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SYMMETRIES AND STABILITY OF MOTIONS IN THE NEWTONIAN AND THE HOOKEAN POTENTIALS

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ABSTRACT. A new way of looking at symmetries is proposed, especially regarding their role in the stability of two-body motions in the Newtonian and the Hookean potentials, the two selected by Bertrand's theorem. The role of the number of spatial dimensions is also addressed.

1. Introduction

In this article, we focus on the symmetries of two well-known super-integrable systems, namely those of a classical, non-relativistic point particle P moving in an inertial frame under the action of either Newtonian or Hookean (harmonic) forces produced by a point source located at the origin O of the frame. According to Joseph Bertrand's theorem [1], the two Newtonian and Hookean cases are the only ones providing bound closed orbits at any distance r from the source. In a recent article, we have shown how dynamical symmetries are strongly involved in this result [2]. Apart from its rather trivial symmetries, space rotationnal invariance and time invariance, each of these systems has an extra symmetry, due to the existence of a specific conserved 3-vector, currently called the eccentricity vector, which is perpendicular to the angular momentum and clearly related to the symmetry of orbits. The components of this vector are first integrals, which, added to the usual ones, energy and angular momentum, yields a number of independent first integrals greater than is necessary to solve the problem. This is why the two systems are said to be super-integrable as reminded in [3]. Associating the eccentricity vector with the angular momentum one can then define a group of transformations under which the Hamiltonian of the system is invariant. This is called a dynamical symmetry, hereafter called S_1 symmetry. In the Newtonian case, the eccentricity vector is the well-known Laplace–Runge–Lenz vector [4–6] (hereafter called LRL vector). The corresponding dynamical group is currently identified as being $SO(4)$.

The transformations of the dynamical symmetry group associated with the S_1 symmetry, hereafter called the S_1 group, are acting on phase space just like canonical transformations, transforming one solution into another by means of Poisson brackets. Geometrically, this means that they transform a trajectory into another. For the two cases under consideration, bound trajectories are ellipses or circles, the latter being viewed as ellipses of zero eccentricity. We refer the reader to Ref. [7] where, for the Kepler problem, the action of the S_1 group on the space of trajectories is well analyzed. Let us review the necessary parameters defining a given

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elliptic orbit. First, the plane where it lies, which is fixed by the direction of the angular momentum $\boldsymbol{L}$; then, its orientation in that plane and its axis, which are defined, respectively, by the orientation of the eccentricity vector, and by both the magnitude $L = \|\boldsymbol{L}\|$ of the angular momentum and the value E of the energy. With the said transformations, one is able to vary continuously the orientation of the plane and the orientation and the dimensions of the ellipse in that plane. But, since their generators are first integrals, they do not change the value of energy, and consequently can only connect ellipses with the same value of energy. To be specific, the respective eccentricities e_N and e_H of the ellipses for the Newtonian potential $V(r) = -\frac{K}{r}$ and for the Hookean potential $V(r) = \frac{Kr^2}{2}$, both with $K > 0$, are given by

$$e_N = \sqrt{1 + \frac{2EL^2}{mK^2}}, \quad \text{with} \quad E < 0,\ |E| < \frac{mK^2}{2L^2}, \quad \text{and} \tag{1.1}$$
$$e_H = \frac{L}{E + \sqrt{E^2 - \frac{L^2 K}{m}}}\sqrt{\frac{K}{m}}, \quad \text{with} \quad E > L\sqrt{\frac{K}{m}},$$

m being the mass of the particle. From these formulas, it is clear that, the energy being given, ellipses with the same eccentricities cannot be connected by these transformations. Hence, the S_1 group alone fails to connect all possible orbits. Hopefully, it appears that in addition to the S_1 symmetries, both the Newtonian and the Hookean motions possess another kind of symmetry that can remedy this problem.

2. Another symmetry: the mechanical similarity

Canonical transformations are generally viewed as transformations in phase space which preserve the form of the canonical equations, time being furthermore fixed. As there is an infinity of such transformations, one may say that any Hamiltonian system has an infinite number of symmetries. But, only a few of them could also preserve the very form of the Hamitonian and as such could be considered as generating an underlying dynamical symmetry. In this case, their infinitesimal generators are found to be first integrals. However, searching for new first integrals is of no help for our problem, as their Poisson brackets with the Hamiltonian are zero. At this point, it should be remembered that, as already pointed out in Ref. [8], a symmetry of a physical system means, fundamentally, invariance of its equation of motion which is currently a differential equation.

From Sophus Lie's pioneering works on the subject at the end of the 19$^{\text{th}}$ century [9], we know how searching for possible symmetries of a differential equation may prove efficient in solving it. For that purpose, one applies to the equation the so-called Lie's transformations of all the variables involved (dependent and independent ones). A very pedagogical presentation of the method can be found in Ref. [10].

In this spirit, it is natural to ask whether the S_1 symmetries of the two systems under consideration could be found applying Lie's method to their respective equations of motion. This is possible in principle, but proves uneasy in practice. We sketch the method by considering the second-order differential equations

$$\ddot{x}_i = F_i(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, t) \tag{2.1}$$

where the x_i's are the Cartesian coordinates at time t of a moving point P, the dot denoting time-derivative. The coordinates, their time-derivatives (velocities) $v_i = \dot{x}_i$ and time being considered as 7 independent variables, apply to these equations the general Lie infinitesimal transformation

$$t \to t' = t + \epsilon\xi, \quad x_i \to x'_i = x_i + \epsilon\eta_i, \quad v_i \to v'_i = v_i + \epsilon\beta_i \tag{2.2}$$

where ξ, η_i and β_i are functions of the coordinates, the velocities and time, and where ϵ is a dimensionless infinitesimal quantity. Then, Eqs. (2.1) become

$$\ddot{x}'_i = F'_i(x'_1, x'_2, x'_3, v'_1, v'_2, v'_3, t')$$

and the transformation is said to be a symmetry of the system if and only if $F'_i = F_i$. Note that the transformation Eq. (2.2) includes also a transformation of time and thus differs notably from a usual canonical transformation corresponding to $\xi = 0$. It appears that taking $\xi = 0$, the method leads to nothing in the case of the Newtonian motion and that we are thus forced to consider $\xi \neq 0$. Unfortunately, even in this case, the result is disapointing because, as explained in [10, §11.4], applying the general Lie's transformation Eq. (2.2) (contact transformation) to the second-order equation Eq. (2.1) leads to intractable equations, whose solutions, although they certainly exist, are generally impossible to derive. Finally, only the case of a so-called Lie point transformation, for which ξ and η_i depend only on the coordinates and time, prove efficient in revealing a special symmetry: whenever the potential in action is an homogeneous function of degree ν relative to coordinates, the equations of motion are invariant under the (finite) substitution

$$x_i \to x'_i = a\, x_i, \quad t \to t' = bt, \quad \text{with} \quad b = a^{1-\nu/2} \tag{2.3}$$

see Ref. [11, 12]. This is actually a well-known fact sometimes called "mechanical similarity" [13]. For such potentials, here $V(r) \propto r^\nu$, it allows one to connect the motion on a given admissible trajectory where the coordinates of the moving point P are $x_i(t)$ at time t to that on another admissible trajectory where the coordinates of the corresponding moving point P' are $x'_i(t')$ at a different time t', making the two trajectories somewhat similar geometrically speaking. If ℓ and ℓ' are some linear dimensions of the corresponding trajectories, the relation

$$\frac{t'}{t} = \left(\frac{\ell'}{\ell}\right)^{1-\nu/2} \tag{2.4}$$

gives the ratio of the corresponding travel times. In the case of the Newtonian potential, $\nu = -1$ and Eq. (2.4) leads to the Kepler's third law. For the Hookean potential, $\nu = 2$ and thus $t = t'$. But in this special case the equation of motion is linear and the above result simply means that at any time, the equation is invariant when multiplying the solution by any constant. Under the transformation Eq. (2.3), the Hamiltonian and the Lagrangian are not invariant but are both multiplied by the factor a^ν. Of course, this operation, which may also be called "Lagrangian rescaling" [14], leaves invariant the canonical equations. The angular momentum L is also changed into $L' = La^{1+\nu/2}$, and we have

$$EL^2 \to a^{2(1+\nu)} EL^2, \quad E/L \to a^{\nu/2-1} E/L \tag{2.5}$$

From Eq. (1.1) and Eq. (2.5), we see that the transformation Eq. (2.3) connects ellipses of the same eccentricities in both cases, Newtonian ($\nu = -1$) and Hookean ($\nu = 2$), and thus provides us the way to complete the S_1 symmetries.

3. A complete mapping of trajectories

To our knowledge, the first attempt to define a complete symmetry group for the classical Kepler problem has been made by J. Krause [15], who called it the Kepler group. Fundamentally, his aim was to define a specific symmetry group of the Kepler equation providing a full mapping of its solutions, which is the topic we are interested in. For this purpose, Krause also used the mechanical similarity, as is done below but in a different way.

The first step is to define the space on which a global symmetry group will be acting. This may be problematic as time is also involved in the transformation

Eq. (2.3). The choice of J. Krause has been the four-dimensional Newtonian space-time. However, there are reasons to consider it as the six-dimensional space of canonical coordinates $\boldsymbol{r}$ and $\boldsymbol{p}$ (phase space). One is that the S_1 group acts already on this space and on functions defined on it. Another one is that the mechanical similarity is also a symmetry of canonical equations. Finally, a global symmetry group is supposed to provide a mapping of trajectories, which amounts to a transformation of their geometrical parameters. But, the latter which are first integrals are generally expressed only in terms of canonical variables as they are independent of time. Hence, all we have to do is reexpress the transformation Eq. (2.3) as a transformation of the canonical variables $\boldsymbol{r}$ and $\boldsymbol{p}$. Obviously, this is achieved as follows:

$$x_i \to a x_i, \quad p_i \to \frac{a}{b} p_i = a^{\nu/2} p_i \tag{3.1}$$

The infinitesimal generator of this operation is

$$\mathcal{M} = x_k \frac{\partial}{\partial x_k} + \frac{\nu}{2} p_k \frac{\partial}{\partial p_k} \tag{3.2}$$

where summation on repeated indices is assumed, as will be done hereafter. The actions of this generator on the Hamiltonian $H = \frac{\boldsymbol{p}^2}{2m} + V(r)$, with $V(r) \propto r^\nu$, on the components of the angular momentum $L_i = \epsilon_{ijk} x_j p_k$ and on the magnitude L of the latter are easily found to be

$$\mathcal{M}(H) = \nu H, \quad \mathcal{M}(L_i) = \Big(1 + \frac{\nu}{2}\Big) L_i, \quad \mathcal{M}(L) = \Big(1 + \frac{\nu}{2}\Big) L \tag{3.3}$$

According to Eq. (2.5), we also have

$$\mathcal{M}(HL^2) = 2(1+\nu) HL^2, \quad \mathcal{M}(H/L) = \Big(\frac{\nu}{2} - 1\Big) H/L$$

From these formulas we deduce again the invariance of the eccentricity e of ellipses under the action of $\mathcal{M}$, in the Newtonian case, taking $\nu = -1$ in the first formula, and in the Hookean case, taking $\nu = 2$ in the second formula.

The action of the generators of the S_1 group on functions of canonical coordinates is defined through Poisson brackets. For example, that of the components of the angular momentum on any function $f(\boldsymbol{r}, \boldsymbol{p})$ is

$$\mathcal{L}_i(f) = \{L_i, f\} = \frac{\partial L_i}{\partial x_k}\frac{\partial f}{\partial p_k} - \frac{\partial L_i}{\partial p_k}\frac{\partial f}{\partial x_k} = \mathcal{L}_{xi}(f) + \mathcal{L}_{pi}(f), \quad \text{where}$$
$$\mathcal{L}_{xi} = -\epsilon_{i\ell k} x_\ell \frac{\partial}{\partial x_k}, \quad \mathcal{L}_{pi} = -\epsilon_{i\ell k} p_\ell \frac{\partial}{\partial p_k},$$

are the infinitesimal generators of rotations acting separately on $\boldsymbol{r}$-space and $\boldsymbol{p}$-space, respectively. Let A and B be two functions of canonical variables. A short calculation leads to

$$\mathcal{M}(\{A, B\}) = \{\mathcal{M}(A), B\} + \{A, \mathcal{M}(B)\} - \Big(1 + \frac{\nu}{2}\Big)\{A, B\} \tag{3.4}$$

Hence, defining the operator $\mathcal{A}$ by $\mathcal{A}(B) = \{A, B\}$, Eq. (3.4) is written as

$$\begin{aligned} \mathcal{M}(\mathcal{A}(B)) &= \{\mathcal{M}(A), B\} + \mathcal{A}(\mathcal{M}(B)) - \Big(1 + \frac{\nu}{2}\Big)\{A, B\}, \quad \text{or} \\ [\mathcal{M}, \mathcal{A}](B) &= \{\mathcal{M}(A), B\} - \Big(1 + \frac{\nu}{2}\Big)\{A, B\} \end{aligned} \tag{3.5}$$

Taking $A = L_i$ and using Eq. (3.3), we find $[\mathcal{M}, \mathcal{L}_i](B) = 0$. This result being true for any B, we infer that

$$[\mathcal{M}, \mathcal{L}_i] = 0$$

i.e. the "similarity" operator $\mathcal{M}$ and the operator angular momentum $\mathcal{L}_i$ are commuting. Let us then apply the operator $\mathcal{M}$ to the LRL vector

$$\boldsymbol{A} = \frac{1}{m}\boldsymbol{p} \wedge \boldsymbol{L} - K\frac{\boldsymbol{r}}{r}$$

The vector $\boldsymbol{r}/r$ being unitary and so dimensionless, it is clear that $\mathcal{M}(\boldsymbol{r}/r) = 0$. Using Eqs. (3.2) and (3.3), we get $\mathcal{M}(\boldsymbol{p} \wedge \boldsymbol{L}) = (1+\nu)\boldsymbol{p} \wedge \boldsymbol{L}$. It thus follows that

$$\mathcal{M}(\boldsymbol{A}) = \frac{(1+\nu)}{m}\boldsymbol{p} \wedge \boldsymbol{L}$$

But, in the Newtonian case we have $\nu = -1$ and hence

$$\mathcal{M}(\boldsymbol{A}) = 0 \tag{3.6}$$

From Eqs. (3.5) and (3.6) we obtain in this case the commutation property

$$[\mathcal{M}, \mathcal{A}_i] = -\frac{1}{2}\mathcal{A}_i$$

$\mathcal{A}_i$ being the operator associated with the component A_i of the LRL vector. In the same case and for $E < 0$, the vector

$$\boldsymbol{R} = \frac{\boldsymbol{A}}{\sqrt{-2mH}} \tag{3.7}$$

is usually used in order to identify the S_1 group as $SO(4)$. From Eqs. (3.3), (3.5) and (3.6) we find ($\nu = -1$)

$$\mathcal{M}(\boldsymbol{R}) = \tfrac{1}{2}\boldsymbol{R}, \quad \text{and} \quad [\mathcal{M}, \mathcal{R}_i] = 0$$

$\mathcal{R}_i$ being the operator associated with the component R_i of the vector Eq. (3.7), which is thus found to commute with $\mathcal{M}$. For $E > 0$ one uses instead the vector

$$\boldsymbol{R}' = \frac{\boldsymbol{A}}{\sqrt{2mH}} \tag{3.8}$$

to identify the S_1 group as $SO(3,1)$. In this case, we get the analogous result

$$\mathcal{M}(\boldsymbol{R}') = \tfrac{1}{2}\boldsymbol{R}', \quad \text{and} \quad [\mathcal{M}, \mathcal{R}'_i] = 0$$

We thus arrive at the conclusion that, in both cases $E < 0$ and $E > 0$, the operator $\mathcal{M}$ is commuting with the corresponding S_1 group, both viewed as acting on functions of canonical variables.

In the Hookean case ($\nu = 2$), we obtain a similar simple result. In Ref. [2] we have defined a vector similar to the LRL vector:

$$\boldsymbol{R}_H = L\,\boldsymbol{t}_- \tag{3.9}$$

where $\boldsymbol{t}_-$ is the unitary vector defining the short axes of ellipses. Associated with the angular momentum, this vector allows one to identify the S_1 group as $SO(3,1)$. Here again we have obviously $\mathcal{M}(\boldsymbol{t}_-) = 0$ and, from Eq. (3.3) with $\nu = 2$, $\mathcal{M}(L) = 2L$. Hence,

$$\mathcal{M}(\boldsymbol{R}_H) = 2\boldsymbol{R}_H, \quad \text{and} \quad [\mathcal{M}, \mathcal{R}_{Hi}] = 0$$

i.e. in the Hookean case too, the corresponding operator $\mathcal{M}$ and S_1 group are commuting.

The interesting conclusion is that, according to their definitions given above, the one-parameter mechanical similarity group generated by the operator Eq. (3.2) (with adapted values of ν) and the S_1 group are commuting, in both the Newtonian and the Hookean cases. In either case, their association yields the complete symmetry group we are looking for, with a S_1 group being either $SO(4)$ or $SO(3,1)$. The commutation rules of its generators are thus the following:

$$\begin{aligned} &[\mathcal{M}, \mathcal{L}_i] = [\mathcal{M}, \mathcal{R}_i] = 0, && [\mathcal{L}_i, \mathcal{L}_j] = \epsilon_{ijk}\mathcal{L}_k, \\ &[\mathcal{L}_i, \mathcal{R}_j] = \epsilon_{ijk}\mathcal{R}_k, && [\mathcal{R}_i, \mathcal{R}_j] = \pm\epsilon_{ijk}\mathcal{L}_k \end{aligned} \tag{3.10}$$

with a plus sign for the $SO(4)$ group, a minus sign for the $SO(3,1)$ group, and with a suitable eccentricity vector $\boldsymbol{R}$.

In any case, let $\boldsymbol{u}$ be the unit vector defining one or the only one symmetry axis of a trajectory. In the Newtonian case for example, we know that the LRL vector can be written as

$$\boldsymbol{A} = Ke\boldsymbol{u}$$

where $\boldsymbol{u}$ is the unit vector along the long axis of an ellipse for $e < 1$, or along the single axis of symmetry of a hyperbola or a parabola when $e \geq 1$. In the Hookean case, the eccentricity vector Eq. (3.9) is also written in this form. Thus, we could choose this unit vector as our "eccentricity" vector. This would unify all possible cases in a way even more economical than that leading to Eq. (3.10), including obviously the Hookean case but also the Newtonian case whatever the value of the eccentricity, in particular the apparently critical case $H = E = 0$ (that of parabolas) in formulas Eqs. (3.7) and (3.8). This is so because, as shown in the Appendix of Ref. [2], the components of any unit vector perpendicular to the angular momentum have zero Poisson brackets between them. Thus, taking $\boldsymbol{R} = \boldsymbol{u}$ leads to the simpler commutation relations

$$[\mathcal{M}, \mathcal{L}_i] = 0, \quad [\mathcal{M}, \mathcal{R}_i] = 0, \quad [\mathcal{L}_i, \mathcal{L}_j] = \epsilon_{ijk}\mathcal{L}_k, \quad [\mathcal{L}_i, \mathcal{R}_j] = \epsilon_{ijk}\mathcal{R}_k, \quad [\mathcal{R}_i, \mathcal{R}_j] = 0$$

Of course, for circular orbits, we must take $\boldsymbol{R} = 0$ and subsequently $\mathcal{R}_i = 0$.

In the above static representation of transformations in phase space, time is useless but can be reintroduced as a simple parameter homogeneous to the ratio mr/p: under the transformation Eq. (3.1), a point P with coordinates $(\boldsymbol{r}, \boldsymbol{p})$ and parameter t is transformed into a point P' with coordinates $(a\boldsymbol{r}, a^{\nu/2}\boldsymbol{p})$ having the parameter $ta^{1-\nu/2}$. Of course, this parameter "time" can be computed using well-known formulas. In the configuration space, a point $P_1(\boldsymbol{r}_1, \boldsymbol{p}_1)$ is on a trajectory which lies in the plane perpendicular to $\boldsymbol{L} = \boldsymbol{r}_1 \times \boldsymbol{p}_1$, has its symmetry axis along $\boldsymbol{u}(\boldsymbol{r}_1, \boldsymbol{p}_1)$ and corresponds to energy $E = \frac{p_1^2}{2m} + V(r_1)$. The time of evolution from point P_1 to another point $P(\boldsymbol{r}, \boldsymbol{p})$ of the same trajectory is then given by the formula

$$t = \int_{r_1}^{r} \frac{dR}{\sqrt{\frac{2}{m}(E - U(R))}}, \quad \text{where} \quad U(R) = \frac{L^2}{2mR^2} + V(R)$$

For an elliptic orbit in the Newtonian field, we obtain the known result

$$t = \frac{T}{2\pi}(\chi - e\sin\chi) + \text{cst}$$

where T is the period of revolution on the ellipse, e the eccentricity of the ellipse and χ the eccentric anomaly at point P [16].

4. Stabilities of trajectories

We now address the question: ultimately, what is the exact role of symmetries? More specifically, what about those described by continuous groups? In the cases under consideration, they allow one to connect continuously any admissible solution into another, in a one-to-one correspondance. They can do that here because the set of solutions is infinite. The said groups act on all the parameters defining a solution in an unique way. Suppose now that some parameters are changed by any external cause and that the system can then evolve freely. Of course, it is assumed that this perturbation does not destroy the fundamental structure of the system but only changes the parameter of a trajectory. Then, system will obviously evolve according to the admissible solution defined by the new set of parameters, which new solution can be connected to the preceding one by a continuous transformation. In this way, the existence of the group ensures the integrity of the whole set of solutions. Hence, due to the existence of the symmetry and its associated group, the set of solutions is stable: the motion resulting from an infinitesimal perturbation of a given motion stays in the vicinity of the latter, and, moreover, the perturbed motion remains in the same category, for example, that of closed orbits. In particular, it is easy to see that perturbing infinitesimaly a circular orbit while remaining in its plane leads to an ellipse of eccentricity close to zero. This is a structural stability, in line with its definition given by the mathematical theory of stability [17, 18].

Let us add the following. In our cases, the circular orbits are special solutions because they are stable in the ordinary sense: for a given value of the angular momentum, each corresponds to a minimum of the effective potential $U(r) = \frac{L^2}{2mr^2} + V(r)$, $V(r)$ being either the Newtonian potential or the Hookean one. For the potential $V(r) = -K/r^\alpha$ the conditions to obtain such orbit at a distance r are

$$\frac{dU}{dr} = \frac{K\alpha}{r^{\alpha+1}} - \frac{L^2}{mr^3} = 0, \quad \text{and}$$
$$\frac{d^2U}{dr^2} = -\frac{3}{r}\frac{dU}{dr} + \frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{dV}{dr}\right) = \frac{K\alpha(2-\alpha)}{r^{\alpha+2}} > 0$$

and they can be satisfied only if $K\alpha > 0$ and $\alpha < 2$. It can be shown that this applies as well to a n-dimentional space, where the equivalent of the Newtonian potential is $V(r) = -K/r^{n-2}$, i.e. $\alpha = n-2$. The above conditions then give $K(n-2) > 0$ and $n < 4$. This is an important result: as far as the Newtonian potential produced by a point source is concerned, a circular orbit cannot be stable in a space of dimension n higher than 3. This could be generalized to extended Newtonian sources as is strongly suggested by the results obtained in Ref. [19]. Note here that far away from the source, the latter can often be considered as a monopole at first order (this is always the case for Gravitation), giving a potential proportional to $1/r^{n-2}$ which, as said above, does not provide stable circular orbits. In contrast, the Hookean potential $Kr^2/2$ still produces stable circular orbits at any distance, whatever the dimension. This short discussion reveals the crucial role of space dimension regarding the stability of circular orbits provided by the generalized Newtonian potential. Note that in a space with $n > 3$ dimensions, the potential $-K/r$ with $K > 0$ provides stable circular orbits but in this case would not have a clear physical meaning.

5. Conclusion

Three main points have been addressed in this article. First, a new scheme to describe the complete symmetries of the Newtonian and Hookean two-body motions has been proposed, taking into account their well-known "dynamical symmetries" and "mechanical similarity". We have shown that these symmetries are most naturally expressed as transformations in phase space, forming a global Lie symmetry group having a very simple and economical structure. This global structure is necessary to provide a way to connect all possible trajectories, a task that cannot be accomplished by the dynamical symmetries alone. Second, we have suggested a new look at this overall symmetry group, arguing that its existence is necessary to ensure the structural stability of the whole set of trajectories, with the function of providing a continuous connection of all trajectories of a same species. The third point is certainly the most intriguing, as it concerns the role of the number of spatial dimensions in the stability of circular orbits in the Newtonian motion, an important topic for all of us. Using a very familiar method, we have shown that this stability can be realized only in a three-dimensional space. This result seems very satisfactory to us, because it links the number of dimensions to a stability problem. The understanding of the physical nature of a spatial dimension is out of reach in the present state of our knowledge, but at least we should try to explain why 3 would be the best number of spatial dimensions for our Universe. In our opinion, there must exit a yet unknown overall principle of stability, which in particular is at work to produce the above-mentioned result. This principle would also have an economic aspect: for example, it is well-known that physical quantities characterized by 2-rank skew-symmetric tensors can be represented by field (pseudo)vectors in the basic configuration space if and only if the dimension of the latter is three. This is the case for the angular momentum and the magnetic field. Many future studies need to be carried out on this topic.

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СИМЕТРИЈЕ И СТАБИЛНОСТ КРЕТАЊА У ЊУТНОВОМ И ХУКОВОМ ПОТЕНЦИЈАЛУ

Резиме. Предлаже се нови начин посматрања симетрија, посебно у погледу њихове улоге у стабилности кретања два тела у Њутновом и Хуковом потенцијалу, два потенцијала издвојена Бертрандовом теоремом. Такође се разматра и улога броја просторних димензија.

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