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A NOTE ON THE MHD FLOW IN A POROUS CHANNEL

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ABSTRACT. In this paper we study the flow of a viscous incompressible conducting fluid through a corrugated channel filled with a porous medium. The fluid flow in the channel is under the action of the transverse magnetic field and driven by the pressure drop between the channel's edges. Using boundary-layer analysis, we derive a higher-order asymptotic model taking into account the inertia and roughness-induced effects on the filtration velocity.

1. Introduction

The magnetohydrodynamic (MHD) flow of a viscous incompressible conducting fluid through a porous medium naturally arises both in natural settings (e.g. plasma dynamics and astrophysics [1]), as well as in the engineering applications (e.g. nuclear engineering and metallurgy [2, 3]). For that reason, these flows have been extensively investigated over the past few decades. We refer the reader to [4] (and the references therein), where a literature overview of theoretical and experimental results on the subject can be found.

Numerous models have been proposed throughout the literature to describe a porous medium flow. Along with classical (and most popular) Darcy [5] and Darcy–Brinkman [6] laws, if the inertia effects are not to be neglected in the filtration process, the right choice becomes the so-called Darcy–Lapwood–Brinkman system originally proposed in [7]. It incorporates the convective inertial term in the momentum equation leading to:

$$\frac{\rho}{\phi^2}(\mathbf{u}^* \cdot \nabla)\mathbf{u}^* = -\nabla p^* - \frac{\mu}{K}\mathbf{u}^* + \mu_e \Delta \mathbf{u}^*. \tag{1.1}$$

Here $\mathbf{u}^* = u^*_{x*}\mathbf{i} + u^*_{y*}\mathbf{j}$ is the filtration velocity, p^* denotes the fluid pressure, K represents the permeability of the porous medium, μ and μ_e stand for the (dynamic) viscosity coefficient and the effective viscosity of the fluid in the porous medium respectively, ρ denotes the fluid density, whereas ϕ stands for the porosity. Hereinafter, we denote by $(\mathbf{i}, \mathbf{j})$ the standard Cartesian basis. The nonlinear equation

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(1.1) turns out to be an essential generalization of the Darcy law. Indeed, (1.1) takes into account the effects due to the flow inertia, and is, in addition, applicable in various physically relevant settings, such as those where the non-slip boundary condition is imposed on an impermeable boundary.

In practice, the boundary of the flow domain almost always contains various small rugosities and dents and, thus, we consider a flow domain given by (see Fig. 1):

$$\Omega_\delta = \left\{ (x^*, y^*) \in \mathbf{R}^2 \,:\, 0 < x^* < \ell,\ |y^*| < R + \delta h\Big(\frac{x^*}{\delta}\Big) \right\}. \tag{1.2}$$

The roughness of the channel is described by a given periodic function h, whereas the channel's length ℓ and R are positive constants. The roughness parameter δ is assumed to be small compared to R, i.e. $0 < \delta \ll R$.

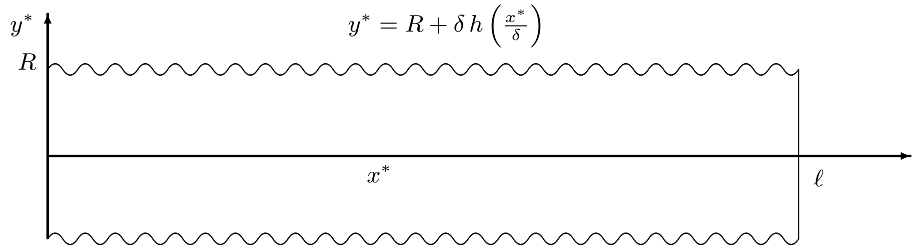

FIGURE 1. Flow domain

We assume that the flow in the channel Ω_δ is subjected to an uniform magnetic field B_0 in the direction normal to the oscillating boundaries. In view of that, we study the following system:

$$\rho(\mathbf{u}^* \cdot \nabla)\mathbf{u}^* = -\nabla p^* - \frac{\mu}{K}\mathbf{u}^* - \sigma B_0^2 u^*_{x^*}\mathbf{i} + \mu_e \Delta \mathbf{u}^* \quad \text{in} \quad \Omega_\delta, \tag{1.3}$$

$$\operatorname{div} \mathbf{u}^* = 0 \quad \text{in} \quad \Omega_\delta, \tag{1.4}$$

where σ represents the conductivity of the fluid. To simplify the notation, we set the porosity ϕ equal to 1 and take $\mu_e = \mu$ (see e.g. [8]).

Obviously, the above problem cannot be exactly solved due to the nonlinearity of the equation (1.3) and the oscillating behavior of the domain (1.2). Therefore, we make use of the asymptotic approach recently proposed in [20, 23] for the flows without magnetic effects and derive the approximate solution of the governing problem. As expected, the zero-order approximation of the solution does not feel the inertia and roughness-induced effects, so we employ the delicate boundary-layer analysis and construct the correctors which clearly acknowledge those effects. As a result, we obtain an asymptotic approximation of higher-order of accuracy which could prove useful a wide range of filtration processes naturally appearing in the practice.

Roughness-induced effects on the fluid flow (without porous structure) have been considered in numerous papers, see e.g. [9–13]. The effects of the boundary irregularities on the porous medium flows have been analytically investigated in

[14–18]. The inertia effects have been taken into account in [19,20] for the Navier–Stokes flow, whereas the Darcy–Lapwood–Brinkman flow has been addressed in [21–23]. Interesting analytical results on the MHD flows can be found in [24–30]. For recent numerical studies of the MHD flow through a porous medium we refer the reader to [31–34]. To our knowledge, [30] contains one of the rare attempts to build analytically the solution of the MHD flow within a domain filled with a porous medium. Starting from the (linear) Darcy–Brinkman system and neglecting the flow inertia, the authors in [30] construct an exact solution in the case of the channel with perfectly plane walls. This, of course, is not feasible in our setting, since we aim to tackle the situation being relevant from the point of view of the applications, and, as such far from the ideal one considered in [30].

2. The problem in dimensionless form

Before writing the problem in dimensionless form, let us complete the system (1.3)–(1.4) with the appropriate boundary conditions. To be in line with the applications, we assume that the fluid flow inside the channel is governed by the prescribed pressure drop between the channel's ends, whereas for the velocity we employ the standard no-slip condition on the upper and the lower wall. Thus, we impose:

$$\mathbf{u}^* = \mathbf{0} \quad \text{on} \quad \Gamma_\delta = \Big\{ (x^*, y^*) \in \mathbf{R}^2 \, : \, 0 < x^* < \ell, \ |y^*| = R + \delta h\Big(\frac{x^*}{\delta}\Big) \Big\}, \tag{2.1}$$

$$\mathbf{u}^* \times \mathbf{i} = \mathbf{0} \quad \text{for} \quad x^* = 0, \ell,$$

$$p^* = p_i \quad \text{for} \quad x^* = i, \ i \in \{0, \ell\}, \tag{2.2}$$

for given p_i ($p_0 > p_\ell > 0$).

To work in a non-dimensional setting, we need to precise the characteristic velocity. We choose the characteristic velocity V as the mean value of the exact solution of (1.3)–(2.2) with $h = 0$. Due to the incompressibility of the flow, we take $\mathbf{u}^* = u^*(y^*)\mathbf{i}$ leading to

$$\frac{\partial p^*}{\partial y^*} = 0 \quad \Rightarrow \quad p^* = p^*(x^*) = p_0 + \frac{p_\ell - p_0}{\ell} x^*. \tag{2.3}$$

$$\mu \frac{d^2 u^*}{dy*^2} - \Big(\frac{\mu}{K} + \sigma B_0^2\Big) u^* - \frac{\partial p^*}{\partial x^*} = 0, \tag{2.4}$$

Since $u^*(-R) = u^*(R) = 0$, we get

$$u^*(y^*) = \frac{K(p_0 - p_\ell)}{\ell(\mu + K\sigma B_0^2)} \left\{ 1 - \frac{\cosh\left(\sqrt{\frac{1}{K} + \frac{\sigma B_0^2}{\mu}}\, y^*\right)}{\cosh\left(\sqrt{\frac{1}{K} + \frac{\sigma B_0^2}{\mu}}\, R\right)} \right\}. \tag{2.5}$$

Now, we rewrite (1.3)–(2.2) in the dimensionless form by introducing:

$$x = \frac{x^*}{2R}, \quad y = \frac{y^*}{2R}, \quad L = \frac{\ell}{2R}, \quad \varepsilon = \frac{\delta}{2R}, \quad k = \frac{K}{4R^2}, \tag{2.6}$$

$$\mathbf{u}^\varepsilon = \frac{\mathbf{u}^*}{V}, \quad p^\varepsilon = \frac{2Rp^*}{V\mu}, \quad \mathrm{Re} = \frac{2RV\rho}{\mu}. \tag{2.7}$$

Along to the Reynolds number Re, another non-dimensional parameter appears in the dimensionless equations characterizing the conductivity of the flow, namely:

$$M = 2RB_0\sqrt{\frac{\sigma}{\mu}} \qquad \text{(Hartmann number)}. \tag{2.8}$$

In view of the above, the velocity (2.5) becomes:

$$u^\varepsilon(y) = \frac{k(q_0 - q_L)}{L(1+M^2k)}\left\{1 - \frac{\cosh\left(\lambda\sqrt{1+M^2k}\,y\right)}{\cosh\left(\frac{\lambda}{2}\sqrt{1+M^2k}\right)}\right\}, \tag{2.9}$$

where we denote by

$$\lambda = \sqrt{\frac{1}{k}} \tag{2.10}$$

the non-dimensional parameter characterizing the porous medium inside the channel. Taking into account (2.6)–(2.7), in what follows we study the following problem:

$$\mathrm{Re}(\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon = -\nabla p^\varepsilon + \Delta \mathbf{u}^\varepsilon - \frac{1}{k}\mathbf{u}^\varepsilon - M^2 u_x^\varepsilon \mathbf{i} \quad \text{in} \quad \Omega_\varepsilon^+, \tag{2.11}$$

$$\operatorname{div} \mathbf{u}^\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon^+, \tag{2.12}$$

$$\mathbf{u}^\varepsilon\Big(x, \frac{1}{2} + \varepsilon h\Big(\frac{x}{\varepsilon}\Big)\Big) = \mathbf{0}, \tag{2.13}$$

$$\frac{\partial \mathbf{u}^\varepsilon}{\partial y}(x,0) = \mathbf{0}, \tag{2.14}$$

$$\mathbf{u}^\varepsilon \times \mathbf{i} = \mathbf{0} \quad \text{for} \quad x = 0, L, \tag{2.15}$$

$$p^\varepsilon = q_i \quad \text{for} \quad x' = i,\ i \in \{0, L\}, \tag{2.16}$$

where $q_0 = \frac{2Rp_0}{V\mu}$, $q_L = \frac{2Rp_\ell}{V\mu}$ and

$$\Omega_\varepsilon^+ = \Big\{(x,y) \in \mathbf{R}^2 \,:\, 0 < x < L,\ 0 < y < \frac{1}{2} + \varepsilon h\Big(\frac{x}{\varepsilon}\Big)\Big\}. \tag{2.17}$$

Note that the condition (2.14) is due to the y-symmetry of the solution. Finally, we put $\varepsilon = \frac{L}{n}$, where $n \gg 1$ is the number of corrugations and assume $\int_0^1 h(\xi)d\xi = 0$ meaning that the parameter ε denotes the average roughness of the wall.

3. The analysis

Before proceeding, it should be noted that we essentially study the effects of the small boundary perturbation of the MHD flow within porous medium. In view of that, it is rational to seek for the solution of the system (2.11)–(2.16) as the perturbation of the Darcy–Brinkman velocity (2.9) with the effects of the applied magnetic field, namely:

$$\mathbf{u}_0^\varepsilon = u_0^\varepsilon \mathbf{i} = \frac{k(q_0 - q_L)}{L(1+M^2k)}\left\{1 - \frac{\cosh\left(\lambda\sqrt{1+M^2k}y\right)}{\cosh\left(\frac{\lambda}{2}\sqrt{1+M^2k}\right)} + t(\varepsilon)\frac{\cosh\left(\lambda\sqrt{1+M^2k}y\right)}{\cosh\left(\frac{\lambda}{2}\sqrt{1+M^2k}\right)}\right\}\mathbf{i}, \tag{3.1}$$

$$p^0 = p^0(x) = q_0 + \frac{q_L - q_0}{L}x, \tag{3.2}$$

where

$$t(\varepsilon) = \varepsilon t_1 + \varepsilon^2 t_2 + \varepsilon^3 t_3 + \cdots . \tag{3.3}$$

Observe that the equations (2.11)–(2.12) and the boundary conditions (2.14)–(2.16) are identically satisfied by $\mathbf{u}_0^\varepsilon$, but

$$\mathbf{u}_0^\varepsilon\Big(x, \frac{1}{2} + \varepsilon h\Big(\frac{x}{\varepsilon}\Big)\Big) = \mathcal{O}(\varepsilon). \tag{3.4}$$

Thus, we need to perform the boundary-layer analysis and correct the above solution. Following [23], the idea is to construct the corresponding boundary-layer correctors so that they vanish exponentially far from the upper boundary. This procedure will give us the values of the constants t_m which will contribute to our approximation far from the upper boundary as well.

Using the standard addition formula for hyperbolic functions and the well-known expansions:

$$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots , \quad \sinh(z) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots ,$$

from $(3.1)_1$ it is straightforward to obtain:

$$\begin{aligned} u_0^\varepsilon\Big(x, \frac{1}{2}+\varepsilon h\Big(\frac{x}{\varepsilon}\Big)\Big) &= \frac{k(q_0 - q_L)}{L(1+M^2k)}\Big\{\varepsilon\Big[t_1 - \lambda\sqrt{1+M^2k}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)h\Big(\frac{x}{\varepsilon}\Big)\Big] \\ &+ \varepsilon^2\Big[t_2 + t_1\lambda\sqrt{1+M^2k}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)h\Big(\frac{x}{\varepsilon}\Big) - \frac{\lambda^2}{2}(1+M^2k)h\Big(\frac{x}{\varepsilon}\Big)^2\Big] \\ &+ \varepsilon^3\Big[t_3 + t_2\lambda\sqrt{1+M^2k}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)h\Big(\frac{x}{\varepsilon}\Big) + t_1\frac{\lambda^2}{2}(1+M^2k)h\Big(\frac{x}{\varepsilon}\Big)^2 \\ &- \frac{\lambda^3}{6}(1+M^2k)^{\frac{3}{2}}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)h\Big(\frac{x}{\varepsilon}\Big)^3\Big] + \cdots\Big\}. \end{aligned} \tag{3.5}$$

3.1. Boundary-layer correctors. Due to (3.4), we need to correct the solution given by $(3.1)_1$. To accomplish that, we introduce the fast variables

$$\xi = \frac{x}{\varepsilon}, \quad \tau = \frac{y - \frac{1}{2}}{\varepsilon}$$

and postulate the following ansatz:

$$\mathbf{u}^\varepsilon = \mathbf{u}_0^\varepsilon + \varepsilon\mathbf{B}^1 + \varepsilon^2\mathbf{B}^2 + \cdots , \quad p^\varepsilon = p^0 + b^1 + \varepsilon b^2 + \cdots . \tag{3.6}$$

Here the boundary-layer correctors

$$\big(\varepsilon\mathbf{B}^1(\xi,\tau) + \varepsilon^2\mathbf{B}^2(\xi,\tau) + \cdots , \ b^1(\xi,\tau) + \varepsilon b^2(\xi,\tau) + \cdots\big)$$

are defined in the semi-infinite strip denoted by

$$G = \{(\xi,\tau) \in \mathbf{R}^2 \ ; \ 0 < \xi < 1, \ \tau < h(\xi)\}.$$

To derive the problems satisfied by the boundary-layer correctors, we plug (3.6) into the system (2.11)–(2.12) and acknowledge (3.5). In the sequel, we embrace the following notation:

$$\Delta_{\xi\tau} = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\tau^2}, \quad \nabla_{\xi\tau} = \frac{\partial}{\partial\xi}\mathbf{i} + \frac{\partial}{\partial\tau}\mathbf{j},$$

$$\operatorname{div}_{\xi\tau} \mathbf{B}^m = \frac{\partial B_\xi^m}{\partial \xi} + \frac{\partial B_\tau^m}{\partial \tau}, \quad \mathbf{B}^m = B_\xi^m \mathbf{i} + B_\tau^m \mathbf{j}.$$

The first problem we obtain is the following:

$$(3.7)\qquad \begin{cases} \frac{1}{\varepsilon} \ : \ -\Delta_{\xi\tau}\mathbf{B}^1 + \nabla_{\xi\tau} b^1 = \mathbf{0} \quad \text{in } \ G, \\ 1 \ : \ \operatorname{div}_{\xi\tau} \mathbf{B}^1 = 0 \quad \text{in } \ G, \\ \varepsilon \ : \ \mathbf{B}^1(\xi, h(\xi)) = -\frac{k(q_0-q_L)}{L(1+M^2k)} \big[t_1 - \lambda\sqrt{1+M^2k}\tanh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)h(\xi)\big]\mathbf{i}, \\ \quad (\mathbf{B}^1, b^1) \quad 1-\text{periodic in } \xi, \quad \nabla_{\xi\tau}\mathbf{B}^1 \in L^2(G). \end{cases}$$

The above problem has the solution decaying exponentially towards some constant, as $\tau \to -\infty$. In fact, there exists a constant B_∞^1 such that (see [19, 23] for details)

$$\lim_{\tau\to-\infty} \mathbf{B}^1 = B_\infty^1 \mathbf{i}.$$

The problem (3.7) can be solved by putting

$$(3.8)\qquad \mathbf{B}^1 = \frac{k(q_0-q_L)}{L(1+M^2k)}\Big(\lambda\sqrt{1+M^2k}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)\mathbf{W} - t_1\,\mathbf{i}\Big),$$

where $\mathbf{W}$ is the solution of the following auxiliary problem:

$$(3.9)\qquad \begin{cases} -\Delta_{\xi\tau}\mathbf{W} + \nabla_{\xi\tau}\pi = \mathbf{0} \quad \text{in } \ G, \\ \operatorname{div}_{\xi\tau} \mathbf{W} = 0 \quad \text{in } G, \\ \mathbf{W}(\xi, h(\xi)) = h(\xi)\mathbf{i}, \\ (\mathbf{W}, \pi) \quad 1-\text{periodic in } \ \xi, \quad \nabla_{\xi\tau}\mathbf{W} \in L^2(G). \end{cases}$$

It can be proved (see [19]) that $\lim_{\tau\to-\infty} \mathbf{W} = W_\infty\,\mathbf{i}$ with

$$(3.10)\qquad W_\infty = -\int_G |\nabla_{\xi\tau}\mathbf{W}|^2.$$

The idea of our construction is to choose the constant t_1 in a way that $B_\infty^1 = 0$ implying

$$(3.11)\qquad t_1 = \lambda\sqrt{1+M^2k}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)W_\infty.$$

The problem for $(\mathbf{B}^2, b^2)$ reads as follows:

$$(3.12)\qquad \begin{cases} 1 \ : \ -\mu\Delta_{\xi\tau}\mathbf{B}^2 + \nabla_{\xi\tau} b^2 = \mathbf{0} \quad \text{in } \ G, \\ \varepsilon \ : \ \operatorname{div}_{\xi\tau} \mathbf{B}^2 = 0 \quad \text{in } \ G, \\ \varepsilon^2 : \ \mathbf{B}^2(\xi, h(\xi)) = -\frac{k(q_0-q_L)}{L(1+M^2k)}\big[t_2 + t_1\lambda\sqrt{1+M^2k}\tanh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)h(\xi) \\ \qquad\qquad - \frac{\lambda^2}{2}(1+M^2k)h(\xi)^2\big]\mathbf{i}, \\ \quad (\mathbf{B}^2, b^2) \quad 1-\text{periodic in } \ \xi, \quad \nabla_{\xi\tau}\mathbf{B}^2 \in L^2(G). \end{cases}$$

The problem (3.12) admits a solution which decays exponentially towards a constant as well, namely:

$$\lim_{\tau\to-\infty} \mathbf{B}^2 = B_\infty^2 \mathbf{i}.$$

To solve (3.12), another auxiliary problem should be introduced:

$$\begin{cases} -\Delta_{\xi\tau}\mathbf{D} + \nabla_{\xi\tau}\eta = \mathbf{0} \quad \text{in} \ \ G, \\ \operatorname{div}_{\xi\tau}\mathbf{D} = 0 \quad \text{in} \ \ G, \\ \mathbf{D}(\xi, h(\xi)) = h(\xi)^2\mathbf{i}, \\ (\mathbf{D}, \eta) \quad 1-\text{periodic in} \ \ \xi, \quad \nabla_{\xi\tau}\mathbf{D} \in L^2(G). \end{cases} \tag{3.13}$$

The above problem has the solution such that

$$\lim_{\tau\to-\infty} \mathbf{D} = D_\infty \mathbf{i},$$

exponentially, where (see [19])

$$D_\infty = \int_0^1 h(\xi)^2 d\xi - 2\int_G \tau \frac{\partial W_\xi}{\partial \tau}. \tag{3.14}$$

We solve (3.12) by taking

$$\mathbf{B}^2 = \frac{k(q_0-q_L)}{L(1+M^2k)}\Big(\frac{\lambda^2}{2}(1+M^2k)\mathbf{D} - t_1\lambda\sqrt{1+M^2k}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)\mathbf{W} - t_2\mathbf{i}\Big), \tag{3.15}$$

Demanding that $B^2_\infty = 0$, we deduce

$$t_2 = -\lambda^2(1+M^2k)\Big(\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)\Big)^2 W^2_\infty + \frac{\lambda^2}{2}(1+M^2k)D_\infty, \tag{3.16}$$

due to (3.11).

In order to capture the inertia effects as well, we need to compute one more boundary-layer corrector in (3.6). We deduce the following problem satisfied by $(\mathbf{B}^3, b^3)$:

$$\begin{cases} \varepsilon \ : \ -\Delta_{\xi\tau}\mathbf{B}^3 + \nabla_{\xi\tau}b^3 + \frac{1}{k}\mathbf{B}^1 + M^2 B^1_\xi \mathbf{i} \\ \qquad +\mathrm{Re}\Big\{(\mathbf{B}^1\cdot\nabla_{\xi\tau})\mathbf{B}^1 - \frac{k(q_0-q_L)}{L(1+M^2k)}\lambda\sqrt{1+M^2k}\tanh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)B^1_\tau\mathbf{i} \\ \qquad + \frac{k(q_0-q_L)}{L(1+M^2k)}\big[-\lambda\sqrt{1+M^2k}\tanh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)\tau + t_1\big]\frac{\partial \mathbf{B}^1}{\partial\xi}\Big\} = \mathbf{0}, \\ \varepsilon^2 : \ \operatorname{div}_{\xi\tau}\mathbf{B}^3 = 0 \quad \text{in} \ \ G, \\ \varepsilon^3 : \ \mathbf{B}^3(\xi, h(\xi)) = -\frac{k(q_0-q_L)}{L(1+M^2k)}\Big[t_3 + t_2\lambda\sqrt{1+M^2k}\tanh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)h(\xi) \\ \qquad + t_1\frac{\lambda^2}{2}(1+M^2k)h(\xi)^2 - \frac{\lambda^3}{6}(1+M^2k)^{\frac{3}{2}}\tanh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)h(\xi)^3\Big], \\ \qquad (\mathbf{B}^3, b^3) \quad 1-\text{periodic in} \ \ \xi, \quad \nabla_{\xi\tau}\mathbf{B}^3 \in L^2(G). \end{cases} \tag{3.17}$$

Recall that $\mathbf{B}^1$ decays exponentially towards zero (as $\tau \to -\infty$), implying that the problem (3.17) admits the solution which decays exponentially towards a constant, namely:

$$\lim_{\tau\to-\infty}\mathbf{B}^3 = B^3_\infty\mathbf{i}.$$

Let us compute the nonlinear term from $(3.17)_1$ using the derived expression for $\mathbf{B}^1$ (see (3.8), (3.11)):

$$\begin{aligned} \mathrm{Re}\Big\{(\mathbf{B}^1\cdot\nabla_{\xi\tau})\mathbf{B}^1 - \frac{k(q_0-q_L)}{L(1+M^2k)}\lambda\sqrt{1+M^2k}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)B^1_\tau\mathbf{i} \\ + \frac{k(q_0-q_L)}{L(1+M^2k)}\Big[-\lambda\sqrt{1+M^2k}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)\tau + t_1\Big]\frac{\partial\mathbf{B}^1}{\partial\xi}\Big\} \end{aligned} \tag{3.18}$$

$$= \mathrm{Re}\Big[\frac{k(q_0-q_L)}{L(1+M^2k)}\lambda\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)\Big]^2(1+M^2k)\Big\{\Big(\mathbf{W}\cdot\nabla_{\xi\tau}\Big)\mathbf{W}-\tau\frac{\partial\mathbf{W}}{\partial\xi}-W_\tau\mathbf{i}\Big\}.$$

To solve (3.17), we introduce three auxiliary problems. The first one takes into account the linear terms from the momentum equation and reads:

$$\begin{cases}-\Delta_{\xi\tau}\mathbf{N}^{3,1}+\nabla_{\xi\tau}n^{3,1}+(1+M^2k)(\mathbf{W}-W_\infty\mathbf{i})-M^2kW_\tau\mathbf{j}=\mathbf{0}, & \text{in } G\\ \mathrm{div}_{\xi\tau}\,\mathbf{N}^{3,1}=0 \quad \text{in } G, \\ \mathbf{N}^{3,1}(\xi,h(\xi))=\mathbf{0}, \\ (\mathbf{N}^{3,1},n^{3,1}) \quad 1-\text{periodic in } \xi, \quad \nabla_{\xi\tau}\mathbf{N}^{3,1}\in L^2(G).\end{cases} \tag{3.19}$$

The above problem is solvable since we established that $\mathbf{W}$ exponentially decays in a way that $\lim_{\tau\to-\infty}\mathbf{W}=W_\infty\,\mathbf{i}$, leading to

$$\lim_{\tau\to-\infty}\mathbf{N}^{3,1}(\xi,\tau)=N_\infty^{3,1}\mathbf{i}.$$

Moreover, it is straightforward to deduce (see [19, 23])

$$N_\infty^{3,1}=(1+M^2k)\int_G\tau(W_\xi-W_\infty)+(1+M^2k)W_\infty\int_G W_\xi-\int_G|W|^2-M^2k\int_G W_\xi^2. \tag{3.20}$$

Taking into account (3.18), we pose the second auxiliary problem as:

$$\begin{cases}-\Delta_{\xi\tau}\mathbf{N}^{3,2}+\nabla_{\xi\tau}n^{3,2}+\big\{(\mathbf{W}\cdot\nabla_{\xi\tau})\mathbf{W}-\tau\frac{\partial\mathbf{W}}{\partial\xi}-W_\tau\mathbf{i}\big\}=\mathbf{0}, \\ \mathrm{div}_{\tau\xi}\,\mathbf{N}^{3,2}=0 \quad \text{in } G, \\ \mathbf{N}^{3,2}(\xi,h(\xi))=0, \\ (\mathbf{N}^{3,2},n^{3,2}) \quad 1-\text{periodic in } \xi, \quad \nabla_{\xi\tau}\mathbf{N}^{3,2}\in L^2(G).\end{cases} \tag{3.21}$$

Again, due to the fact that $\mathbf{W}$ exponentially decays such that $\lim_{\tau\to-\infty}\mathbf{W}=W_\infty\,\mathbf{i}$, the above problem has the solution that decays exponentially towards a constant:

$$\lim_{\tau\to-\infty}\mathbf{N}^{3,2}(\xi,\tau)=N_\infty^{3,2}\mathbf{i}$$

This constant can be computed as (see [19]):

$$N_\infty^{3,2}=-\frac{1}{2}\int_G W_\xi W_\tau. \tag{3.22}$$

The third auxiliary problem related to (3.17) acknowledges the last term from the boundary condition and has the form:

$$\begin{cases}-\Delta_{\xi\tau}\mathbf{E}+\nabla_{\xi\tau}e=\mathbf{0} \quad \text{in } G, \\ \mathrm{div}_{\xi\tau}\,\mathbf{E}=0 \quad \text{in } G, \\ \mathbf{E}(\xi,h(\xi))=h(\xi)^3\mathbf{i}, \\ (\mathbf{E},e) \quad 1-\text{periodic in } \xi, \quad \nabla_{\xi\tau}\mathbf{E}\in L^2(G).\end{cases} \tag{3.23}$$

Similarly as for $\mathbf{D}$, we deduce that

$$\lim_{\tau\to-\infty}\mathbf{E}=E_\infty\mathbf{i}, \quad E_\infty=\int_0^1 h(\xi)^3d\xi-2\int_G\tau\frac{\partial W_\xi}{\partial\tau}.$$

Now, we go back to the problem (3.17) and solve it using the solutions of the auxiliary problems (3.19), (3.21) and (3.23). We get:

$$\mathbf{B}^3 = \frac{k(q_0-q_L)}{L(1+M^2k)}\Big\{\lambda\sqrt{1+M^2k}\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big) \cdot\Big[\frac{\lambda^2}{6}(1+M^2k)\mathbf{E} - \frac{\lambda^2}{2}(1+M^2k)W_\infty\mathbf{D} - \lambda^2(1+M^2k)\Big(\Big(\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)\Big)^2 W_\infty^2 + D_\infty\Big)\mathbf{W} + \mathbf{N}^{3,1} - t_3\mathbf{i}\Big\} + \mathrm{Re}(1+M^2k)\Big[\frac{k(q_0-q_L)}{L(1+M^2k)}\lambda\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)\Big]^2\mathbf{N}^{3,2}. \tag{3.24}$$

We choose the constant t_3 such that $B^3_\infty = 0$ leading to

$$t_3 = \lambda\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)\Big[\frac{\lambda^2}{6}(1+M^2k)^{\frac{3}{2}}E_\infty - \frac{\lambda^2}{2}(1+M^2k)^{\frac{3}{2}}W_\infty D_\infty - \lambda^2(1+M^2k)^{\frac{3}{2}}\Big(\Big(\tanh\Big(\frac{\lambda}{2}\sqrt{1+M^2k}\Big)\Big)^2 W_\infty^2 + D_\infty\Big)W_\infty + \sqrt{1+M^2k}N_\infty^{3,1} + \mathrm{Re}\frac{k(q_0-q_L)}{L}\lambda N_\infty^{3,2}\Big]. \tag{3.25}$$

4. Concluding remarks

In view of the analysis undertaken in Sec. 3, our asymptotic approximation takes the following form:

$$\mathbf{u}^\varepsilon_{\mathrm{approx}} = \mathbf{u}^0_{\varepsilon,3} + \varepsilon\mathbf{B}^1 + \varepsilon^2\mathbf{B}^2 + \varepsilon^3\mathbf{B}^3, \tag{4.1}$$

$$\mathbf{u}^0_{\varepsilon,3} = \frac{k(q_0-q_L)}{L(1+M^2k)}\Big\{1 - \frac{\cosh\big(\lambda\sqrt{1+M^2k}y\big)}{\cosh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)} + \frac{\cosh\big(\lambda\sqrt{1+M^2k}y\big)}{\cosh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)}(\varepsilon t_1 + \varepsilon^2 t_2 + \varepsilon^3 t_3)\Big\}\mathbf{i}.$$

The longitudinal component of $\mathbf{u}^\varepsilon_{\mathrm{approx}}$ reads:

$$u^\varepsilon_{\mathrm{approx}} = \mathbf{u}^\varepsilon_{\mathrm{approx}}\cdot\mathbf{i} = \frac{k(q_0-q_L)}{L(1+M^2k)}\Big\{1 - \frac{\cosh\big(\lambda\sqrt{1+M^2k}y\big)}{\cosh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)}\Big\} + \varepsilon\Big\{B_x^1 + t_1\frac{k(q_0-q_L)}{L(1+M^2k)}\frac{\cosh(\lambda\sqrt{1+M^2k}y)}{\cosh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)}\Big\} + \varepsilon^2\Big\{B_x^2 + t_2\frac{k(q_0-q_L)}{L(1+M^2k)}\frac{\cosh(\lambda\sqrt{1+M^2k}y)}{\cosh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)}\Big\} + \varepsilon^3\Big\{B_x^3 + t_3\frac{k(q_0-q_L)}{L(1+M^2k)}\frac{\cosh(\lambda\sqrt{1+M^2k}y)}{\cosh\big(\frac{\lambda}{2}\sqrt{1+M^2k}\big)}\Big\}. \tag{4.2}$$

The constants t_1, t_2 and t_3 are given by the explicit expressions (3.11), (3.16) and (3.25), respectively. Recall that we had to choose (nonzero) values of constants t_m to ensure that the proposed boundary-layer correctors vanish exponentially. Consequently, as we can clearly see from (4.2), those constants end up contributing to the velocity approximation far from the corrugated boundary as well. All the constants are computed in terms of the stabilizations constants (namely, W_∞, D_∞, $N^{3,1}_\infty$, $N^{3,2}_\infty$ and E_∞) which have all been computed in the explicit form. This means that the proposed asymptotic solution is amenable for numerical simulations and could improve the current engineering practice.

The higher-order approximation (4.2) acknowledges all the effects relevant for the considered filtration process. The effects of the oscillating boundary are clearly seen through the average boundary roughness ε and the constants t_m $(m = 1, 2, 3)$. It should be observed that the constant t_3 depends on the prescribed pressure drop and on the Reynolds number as well (see (3.25)). This means that, if the magnitude of the Reynolds number is significant, i.e. $\mathrm{Re} \approx 2 \cdot 10^3$, then the term $\varepsilon^3 t_3$ does not have to be negligible with respect to the remaining part of the asymptotic solution. Consequently, if ε is not too small (e.g. $\varepsilon = 10^{-1}$), the inertia effects should definitely be taken into account. Finally, the effects of the applied magnetic field are clearly observed already in the zero-order approximation, through the appearance of the Hartmann number M. Putting $M = 0$, we get the approximate solution proposed in [23].

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О МХД ТОКУ У ПОРОЗНОМ КАНАЛУ

Резиме. У овом раду проучавамо ток вискозног нестишљивог проводног флуида кроз ребрасти канал испуњен порозном средином. Ток флуида у каналу је под дејством попречног магнетног поља и вођен је падом притиска између ивица канала. Користећи анализу граничног слоја, изводимо асимптотски модел вишег реда узимајући у обзир утицај инерције и храпавости на брзину филтрације.

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