpha + \beta)(\tilde{\mu}\Delta + \rho\omega^2) - {K^*}^2]\,\mathrm{grad}\ \Psi_6 \\ + (\tilde{\mu}\Delta + \rho\omega^2)(\Delta + \lambda_8^2)\mathbf{V} - K^*(\Delta + \lambda_8^2)\,\mathrm{curl}\,\mathbf{H}\}. \end{aligned} \tag{4.21}$$

From equations (4.7), (4.13) and (4.20), we obtain

$$\mathbf{\Theta}(\Delta)\mathbf{U}(\mathbf{x}) = \hat{\mathbf{\Psi}}(\mathbf{x}), \tag{4.22}$$

where $\hat{\mathbf{\Psi}} = (\mathbf{\Psi}', \mathbf{\Psi}'', \Psi_2, \Psi_3, \Psi_4, \Psi_5)$ and

$$\mathbf{\Theta}(\Delta) = \big(\Theta_{gh}(\Delta)\big)_{10\times 10}$$

$$\Theta_{pp}(\Delta) = \Gamma_1(\Delta)\Gamma_2(\Delta) = \prod_{i=1}^{7}(\Delta + \lambda_i^2),$$

$$\Theta_{p+3;p+3}(\Delta) = \Gamma_2(\Delta)(\Delta + \lambda_8^2) = \prod_{i=6}^{8}(\Delta + \lambda_i^2),$$

$$\Theta_{gh}(\Delta) = 0,\ \Theta_{jj}(\Delta) = \Gamma_1(\Delta) = \prod_{i=1}^{5}(\Delta + \lambda_i^2),$$

$$p = 1,2,3 \quad g,h = 1,\dots,10 \quad j = 7,8,9,10 \quad g \neq h$$

The equations (4.8), (4.14) and (4.21) can be rewritten in the form

$$\mathbf{\Psi}' = \Big[\frac{1}{N^*}(\gamma\Delta + \tilde{K})\Gamma_1(\Delta)\mathbf{J} + w_{11}(\Delta)\,\mathrm{grad\ div}\Big]\mathbf{H} + w_{21}(\Delta)\,\mathrm{curl}\ \mathbf{V} + \sum_{i=2}^{5} w_{i+1;1}(\Delta)\,\mathrm{grad}\ w_i,$$

$$\mathbf{\Psi}'' = w_{12}(\Delta)\,\mathrm{curl}\,\mathbf{H} + \Big[\frac{1}{N^*}(\Delta + \lambda_8^2)(\tilde{\mu}\Delta + \rho\omega^2)\mathbf{J} + w_{22}(\Delta)\,\mathrm{grad\,div}\Big]\mathbf{V}, \tag{4.23}$$

$$\Psi_p = w_{1;p+1}(\Delta)\,\mathrm{div}\,\mathbf{H} + \sum_{i=2}^{5} w_{i+1;p+1}(\Delta) w_i \quad p = 2,3,4,5,$$

where $\mathbf{J} = \big(\delta_{gh}\big)_{3\times 3}$ is the unit matrix.

In the equations (4.23), the following notations have been used:

$$w_{11}(\Delta) = -\frac{1}{M^*N^*}\{[(\lambda' + \mu)(\gamma\Delta + \tilde{K}) - K^{*^2}]N_{11}^*(\Delta) + (\gamma\Delta + \tilde{K}) \times [-g_1 N_{12}^*(\Delta) - g_2 N_{13}^*(\Delta) + \tau_1 l_1 T_0 N_{14}^*(\Delta) + \tau^1 l_2 N_{15}^*(\Delta)]\},$$

$$w_{p+1;1}(\Delta) = -\frac{1}{M^*N^*}\{[(\lambda' + \mu)(\gamma\Delta + \tilde{K}) - K^{*^2}]N_{p1}^*(\Delta) + (\gamma\Delta + \tilde{K}) \times [-g_1 N_{p2}^*(\Delta) - g_2 N_{p3}^*(\Delta) + \tau_1 l_1 T_0 N_{p4}^*(\Delta) + \tau^1 l_2 N_{p5}^*(\Delta)]\},$$

$$w_{21}(\Delta) = -\frac{K^*\Gamma_1(\Delta)}{N^*}, \quad w_{12}(\Delta) = -\frac{K^*(\Delta + \lambda_8^2)}{N^*},$$

$$w_{22}(\Delta) = -\frac{(\alpha + \beta)(\tilde{\mu}\Delta + \rho\omega^2) - K^{*^2}}{N^*\tilde{\alpha}},$$

$$w_{1;p+1}(\Delta) = \frac{N_{1p}^*(\Delta)}{M^*}, \quad w_{i+1;p+1}(\Delta) = \frac{N_{ip}^*(\Delta)}{M^*} \qquad i,p = 2,\dots,5.$$

From equations (4.23), we have

$$\hat{\mathbf{\Psi}}(\mathbf{x}) = \mathbf{R}^{tr}(\mathbf{D_x})\mathbf{Q}(\mathbf{x}), \tag{4.24}$$

where

$$\mathbf{R}(\mathbf{D_x}) = \big(R_{gh}(\mathbf{D_x})\big)_{10\times 10} = \begin{pmatrix} \mathbf{R}^{(1)} & \mathbf{R}^{(2)} & \mathbf{R}^{(5)} \\ \mathbf{R}^{(3)} & \mathbf{R}^{(4)} & \mathbf{R}^{(6)} \\ \mathbf{R}^{(7)} & \mathbf{R}^{(8)} & \mathbf{R}^{(9)} \end{pmatrix}_{10\times 10},$$

$$\mathbf{R}^{(i)}(\mathbf{D_x}) = \big(R^{(i)}_{gh}(\mathbf{D_x})\big)_{3\times 3}, \quad \mathbf{R}^{(j)}(\mathbf{D_x}) = \big(R^{(j)}_{gh}(\mathbf{D_x})\big)_{3\times 4},$$

$$\mathbf{R}^{(l)}(\mathbf{D_x}) = \big(R^{(l)}_{gh}(\mathbf{D_x})\big)_{4\times 3}, \quad \mathbf{R}^{(9)}(\mathbf{D_x}) = \big(R^{(9)}_{gh}(\mathbf{D_x})\big)_{4\times 4},$$

$$R^{(1)}_{gh}(\mathbf{D_x}) = \frac{1}{N^*}(\gamma\Delta + \tilde{K})\Gamma_1(\Delta)\delta_{gh} + w_{11}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h}, \tag{4.25}$$

$$R^{(2)}_{gh}(\mathbf{D_x}) = w_{12}(\Delta)\sum_{p=1}^{3}\varepsilon_{gph}\frac{\partial}{\partial x_p}, \quad R^{(3)}_{gh}(\mathbf{D_x}) = w_{21}(\Delta)\sum_{p=1}^{3}\varepsilon_{gph}\frac{\partial}{\partial x_p},$$

$$R^{(4)}_{gh}(\mathbf{D_x}) = \frac{1}{N^*}(\Delta + \lambda_8^2)(\tilde{\mu}\Delta + \rho\omega^2)\delta_{gh} + w_{22}(\Delta)\frac{\partial^2}{\partial x_g \partial x_h},$$

$$R^{(5)}_{gh}(\mathbf{D_x}) = w_{1;h+2}(\Delta)\frac{\partial}{\partial x_g}, \quad R^{(6)}_{gh}(\mathbf{D_x}) = R^{(8)}_{gh}(\mathbf{D_x}) = 0,$$

$$R^{(7)}_{gh}(\mathbf{D_x}) = w_{g+2;1}(\Delta)\frac{\partial}{\partial x_h},$$

$$R^{(9)}_{gh}(\mathbf{D_x}) = w_{g+2;h+2}(\Delta) \quad i = 1,\dots,4 \;\; j = 5,6 \;\; l = 7,8.$$

From equations (4.2), (4.22) and (4.24), we obtain

$$\mathbf{\Theta U} = \mathbf{R}^{tr}\mathbf{F}^{tr}\mathbf{U}.$$

The above relation implies

$$\mathbf{R}^{tr}\mathbf{F}^{tr} = \mathbf{\Theta}.$$

Therefore, we obtain

$$\mathbf{F}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x}) = \mathbf{\Theta}(\Delta). \tag{4.26}$$

We assume that

$$\lambda_p^2 \neq \lambda_q^2 \neq 0 \quad p,q = 1,\dots,8 \;\; p \neq q.$$

Let

$$\mathbf{Y}(\mathbf{x}) = \big(Y_{ij}(\mathbf{x})\big)_{10\times 10}, \quad Y_{pp}(\mathbf{x}) = \sum_{g=1}^{7} r_{1g}\varsigma_g(\mathbf{x}), \quad Y_{p+3;p+3}(\mathbf{x}) = \sum_{g=6}^{8} r_{2g}\varsigma_g(\mathbf{x}),$$

$$Y_{ll}(\mathbf{x}) = \sum_{g=1}^{5} r_{3g}\varsigma_g(\mathbf{x}), Y_{qh}(\mathbf{x}) = 0 \quad p = 1,2,3 \;\; l = 7,\dots,10 \;\; q,h = 1,\dots,10 \;\; q \neq h,$$

where

$$\varsigma_g(\mathbf{x}) = -\frac{e^{\iota\lambda_g|\mathbf{x}|}}{4\pi|\mathbf{x}|}, \quad r_{1l} = \prod_{i=1, i\neq l}^{7}(\lambda_i^2 - \lambda_l^2)^{-1}, \quad r_{2h} = \prod_{i=6, i\neq h}^{8}(\lambda_i^2 - \lambda_h^2)^{-1}, \tag{4.27}$$

$$r_{3e} = \prod_{i=1, i\neq e}^{5}(\lambda_i^2 - \lambda_e^2)^{-1} \quad g = 1,\dots,8 \;\; l = 1,\dots,7 \;\; h = 6,7,8 \;\; e = 1,\dots,5.$$

LEMMA 4.1. *The matrix* $\mathbf{Y}$ *defined above is the fundamental matrix of operator* $\mathbf{\Theta}(\Delta)$*, i.e.*

$$\mathbf{\Theta}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{I}(\mathbf{x}) \tag{4.28}$$

PROOF. To prove the lemma, it is sufficient to prove that

$$\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{11}(\mathbf{x}) = \delta(\mathbf{x}), \quad \Gamma_2(\Delta)(\Delta+\lambda_8^2)Y_{44}(\mathbf{x}) = \delta(\mathbf{x}), \tag{4.29}$$
$$\Gamma_1(\Delta)Y_{77}(\mathbf{x}) = \delta(\mathbf{x}).$$

Consider

$$\sum_{i=1}^{7} r_{1i} = \frac{\sum_{j=1}^{7}(-1)^{j+1}z_j}{z_8},$$

where

$$z_1 = \prod_{i=3}^{7}(\lambda_2^2-\lambda_i^2)\prod_{j=4}^{7}(\lambda_3^2-\lambda_j^2)\prod_{l=5}^{7}(\lambda_4^2-\lambda_l^2)\prod_{p=6}^{7}(\lambda_5^2-\lambda_p^2)(\lambda_6^2-\lambda_7^2),$$

$$z_2 = \prod_{i=3}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{7}(\lambda_3^2-\lambda_j^2)\prod_{l=5}^{7}(\lambda_4^2-\lambda_l^2)\prod_{p=6}^{7}(\lambda_5^2-\lambda_p^2)(\lambda_6^2-\lambda_7^2),$$

$$z_3 = \prod_{i=2,i\neq3}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=4}^{7}(\lambda_2^2-\lambda_j^2)\prod_{l=5}^{7}(\lambda_4^2-\lambda_l^2)\prod_{p=6}^{7}(\lambda_5^2-\lambda_p^2)(\lambda_6^2-\lambda_7^2),$$

$$z_4 = \prod_{i=2,i\neq4}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq4}^{7}(\lambda_2^2-\lambda_j^2)\prod_{l=5}^{7}(\lambda_3^2-\lambda_l^2)\prod_{p=6}^{7}(\lambda_5^2-\lambda_p^2)(\lambda_6^2-\lambda_7^2),$$

$$z_5 = \prod_{i=2,i\neq5}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq5}^{7}(\lambda_2^2-\lambda_j^2)\prod_{l=4,l\neq5}^{7}(\lambda_3^2-\lambda_l^2)\prod_{p=6}^{7}(\lambda_4^2-\lambda_p^2)(\lambda_6^2-\lambda_7^2),$$

$$z_6 = \prod_{i=2,i\neq6}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=3,j\neq6}^{7}(\lambda_2^2-\lambda_j^2)\prod_{l=4,l\neq6}^{7}(\lambda_3^2-\lambda_l^2)\prod_{p=5,p\neq6}^{7}(\lambda_4^2-\lambda_p^2)(\lambda_5^2-\lambda_7^2),$$

$$z_7 = \prod_{i=2}^{6}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{6}(\lambda_2^2-\lambda_j^2)\prod_{l=4}^{6}(\lambda_3^2-\lambda_l^2)\prod_{p=5}^{6}(\lambda_4^2-\lambda_p^2)(\lambda_5^2-\lambda_6^2),$$

$$z_8 = \prod_{i=2}^{7}(\lambda_1^2-\lambda_i^2)\prod_{j=3}^{7}(\lambda_2^2-\lambda_j^2)\prod_{l=4}^{7}(\lambda_3^2-\lambda_l^2)\prod_{p=5}^{7}(\lambda_4^2-\lambda_p^2)\prod_{e=6}^{7}(\lambda_5^2-\lambda_e^2)(\lambda_6^2-\lambda_7^2).$$

On simplifying the right hand side of above relation, we obtain

$$\sum_{i=1}^{7} r_{1i} = 0. \tag{4.30}$$

Similarly, we find that

$$\sum_{i=2}^{7} r_{1i}(\lambda_1^2-\lambda_i^2) = 0, \quad \sum_{i=3}^{7} r_{1i}\Big[\prod_{j=1}^{2}(\lambda_j^2-\lambda_i^2)\Big] = 0,$$

$$\sum_{i=4}^{7} r_{1i}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_i^2)\right]=0, \quad \sum_{i=5}^{7} r_{1i}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_i^2)\right]=0, \tag{4.31}$$

$$\sum_{i=6}^{7} r_{1i}\left[\prod_{j=1}^{5}(\lambda_j^2-\lambda_i^2)\right]=0, \quad \prod_{j=1}^{6} r_{17}(\lambda_j^2-\lambda_7^2)=1.$$

Also,

$$(\Delta+\lambda_p^2)\varsigma_g(\mathbf{x})=\delta(\mathbf{x})+(\lambda_p^2-\lambda_g^2)\varsigma_g(\mathbf{x}) \quad p,g=1,\dots,8. \tag{4.32}$$

Now consider

$$\begin{aligned}\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{11}(\mathbf{x}) &= \prod_{i=1}^{7}(\Delta+\lambda_i^2)\sum_{g=1}^{7} r_{1g}\varsigma_g(\mathbf{x})\\ &= \prod_{i=2}^{7}(\Delta+\lambda_i^2)\sum_{g=1}^{7} r_{1g}[\delta(\mathbf{x})+(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})]\\ &= \prod_{i=2}^{7}(\Delta+\lambda_i^2)\left[\delta(\mathbf{x})\sum_{g=1}^{7} r_{1g}+\sum_{g=2}^{7} r_{1g}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right].\end{aligned}$$

Using equations (4.30)–(4.32) in the above relation, we obtain

$$\begin{aligned}\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{11}(\mathbf{x}) &= \prod_{i=2}^{7}(\Delta+\lambda_i^2)\left[\sum_{g=2}^{7} r_{1g}(\lambda_1^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\\ &= \prod_{i=3}^{7}(\Delta+\lambda_i^2)\left[\sum_{g=2}^{7} r_{1g}(\lambda_1^2-\lambda_g^2)\left[\delta(\mathbf{x})+(\lambda_2^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\right]\\ &= \prod_{i=3}^{7}(\Delta+\lambda_i^2)\left[\sum_{g=3}^{7} r_{1g}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x})\right]\\ &= \prod_{i=4}^{7}(\Delta+\lambda_i^2)\left[\sum_{g=3}^{7} r_{1g}\left[\prod_{j=1}^{2}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+(\lambda_3^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\right]\\ &= \prod_{i=4}^{7}(\Delta+\lambda_i^2)\left[\sum_{g=4}^{7} r_{1g}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x})\right]\\ &= \prod_{i=5}^{7}(\Delta+\lambda_i^2)\left[\sum_{g=4}^{7} r_{1g}\left[\prod_{j=1}^{3}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+(\lambda_4^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\right]\\ &= \prod_{i=5}^{7}(\Delta+\lambda_i^2)\left[\sum_{g=5}^{7} r_{1g}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\right]\varsigma_g(\mathbf{x})\right]\\ &= \prod_{i=6}^{7}(\Delta+\lambda_i^2)\left[\sum_{g=5}^{7} r_{1g}\left[\prod_{j=1}^{4}(\lambda_j^2-\lambda_g^2)\right]\left[\delta(\mathbf{x})+(\lambda_5^2-\lambda_g^2)\varsigma_g(\mathbf{x})\right]\right]\end{aligned}$$

$$= \prod_{i=6}^{7}(\Delta + \lambda_i^2)\Big[\sum_{g=6}^{7} r_{1g}\Big[\prod_{j=1}^{5}(\lambda_j^2 - \lambda_g^2)\Big]\varsigma_g(\mathbf{x})\Big]$$
$$= (\Delta + \lambda_7^2)\Big[\sum_{g=6}^{7} r_{1g}\Big[\prod_{j=1}^{5}(\lambda_j^2 - \lambda_g^2)\Big]\Big[\delta(\mathbf{x}) + (\lambda_6^2 - \lambda_g^2)\varsigma_g(\mathbf{x})\Big]\Big]$$
$$= (\Delta + \lambda_7^2)\varsigma_7(\mathbf{x}) = \delta(\mathbf{x}).$$

The equations $(4.29)_2$ and $(4.29)_3$ can be proved in the similar way. □

We introduce the matrix

$$\mathbf{G}(\mathbf{x}) = \mathbf{R}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}). \tag{4.33}$$

From equations (4.26), (4.28) and (4.33), we obtain

$$\mathbf{F}(\mathbf{D_x})\mathbf{G}(\mathbf{x}) = \mathbf{F}(\mathbf{D_x})\mathbf{R}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}) = \mathbf{\Theta}(\Delta)\mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x})\,\mathbf{I}(\mathbf{x}).$$

Hence, $\mathbf{G}(\mathbf{x})$ is a solution to equation (3.4).

Theorem 4.1. *If the condition (3.3) is satisfied, then the matrix $\mathbf{G}(\mathbf{x})$ defined by the equation (4.33) is the fundamental solution of the system of equations (3.2) and the matrix $\mathbf{G}(\mathbf{x})$ is represented in the following form:*

$$\mathbf{G}(\mathbf{x}) = \big(G_{pq}(\mathbf{x})\big)_{10\times 10} = \begin{pmatrix} \mathbf{G}^{(1)} & \mathbf{G}^{(2)} & \mathbf{G}^{(5)} \\ \mathbf{G}^{(3)} & \mathbf{G}^{(4)} & \mathbf{G}^{(6)} \\ \mathbf{G}^{(7)} & \mathbf{G}^{(8)} & \mathbf{G}^{(9)} \end{pmatrix}_{10\times 10},$$
$$\mathbf{G}^{(e)}(\mathbf{x}) = \big(G_{gh}^{(e)}(\mathbf{x})\big)_{3\times 3}, \quad \mathbf{G}^{(y)}(\mathbf{x}) = \big(G_{gh}^{(y)}(\mathbf{x})\big)_{3\times 4},$$
$$\mathbf{G}^{(z)}(\mathbf{x}) = \big(G_{gh}^{(z)}(\mathbf{x})\big)_{4\times 3}, \quad \mathbf{G}^{(9)}(\mathbf{x}) = \big(G_{gh}^{(9)}(\mathbf{x})\big)_{4\times 4},$$
$$\mathbf{G}^{(i)}(\mathbf{x}) = \mathbf{R}^{(i)}(\mathbf{D_x})Y_{11}(\mathbf{x}), \quad \mathbf{G}^{(j)}(\mathbf{x}) = \mathbf{R}^{(j)}(\mathbf{D_x})Y_{44}(\mathbf{x}),$$
$$\mathbf{G}^{(l)}(\mathbf{x}) = \mathbf{R}^{(l)}(\mathbf{D_x})Y_{77}(\mathbf{x}),$$
$$e = 1,2,3,4 \quad y = 5,6 \quad z = 7,8 \quad i = 1,3,7 \quad j = 2,4,8 \quad l = 5,6,9.$$

5. Basic Properties of $\mathbf{G}(\mathbf{x})$

Theorem 5.1. *Each column of the matrix $\mathbf{G}(\mathbf{x})$ is a solution of the system of equations (3.2) at every point $\mathbf{x} \in \mathrm{E}^3$ except the origin.*

Theorem 5.2. *If the condition (3.3) is satisfied, then the fundamental solution of the system $\tilde{\mathbf{F}}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0}$ is the matrix*

$$\mathbf{W}(\mathbf{x}) = \big(W_{gh}(\mathbf{x})\big)_{10\times 10} = \begin{pmatrix} \mathbf{W}^{(1)} & \mathbf{W}^{(2)} & \mathbf{W}^{(5)} \\ \mathbf{W}^{(3)} & \mathbf{W}^{(4)} & \mathbf{W}^{(6)} \\ \mathbf{W}^{(7)} & \mathbf{W}^{(8)} & \mathbf{W}^{(9)} \end{pmatrix}_{10\times 10},$$
$$\mathbf{W}^{(i)}(\mathbf{x}) = \big(W_{gh}^{(i)}(\mathbf{x})\big)_{3\times 3}, \quad \mathbf{W}^{(j)}(\mathbf{x}) = \big(W_{gh}^{(j)}(\mathbf{x})\big)_{3\times 4},$$
$$\mathbf{W}^{(l)}(\mathbf{x}) = \big(W_{gh}^{(l)}(\mathbf{x})\big)_{4\times 3}, \quad \mathbf{W}^{(9)}(\mathbf{x}) = \big(W_{gh}^{(9)}(\mathbf{x})\big)_{4\times 4},$$
$$W_{gh}^{(1)}(\mathbf{x}) = \Big[\frac{1}{\tilde{\lambda}}\,\mathrm{grad\,div} - \frac{1}{\tilde{\mu}}\,\mathrm{curl\,curl}\Big]\tilde{\varsigma}_1(\mathbf{x}),$$

$$W_{gh}^{(2)}(\mathbf{x}) = W_{gh}^{(3)}(\mathbf{x}) = 0, W_{gh}^{(4)}(\mathbf{x}) = \Big[\frac{1}{\tilde{\alpha}}\,\mathrm{grad}\,\mathrm{div} - \frac{1}{\gamma}\,\mathrm{curl}\,\mathrm{curl}\Big]\tilde{\varsigma}_1(\mathbf{x}),$$

$$W_{gh}^{(j)}(\mathbf{x}) = W_{gh}^{(l)}(\mathbf{x}) = 0, \quad W_{11}^{(9)}(\mathbf{x}) = \frac{t_2}{\sigma}\tilde{\varsigma}_2(\mathbf{x}), \tag{5.1}$$

$$W_{12}^{(9)}(\mathbf{x}) = W_{21}^{(9)}(\mathbf{x}) = -\frac{r_1}{\sigma}\tilde{\varsigma}_2(\mathbf{x}), \quad W_{22}^{(9)}(\mathbf{x}) = \frac{t_1}{\sigma}\tilde{\varsigma}_2(\mathbf{x}),$$

$$W_{33}^{(9)}(\mathbf{x}) = \frac{1}{K}\tilde{\varsigma}_2(\mathbf{x}), \quad W_{44}^{(9)}(\mathbf{x}) = \frac{1}{D}\tilde{\varsigma}_2(\mathbf{x}), \quad W_{ey}^{(9)}(\mathbf{x}) = W_{ye}^{(9)}(\mathbf{x}) = 0,$$

$$W_{34}^{(9)}(\mathbf{x}) = W_{43}^{(9)}(\mathbf{x}) = 0, \quad \tilde{\varsigma}_1(\mathbf{x}) = -\frac{|\mathbf{x}|}{8\pi}, \quad \tilde{\varsigma}_2(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|},$$

$$i = 1, \dots, 4 \quad j = 5, 6 \quad l = 7, 8 \quad e = 1, 2 \quad y = 3, 4.$$

Lemma 5.1. *If condition* (3.3) *is satisfied, then*

$$\begin{aligned} \Delta w_{11}(\Delta) &= \frac{1}{M^*}\Gamma_2(\Delta)N_{11}^*(\Delta) - \frac{1}{N^*}\Gamma_1(\Delta)(\gamma\Delta + \tilde{K}), \\ \Delta w_{22}(\Delta) &= \frac{1}{\tilde{\alpha}}\Big[\Gamma_2(\Delta) - \frac{1}{N^*}(\tilde{\alpha}\Delta + \tilde{K})(\tilde{\mu}\Delta + \rho\omega^2)\Big]. \end{aligned} \tag{5.2}$$

Proof. Consider

$$\begin{aligned} w_{11}(\Delta) = -\frac{1}{M^*N^*}\{[(\lambda' + \mu)(\gamma\Delta + \tilde{K}) - {K^*}^2]N_{11}^*(\Delta) + (\gamma\Delta + \tilde{K}) \\ \times [-g_1N_{12}^*(\Delta) - g_2N_{13}^*(\Delta) + \tau_1 l_1 T_0 N_{14}^*(\Delta) + \tau^1 l_2 N_{15}^*(\Delta)]\}. \end{aligned}$$

Now

$$\begin{aligned} \Gamma_1(\Delta) = \frac{1}{M^*}\,|\mathbf{N}(\Delta)| = \frac{1}{M^*}\{[\tilde{\lambda}\Delta + \rho\omega^2]N_{11}^* - g_1\Delta N_{12}^* \\ - g_2\Delta N_{13}^* + \tau_1 l_1 T_0 \Delta N_{14}^*(\Delta) + \tau^1 l_2 \Delta N_{15}^*(\Delta)\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta w_{11}(\Delta) = -\frac{1}{M^*N^*}\{[(\lambda' + \mu)(\gamma\Delta + \tilde{K}) - {K^*}^2]\Delta N_{11}^*(\Delta) + (\gamma\Delta + \tilde{K}) \\ \times [-g_1\Delta N_{12}^*(\Delta) - g_2\Delta N_{13}^*(\Delta) + \tau_1 l_1 T_0 \Delta N_{14}^*(\Delta) + \tau^1 l_2 \Delta N_{15}^*(\Delta)]\} \\ = -\frac{1}{M^*N^*}\{[M^*\Gamma_1(\Delta) - (\tilde{\lambda}\Delta + \rho\omega^2)N_{11}^*](\gamma\Delta + \tilde{K}) \\ + [(\lambda' + \mu)(\gamma\Delta + \tilde{K}) - {K^*}^2]\Delta N_{11}^*\} \\ = \frac{1}{M^*}\Gamma_2(\Delta)N_{11}^*(\Delta) - \frac{1}{N^*}\Gamma_1(\Delta)(\gamma\Delta + \tilde{K}). \end{aligned}$$

Also taking the R.H.S. of equation $(5.2)_2$

$$\begin{aligned} \frac{1}{N^*\tilde{\alpha}}\{[(\tilde{\mu}\Delta + \rho\omega^2)(\gamma\Delta + \tilde{K}) + {K^*}^2\Delta] - (\tilde{\alpha}\Delta + \tilde{K})(\tilde{\mu}\Delta + \rho\omega^2)\} \\ = -\frac{1}{N^*\tilde{\alpha}}\Delta[(\alpha + \beta)(\tilde{\mu}\Delta + \rho\omega^2) - {K^*}^2] = \Delta w_{22}(\Delta). \quad \square \end{aligned}$$

THEOREM 5.3. *If condition* (3.3) *is satisfied and* $\mathbf{x} \in \mathrm{E}^3 - \{\mathbf{0}\}$, *then*

$$\mathbf{G}^{(1)}(\mathbf{x}) = \operatorname{grad}\operatorname{div} \sum_{j=1}^{5} x_{1j}\varsigma_j(\mathbf{x}) - \operatorname{curl}\operatorname{curl} \sum_{e=6}^{7} x_{1e}\varsigma_e(\mathbf{x}),$$

$$\mathbf{G}^{(2)}(\mathbf{x}) = \mathbf{G}^{(3)}(\mathbf{x}) = \operatorname{curl} \sum_{e=6}^{7} x_{2e}\varsigma_e(\mathbf{x}),$$

$$\mathbf{G}^{(4)}(\mathbf{x}) = \operatorname{grad}\operatorname{div}\,[x_{48}\varsigma_8(\mathbf{x})] - \operatorname{curl}\operatorname{curl} \sum_{e=6}^{7} x_{4e}\varsigma_e(\mathbf{x}),$$

$$G^{(5)}_{iq}(\mathbf{x}) = \frac{\partial}{\partial x_i}\sum_{j=1}^{5} x_{5qj}\varsigma_j(\mathbf{x}), \qquad \mathbf{G}^{(6)}(\mathbf{x}) = \mathbf{G}^{(8)}(\mathbf{x}) = \mathbf{0},$$

$$G^{(7)}_{qi}(\mathbf{x}) = \frac{\partial}{\partial x_i}\sum_{j=1}^{5} x_{7qj}\varsigma_j(\mathbf{x}), \quad G^{(9)}_{ql}(\mathbf{x}) = \sum_{j=1}^{5} x_{9qlj}\varsigma_j(\mathbf{x}) \quad i = 1,2,3 \;\; q,l = 1,2,3,4,$$

where

$$x_{1j} = -\frac{r_{3j}}{M^*\lambda_j^2}N^*_{11}(-\lambda_j^2), \quad x_{1e} = \frac{(-1)^e(\gamma\lambda_e^2 - \tilde{K})}{N^*\lambda_e^2(\lambda_7^2-\lambda_6^2)}, \quad x_{2e} = \frac{(-1)^{e+1}K^*}{N^*(\lambda_7^2-\lambda_6^2)},$$

$$x_{4e} = \frac{(-1)^e(\tilde{\mu}\lambda_e^2 - \rho\omega^2)}{N^*\lambda_e^2(\lambda_7^2-\lambda_6^2)}, \quad x_{48} = -\frac{1}{\tilde{\alpha}\lambda_8^2}, \quad x_{5qj} = \frac{r_{3j}}{M^*}N^*_{1;q+1}(-\lambda_j^2), \tag{5.3}$$

$$x_{7qj} = -\frac{r_{3j}}{M^*\lambda_j^2}N^*_{q+1;1}(-\lambda_j^2), \quad x_{9qlj} = \frac{r_{3j}}{M^*}N^*_{q+1;l+1}(-\lambda_j^2),$$
$$j = 1,\dots,5 \quad q,l = 1,\dots,4 \quad e = 6,7.$$

PROOF. From equation (4.32),

$$\Delta\varsigma_j(\mathbf{x}) = -\lambda_j^2\varsigma_j(\mathbf{x}) \quad j = 1,\dots,8. \tag{5.4}$$

Thus, we have

$$-\frac{1}{\lambda_j^2}(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})\varsigma_j(\mathbf{x}) = \mathbf{J}\,\varsigma_j(\mathbf{x}), \quad \mathbf{x} \neq \mathbf{0}. \tag{5.5}$$

Consider

$$\begin{aligned}\mathbf{G}^{(1)}(\mathbf{x}) &= \mathbf{R}^{(1)}(\mathbf{D}_\mathbf{x})Y_{11}(\mathbf{x}) \\ &= \Big\{\frac{1}{N^*}(\gamma\Delta + \tilde{K})\mathbf{J}\,\Gamma_1(\Delta) + w_{11}(\Delta)\operatorname{grad}\operatorname{div}\Big\}\sum_{j=1}^{7} r_{1j}\varsigma_j(\mathbf{x}) \\ &= \sum_{j=1}^{7} r_{1j}\Big\{\Big[-\frac{1}{N^*\lambda_j^2}(-\gamma\lambda_j^2 + \tilde{K})\Gamma_1(-\lambda_j^2) + w_{11}(-\lambda_j^2)\Big]\operatorname{grad}\operatorname{div} \\ &\qquad + \frac{1}{N^*\lambda_j^2}(-\gamma\lambda_j^2 + \tilde{K})\Gamma_1(-\lambda_j^2)\operatorname{curl}\operatorname{curl}\Big\}\varsigma_j(\mathbf{x}).\end{aligned} \tag{5.6}$$

From equation $(5.2)_1$, we have

$$w_{11}(-\lambda_j^2) = -\frac{1}{M^*\lambda_j^2}\Gamma_2(-\lambda_j^2)N_{11}^*(-\lambda_j^2) + \frac{1}{N^*\lambda_j^2}\Gamma_1(-\lambda_j^2)(-\gamma\lambda_j^2 + \tilde{K}). \tag{5.7}$$

Using above equation in equation (5.6), we get

$$\mathbf{G}^{(1)}(\mathbf{x}) = \sum_{j=1}^{7} r_{1j}\Big\{\Big[-\frac{1}{M^*\lambda_j^2}\Gamma_2(-\lambda_j^2)N_{11}^*(-\lambda_j^2)\Big]\operatorname{grad}\operatorname{div} + \frac{1}{N^*\lambda_j^2}(-\gamma\lambda_j^2 + \tilde{K})\Gamma_1(-\lambda_j^2)\operatorname{curl}\operatorname{curl}\Big\}\varsigma_j(\mathbf{x}). \tag{5.8}$$

Now,

$$\begin{aligned} \Gamma_1(-\lambda_j^2)r_{1j} &= 0, \quad j = 1,\ldots,5,\\ \Gamma_1(-\lambda_j^2)r_{1j} &= \frac{(-1)^j}{\lambda_7^2 - \lambda_6^2}, \quad j = 6,7,\\ \Gamma_2(-\lambda_j^2)r_{1j} &= r_{3j}, \quad j = 1,\ldots,5,\\ \Gamma_2(-\lambda_j^2)r_{1j} &= 0, \quad j = 6,7. \end{aligned} \tag{5.9}$$

By virtue of equation (5.9), equation (5.8) becomes

$$\begin{aligned} \mathbf{G}^{(1)}(\mathbf{x}) &= \operatorname{grad}\operatorname{div}\sum_{j=1}^{5}\Big[-\frac{1}{M^*\lambda_j^2}r_{3j}N_{11}^*(-\lambda_j^2)\Big]\varsigma_j(\mathbf{x}) \\ &\quad + \operatorname{curl}\operatorname{curl}\sum_{e=6}^{7}\frac{(-1)^e(-\gamma\lambda_e^2 + \tilde{K})}{N^*\lambda_e^2(\lambda_7^2 - \lambda_6^2)}\varsigma_e(\mathbf{x}) \\ &= \operatorname{grad}\operatorname{div}\sum_{j=1}^{5}x_{1j}\varsigma_j(\mathbf{x}) - \operatorname{curl}\operatorname{curl}\sum_{e=6}^{7}x_{1e}\varsigma_e(\mathbf{x}). \end{aligned}$$

The remaining formulae of above theorem can be proved in the similar way. □

LEMMA 5.2. *If the condition (3.3) is satisfied, then*

$$\begin{gathered} \sum_{j=1}^{5} r_{3j} = \sum_{j=1}^{5} r_{3j}\lambda_j^2 = \sum_{j=1}^{5} r_{3j}\lambda_j^4 = \sum_{j=1}^{5} r_{3j}\lambda_j^6 = 0, \\ \sum_{j=1}^{5} r_{3j}\lambda_j^8 = 1, \quad \sum_{j=1}^{5}\frac{r_{3j}}{\lambda_j^2} = \prod_{i=1}^{5}\lambda_i^{-2} = \frac{M^*}{\rho\omega^2 N_{11}^*(0)}, \end{gathered} \tag{5.10}$$

and

$$\sum_{j=1}^{5} x_{1j} = -(\rho\omega^2)^{-1}, \quad \sum_{j=1}^{5} x_{1j}\lambda_j^2 = -\tilde{\lambda}^{-1}, \quad \sum_{e=6}^{7} x_{1e}\lambda_e^2 = -\tilde{\mu}^{-1}. \tag{5.11}$$

PROOF. Consider

$$N^*_{11}(-\lambda_j^2) = KD\sigma\lambda_j^8 + M^*_1\lambda_j^6 + M^*_2\lambda_j^4 + M^*_3\lambda_j^2 + N^*_{11}(0), \quad j = 1, \dots, 6 \tag{5.12}$$

where M^*_p, $p = 1, 2, 3$ are coefficients, independent of λ_j and skipped due to lengthy calculations.

Using equation (4.27), relations (5.10) can be proved by direct calculations.

From equations (5.10) and (5.12), we get

$$\sum_{j=1}^{5} \frac{r_{3j}}{\lambda_j^2} N^*_{11}(-\lambda_j^2) = \sum_{j=1}^{5} r_{3j}[KD\sigma\lambda_j^6 + M^*_1\lambda_j^4 + M^*_2\lambda_j^2 + M^*_3 + N^*_{11}(0)\lambda_j^{-2}]$$
$$= N^*_{11}(0)\sum_{j=1}^{5}\frac{r_{3j}}{\lambda_j^2} = \frac{M^*}{\rho\omega^2}$$

and

$$\sum_{j=1}^{5} r_{3j} N^*_{11}(-\lambda_j^2) = \sum_{j=1}^{5} r_{3j}\left[KD\sigma\lambda_j^8 + M^*_1\lambda_j^6 + M^*_2\lambda_j^4 + M^*_3\lambda_j^2 + N^*_{11}(0)\right] = KD\sigma.$$

Therefore, from equation (5.3), we have

$$\sum_{j=1}^{5} x_{1j} = -\sum_{j=1}^{5} \frac{r_{3j}}{M^*\lambda_j^2} N^*_{11}(-\lambda_j^2) = -(\rho\omega^2)^{-1},$$
$$\sum_{j=1}^{5} x_{1j}\lambda_j^2 = -\sum_{j=1}^{5} \frac{r_{3j}}{M^*} N^*_{11}(-\lambda_j^2) = -\frac{KD\sigma}{M^*} = -\tilde{\lambda}^{-1},$$

Also, we obtain

$$\sum_{e=6}^{7} x_{1e}\lambda_e^2 = -\frac{\gamma}{N^*} = -\tilde{\mu}^{-1}.$$ □

THEOREM 5.4. *The relations*

$$G_{pq}(\mathbf{x}) - W_{pq}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|), \quad p, q = 1, \dots, 10 \tag{5.13}$$

hold in the neighborhood of the origin.

PROOF. Consider

$$\mathbf{G}^{(1)}(\mathbf{x}) - \mathbf{W}^{(1)}(\mathbf{x}) = \operatorname{grad}\operatorname{div}\tilde{Y}_{11}(\mathbf{x}) - \operatorname{curl}\operatorname{curl}\tilde{Y}_{22}(\mathbf{x}), \tag{5.14}$$

where

$$\tilde{Y}_{11}(\mathbf{x}) = \sum_{j=1}^{5} x_{1j}\varsigma_j(\mathbf{x}) - \frac{\tilde{\varsigma}_1(\mathbf{x})}{\tilde{\lambda}}, \tag{5.15}$$
$$\tilde{Y}_{22}(\mathbf{x}) = \sum_{e=6}^{7} x_{1e}\varsigma_e(\mathbf{x}) - \frac{\tilde{\varsigma}_1(\mathbf{x})}{\tilde{\mu}}.$$

From equation $(5.15)_1$, we have

$$(5.16)\quad \tilde{Y}_{11}(\mathbf{x}) = \sum_{j=1}^{5} \frac{-x_{1j}}{4\pi} \sum_{g=0}^{\infty} \frac{\iota^g \lambda_j^g}{g!} |\mathbf{x}|^{g-1} + \frac{|\mathbf{x}|}{8\pi\tilde{\lambda}}$$

$$= -\frac{1}{8\pi}\left[2\sum_{j=1}^{5} x_{1j} \sum_{g=0}^{\infty} \frac{\iota^g \lambda_j^g}{g!} |\mathbf{x}|^{g-1} - \frac{|\mathbf{x}|}{\tilde{\lambda}}\right]$$

$$= -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{j=1}^{5} x_{1j} - |\mathbf{x}|\left(\sum_{j=1}^{5} x_{1j}\lambda_j^2 + \frac{1}{\tilde{\lambda}}\right)\right] - \frac{\iota}{4\pi}\sum_{j=1}^{5} x_{1j}\lambda_j + \tilde{Y}_{33}(\mathbf{x}).$$

Similarly,

$$(5.17)\quad \tilde{Y}_{22}(\mathbf{x}) = -\frac{1}{8\pi}\left[\frac{2}{|\mathbf{x}|}\sum_{e=6}^{7} x_{1e} - |\mathbf{x}|\left(\sum_{e=6}^{7} x_{1e}\lambda_e^2 + \frac{1}{\tilde{\mu}}\right)\right] - \frac{\iota}{4\pi}\sum_{e=6}^{7} x_{1e}\lambda_e + \tilde{Y}_{44}(\mathbf{x}),$$

where

$$(5.18)\quad \tilde{Y}_{33}(\mathbf{x}) = -\frac{1}{4\pi}\sum_{j=1}^{5} x_{1j} \sum_{g=3}^{\infty} \frac{\iota^g \lambda_j^g}{g!} |\mathbf{x}|^{g-1},$$

$$\tilde{Y}_{44}(\mathbf{x}) = -\frac{1}{4\pi}\sum_{e=6}^{7} x_{1e} \sum_{g=3}^{\infty} \frac{\iota^g \lambda_e^g}{g!} |\mathbf{x}|^{g-1}.$$

Clearly

$$(5.19)\quad \tilde{Y}_{qq}(\mathbf{x}) = O(|\mathbf{x}|^2), \quad \frac{\partial}{\partial x_e}\tilde{Y}_{qq}(\mathbf{x}) = O(|\mathbf{x}|),$$

$$\frac{\partial^2}{\partial x_e \partial x_i}\tilde{Y}_{qq}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|), \quad e, i = 1, 2, 3 \quad q = 3, 4.$$

Consider

$$\frac{\partial}{\partial x_i}\left(\frac{1}{|\mathbf{x}|}\right) = -\frac{x_i}{|\mathbf{x}|^3}, \quad \frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right) = \left[\frac{3x_i^2}{|\mathbf{x}|^5} - \frac{1}{|\mathbf{x}|^3}\right].$$

Hence,

$$\Delta\frac{1}{|\mathbf{x}|} = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\left(\frac{1}{|\mathbf{x}|}\right) = 0.$$

Therefore,

$$(5.20)\quad (\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})\frac{1}{|\mathbf{x}|} = \mathbf{J}\Delta\frac{1}{|\mathbf{x}|} = \mathbf{0}.$$

Equation (5.14) with the aid of equations (5.11) and (5.16)–(5.20) becomes

$$\mathbf{G}^{(1)}(\mathbf{x}) - \mathbf{W}^{(1)}(\mathbf{x}) = \operatorname{grad}\operatorname{div}\,\tilde{Y}_{33}(\mathbf{x}) - \operatorname{curl}\operatorname{curl}\,\tilde{Y}_{44}(\mathbf{x}) = \text{constant} + O(|\mathbf{x}|).$$

Similarly other formulae of equation (5.13) can be proved. □

Therefore, matrix $\mathbf{W}(\mathbf{x})$ is the singular part of the fundamental matrix $\mathbf{G}(\mathbf{x})$ in the neighborhood of the origin.

6. Particular Cases

From the derived basic equations, the following subcases can be obtained:

1. If we skip diffusion effect, the governing equations of generalized micropolar thermoelasticity with double porosity can be obtained.
2. If double porosity effect is overlooked, the equations given by Aouadi [14] are acquired.
3. If both microrotation and diffusion effects are neglected, we get the same equations as derived by Iesan and Quintanilla [22].
4. If both microrotation and double porosity effects are excluded, the results are same as obtained by Sherief et al. [11].
5. If we omit double porosity and diffusion effects, then the obtained equations are same as given by Eringen [1, 2].

7. Conclusions

The current paper gives the following outcomes:

1. Without utilizing Darcy's law, the linear theory of micropolar thermoelastic diffusion with double porosity are derived. This theory can be useful for finding fundamental solutions, studying wave phenomenon etc.
2. After reducing the governing equations in isotropic medium, the fundamental matrix of system of equations in case of steady oscillations is obtained and properties of fundamental matrix are discussed.
3. Finally, some special cases are also discussed.

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ТЕОРИЈА УОПШТЕНЕ МИКРОПОЛАРНЕ ТЕРМОЕЛАСТИЧНЕ ДИФУЗИЈЕ СА ДВОСТРУКОМ ПОРОЗНОШЋУ

Резиме. Основна сврха рада је да се изведу конститутивне релације и једначина поља за анизотропну микрополарну термоеластичну средину са дифузијом масе и двоструком порозношћу. Поред тога, конструисано је и фундаментално решење система једначина у случају постојаних осцилација.

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