THEORETICAL AND APPLIED MECHANICS
Volume 49 (2022) Issue 1, 29–48 DOI: https://doi.org/10.2298/TAM211101003V

MODELLING AND STABILITY ANALYSIS OF THE NONLINEAR SYSTEM

Mitra Vesović and Radoslav Radulović

Abstract. The production industries have repeatedly combated the problem of system modelling. Successful control of a system depends mainly on the exactness of the mathematical model that predicts its dynamic. Different types of studies are very common in the complicated challenges involving the estimations and approximations in describing nonlinear machines are based on a variety of studies. This article examines the behaviour and stability of holonomic mechanical system in the the arbitrary parameter sets and functional configuration of forces. Differential equations of the behaviour are obtained for the proposed system on the ground of general mechanical theorems, kinetic and potential energies of the system. Lagrange's equations of the first and second kind are introduced, as well as the representation of the system in the generalized coordinates and in Hamilton's equations. In addition to the numerical calculations applied the system, the theoretical structures and clarifications on which all of the methods rely on are also presented. Furthermore, static equilibriums are found via two different approaches: graphical and numerical. Above all, stability of motion of undisturbed system and, later, the system that works under the action of an external disturbance was inspected. Finally, the stability of motion is reviewed through Lagrange–Dirichlet theorem, and Routh and Hurwitz criteria. Linearized equations are obtained from the nonlinear ones, and previous conclusions for the stability were proved.

1. Introduction

Systems of nonlinear differential equations play an important part in diverse subjects. Understanding nonlinear behaviour is essential, because these dynamical systems are rarely amenable to obtain the exact analytical solution, and numerical modelling is frequently required to supplement experimental research [1]. Literature related to the issue of explicit analysis of analytical mechanics and system motion, controllability and stability can be found in [2–7]. A significant overview of papers related to n-dimensional rigid body dynamics is provided in [8] while the papers in the field of the system stability problems can also be found in [9] and [10].

2020 *Mathematics Subject Classification:* 70E50; 70E55; 70F20; 70H03; 70H05; 70H14.

Key words and phrases: applied mechanics, Hamiltonian function, holonomic system, nonlinear systems, stability of undisturbed and disturbed motion.

This article suggests a different approaches for the modelling predefined multi-body system with the theoretically-based review. The example of a holonomic mechanical system in a plane, with limited reactions of constraints, concerning its energy during motion is considered. Initial positions are predefined. For modelling this precise multi-body system, the codes for simulation and graphical representation are obtained using Wolfram Mathematica program.

1.1. Description of a particular mechanical system. From a dynamical point of view, any material system can be regarded as a collection of material particles [11]. The specific mechanical system is shown in Figure 1. It is made up of material points M_1 and M_2 and of the slider M_3. They have been mutually articulated with light rigid rods. The fixed plane O_{xy} coincides with the vertical plane of motion. The vertical O_y axis is directed upwards. In the configuration space, the position of the mechanical system in relation to the fixed coordinate system O_{xy} is defined by the set of Lagrangian coordinates (q^1, q^2) where $q^1 = \varphi$ and $q^2 = \theta$ are absolute angles. The rods OM_1 and M_1M_2 are of equal length l. The third rod M_1M_3 is of double length $2l$. The spring with stiffness c_1, whose length is in the unstressed state, $l_{01} = \overline{O_1O}$, is tied between the material point M_2 and left fixed wall. Spring with stiffness c_2, whose length in the unstressed state $l_{02} = \overline{O_2A}$, has one end tied to material point M_1, while at the other end it is also connected to the same fixed wall, but in the point O_2, with $A(0, l)$. Slider M_3, which moves horizontally in O_x axis, is connected with a damper, with a coefficient of proportionality β. The second end of damper is attached to a right fixed wall. On material point M_2 acts force $F = F_0e^{-\alpha t}$. All necessary numerical data are given in the Appendix.

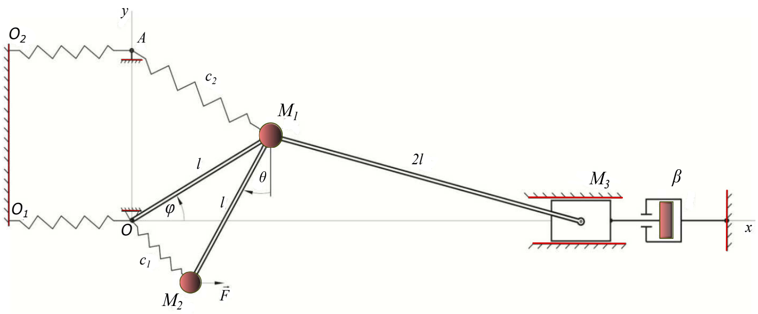


FIGURE 1. Mechanical system with stationary holonomic constrains

1.2. Constraints and Lagrange equations of the first kind. The mechanical system of N material points M_ν $(\nu = 1, 2, \dots N)$ has a certain state which is determined in each moment t by the position and velocities of all its points in the inertial reference system (IRS). When a fixed Cartesian system is introduced into an IRS, the state of the system is determined by variable scalar quantities: coordinates: x_ν, y_ν, z_ν and velocity projections $\dot{x}_\nu$, $\dot{y}_\nu$, $\dot{z}_\nu$, which must satisfy the relations:

$$f_\mu(x_1, y_1, z_1, \dots, x_N, y_N, z_N, \dot{x}_1, \dot{y}_1, \dot{z}_1, \dots, \dot{x}_N, \dot{y}_N, \dot{z}_N; t) = 0, \quad \mu = 1, 2, \dots, m < 3N \tag{1.1}$$

According to the task setting, two material points and a slider could be noticed. The movement of the slider is limited by three sticks of negligible mass that allows the point M_3 to slide along O_x axis. The first rigid rod OM_1 is connected to a fixed support O, so the movement of the point M_1 is on a circle of radius l. The second rigid rod contains points M_1M_2, while the third rigid rod contains points M_1M_3, so from this connection the first, second and third equations of the connection can be obtained in the form of (1.1):

$$f^1 = x_1^2 + y_1^2 - l^2 = 0, \tag{1.2}$$

$$f^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 = 0, \tag{1.3}$$

$$f^3 = (x_3 - x_1)^2 + (y_3 - y_1)^2 - 4l^2 = 0, \tag{1.4}$$

Also, as the movement of the slider is disabled in the vertical direction due to the guides, the slider M_3 moves only in the horizontal direction, so the fourth connection equation has the form:

$$f^4 = y_3 = 0. \tag{1.5}$$

According to the (1.2)–(1.5), it could be noticed that the connections of the material system neither depend on the speed of the material points, nor on the time, so it could be concluded, that system is holonomic and stationary. The number of geometric constrains is $p = 4$, while the number of material points is $N = 3$. The total number of unknown parameters is $2N = 6$, so the number degrees freedom movement of the system can be calculated according to: $n = 2N - p$. To describe the motion of the system, two generalized coordinates are needed, which are also given in the task setting: $q^1 = \varphi$ and $q^2 = \theta$.

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial y_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial y_3} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial y_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial y_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial y_3} \end{bmatrix} \tag{1.6}$$

The values of the elements in Jacobian matrix (1.6) are given in the Appendix. Due to the fact that all trajectories of the points are parallel to the vertical fixed O_{xy} plane matrix $\mathbf{J}$ is full rank, $\operatorname{rank} \mathbf{J} = p = 4$, so all the active constraints are independent. The system of differential equations, which represents Lagrange's equations of the first kind, in general case, is:

$$\begin{gathered} m_\nu \ddot{x}_\nu = X_\nu + \sum_{\alpha=1}^{p} \lambda_\alpha \frac{\partial f_\alpha}{\partial x_\nu}, \quad m_\nu \ddot{y}_\nu = Y_\nu + \sum_{\alpha=1}^{p} \lambda_\alpha \frac{\partial f_\alpha}{\partial y_\nu}, \\ m_\nu \ddot{z}_\nu = Z_\nu + \sum_{\alpha=1}^{p} \lambda_\alpha \frac{\partial f_\alpha}{\partial z_\nu} \end{gathered} \tag{1.7}$$

where λ_i are Lagrange multipliers. For the given system, based on Figure 1 and (1.7), Lagrange's equations of the first kind can be obtained as (1.8):

$$\begin{aligned}
m_1\ddot{x}_1 &= -c_2x_1 + 2\lambda_1x_1 + 2\lambda_2(x_1 - x_2) + 2\lambda_3(x_1 - x_3)\\
m_1\ddot{y}_1 &= -mg + c_2(l - y_1) + 2\lambda_1y_1 + 2\lambda_2(y_1 - y_2) + 2\lambda_3(y_1 - y_3)\\
m_2\ddot{x}_2 &= F - c_1x_2 + 2\lambda_2(x_2 - x_1); \quad m_2\ddot{y}_2 = -mg - c_1y_2 + 2\lambda_2(y_2 - y_1)\\
m_3\ddot{x}_3 &= -\beta\dot{x}_3 + 2\lambda_3(x_3 - x_1) + F_{(fr)x}; \quad m_3\ddot{y}_3 = -mg + 2\lambda_3(y_3 - y_1) + \lambda_4
\end{aligned} \tag{1.8}$$

Projection of the sliding friction force $F_{(fr)x}$ can be calculated using the dynamic coefficient μ_d as: $F_{(fr)x} = -\mu_d|N_3|\,\mathrm{sign}(\dot{x}_3)$ and $N_3 = \lambda_4$ can be determined in advance by applying the D'Alembert principle defined by the force of inertia. As it can be seen from Figure 1, friction was not taken into account in this paper. The slider M_3 is considered to move along smooth horizontal guides and the term $F_{(fr)x}$ in (1.8) was neglected. A more detailed description and friction modelling has been shown by authors in [12], where a mechanical model with a specific challenge was presented in order to provide insight into the functioning of the machine prior to production. Analytical description, as well as insight into Coulumb friction force, are supplied for the suggested system. The static equilibrium conditions are calculated. Finally, the disturbed and undisturbed systems' motion stability was studied. Here we applied similar analysis but for the system which is specified with more springs and less dampers. In addition, we included Lagrange multipliers as four new unknowns and, by calculating the second derivative of the (1.2)–(1.5), ten equations with ten variables are obtained. Solving these equations, the expressions for Lagrange multipliers and equations of motion, i.e. explicit expressions for: x_1, x_2, x_3, y_1, y_2, y_3, λ_1, λ_2, λ_3 and λ_4, are acquired. Constraints (1.2)–(1.4) are checked in Figure 2 and coordinates over time are depicted in Figure 3. Also, the necessary and sufficient condition for a stationary force field to be potential is verified. In contrast to the previous work, here the system under the affection of two different forces is considered. The necessary and sufficient condition for a stationary force field to be potential is verified. System under the affection of two different forces is considered. In the first case, force is constantly decreasing over time. In the second case, force is constant over time. Above all that, another stability criteria (Routh) is checked. Hurwitz and Routh processes are similar, with the Routh test being a but the Routh test avoids direct calculation of determinants.

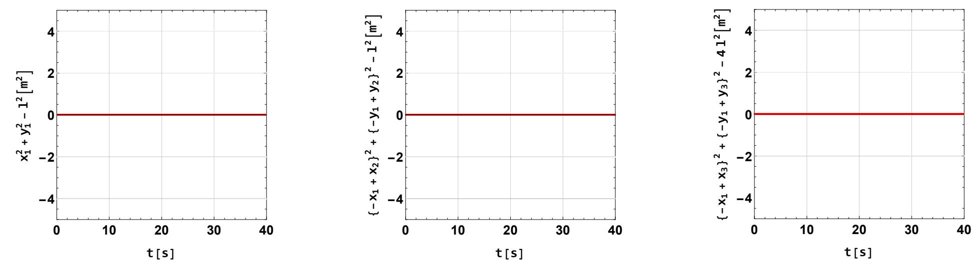


FIGURE 2. Constraints (1.2)–(1.4) check

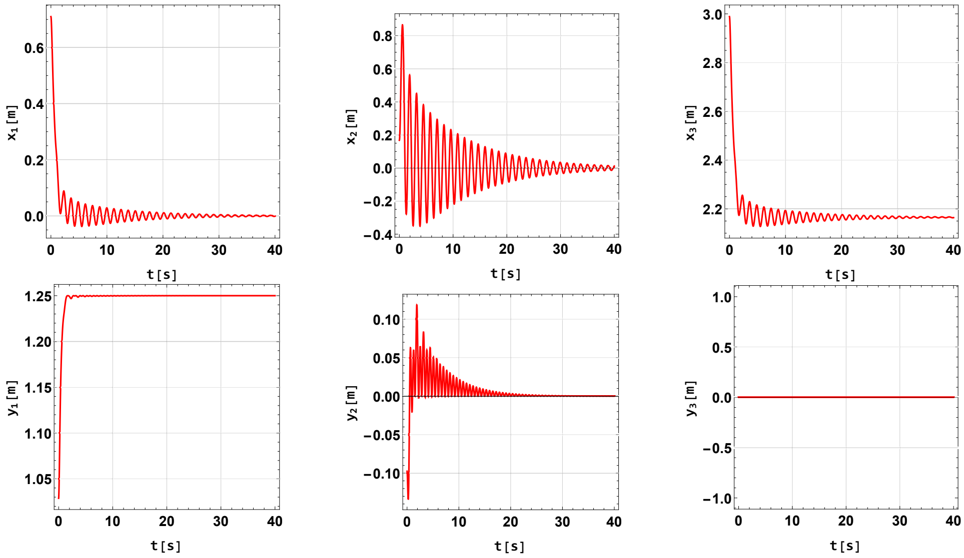


FIGURE 3. Graph of coordinates x_1, x_2, x_3, y_1, y_2 and y_3 over time

As the masses m_1, m_2 and m_3 of all three points are equal (see Appendix), from now on, the same notation m will be used for them.

1.3. Generalized coordinates and Jacobi transformation. So far, the differential equations of motion and the equations of relations, expressed as a function of position coordinates in relation to the adopted coordinate system, have been shown. Generalized Cartesian coordinates, constrains and Lagrangian equations of the first kind are introduced, so the advantages and disadvantages in terms of reaching a solution are shown. Based on the presentation so far, it could be clearly noticed that the procedure of describing the motion of the considered mechanical system with the help of independent Cartesian coordinates is not the most appropriate one. For this reason, instead of independent Cartesian coordinates, independent generalized coordinates are introduced, which also unambiguously determine the position of the mechanical system. Independent generalized coordinates represent the minimum number of geometric parameters that can unambiguously describe the movement of the considered mechanical system in the configuration space. The selected geometric parameters will be denoted by (q^1, q^2), where $q^1 = \varphi$ and $q^2 = \theta$ are the absolute angles shown in Figure 1. By introducing generalized coordinates, all independent Cartesian coordinates can be expressed as: $\xi_{p+j} = \xi_{p+j}(q^1, q^2, \dots, q^n; t)$; $j = 1, 2, \dots n$, where n is the number of degrees of freedom, $n = 3N - p$ and $q^j, j = 1, 2, \dots n$ generalized coordinates. If independent Cartesian coordinates are y_1 and x_2, it can be written: $y_1 = l \sin q^1$ and $x_2 = l \cos q^1 - l \sin q^2$. The coordinates of all points can be expressed via generalized coordinates: $\xi_i = \xi_i(q^1, q^2, \dots, q^n; t)$ $i = 1, 2, \dots 3N$, but only under the condition that the determinant of the Jacobian matrix $\boldsymbol{J}_1$, of the Jacobian matrix $\boldsymbol{J}_1$ is not equal to 0:

$$\mathbf{J_1} = \begin{bmatrix} \frac{\partial y_1}{\partial q^1} & \frac{\partial y_1}{\partial q^2} \\ \frac{\partial x_3}{\partial q^1} & \frac{\partial x_3}{\partial q^2} \end{bmatrix}; \quad |\mathbf{J_1}| = \begin{vmatrix} \frac{\partial \xi_{p+1}}{\partial q^1} & \frac{\partial \xi_{p+1}}{\partial q^2} & \cdots & \frac{\partial \xi_{p+1}}{\partial q^n} \\ \frac{\partial \xi_{p+2}}{\partial q^1} & \frac{\partial \xi_{p+2}}{\partial q^2} & \cdots & \frac{\partial \xi_{p+2}}{\partial q^1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \xi_{3N}}{\partial q^1} & \frac{\partial \xi_{3N}}{\partial q^2} & \cdots & \frac{\partial \xi_{3N}}{\partial q^n} \end{vmatrix} \neq 0 \tag{1.9}$$

The Jacobian matrix of transformation (1.9) and determinant is given in the Appendix. Dependent Cartesian coordinates are expressed through generalized coordinates and by differentiating those expressions velocities over generalized coordinates are easily obtained:

$$\begin{aligned}
x_1 &= l\cos q^1 & \dot{x}_1 &= -l\sin q^1 \dot{q}^1 \\
y_1 &= l\sin q^1 & \dot{y}_1 &= l\cos q^1 \dot{q}^1 \\
x_2 &= l\cos q^1 - l\sin q^2 & \dot{x}_2 &= -l\sin q^1 \dot{q}^1 - l\cos q^2 \dot{q}^2 \\
y_2 &= l\sin q^1 - l\cos q^2 & \dot{y}_2 &= l\cos q^1 \dot{q}^1 + l\sin q^2 \dot{q}^2 \\
x_3 &= l\cos q^1 + l\sqrt{3+\cos^2 q^1} & \dot{x}_3 &= -l\sin q^1 \left(1 + \frac{\cos q^1}{\sqrt{3+\cos^2 q^1}}\right)\dot{q}^1 \\
y_3 &= 0 & \dot{y}_3 &= 0
\end{aligned} \tag{1.10}$$

The velocities intensities of the material points M_1, M_2 and M_3 are calculated according to the 1.10 and following formulas:

$$\begin{aligned}
V_1 &= \sqrt{\dot{x}_1^2 + \dot{y}_1^2} = \sqrt{l^2 \dot{q}^{1^2}} \\
V_2 &= \sqrt{\dot{x}_2^2 + \dot{y}_2^2} = \sqrt{l^2 \dot{q}^{1^2} + 2l^2 \sin(q^1+q^2)\dot{q}^1\dot{q}^2 + l^2\dot{q}^{2^2}} \\
V_3 &= \sqrt{\dot{x}_3^2 + \dot{y}_3^2} = \sqrt{l^2 \sin^2 q^1 \left(1 + \frac{\cos q^1}{\sqrt{3+\cos^2 q^1}}\right)^2 \dot{q}^{1^2}}
\end{aligned} \tag{1.11}$$

Based on expressions in (1.11), the kinetic energy of the system can be calculated:

$$\begin{aligned}
T &= \sum_{\nu=1}^{N} T_\nu = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu v_\nu^2 \\
T &= \frac{1}{2} m l^2 \Bigg(\dot{q}^{1^2} + \sin^2 q^1 \left(1 + \frac{\cos q^1}{\sqrt{3+\cos^2 q^1}}\right)^2 \dot{q}^{1^2} \\
&\qquad + (\dot{q}^{1^2} + 2\dot{q}^1\dot{q}^2 \sin\big(q^1+q^2\big) + \dot{q}^{2^2})\Bigg)
\end{aligned} \tag{1.12}$$

1.4. Generalized forces. Virtual work δA on virtual displacement can be written in the developed form: $\delta A = Q_1\delta q^1 + Q_2\delta q^2 + \cdots + Q_n\delta q^n = \sum_{\alpha=1}^{n} Q_\alpha \delta q^\alpha$, where Q_α represents generalized force corresponding to a generalized coordinate q^α.

$$Q_\alpha = Q_\alpha^{cs} + Q_\alpha^{w} + Q_\alpha^{ncs} \quad \alpha = 1, \dots, n \tag{1.13}$$

Generalized force Q_α per the generalized coordinate q^α represents the sum of generalized conservative forces Q^{cs}_α, generalized damping forces Q^w_α and generalized non–conservative forces Q^{ncs}_α per the same generalized coordinate. Generalized conservative forces Q^{cs}_α are calculated according to: $Q^{cs}_\alpha = -\frac{\partial \Pi}{\partial q^\alpha}$.

The function Π represents the potential energy of the system and for our task is:

$$\begin{gathered}\Pi = mg(y_1 + y_2 + y_3) + \frac{c_1}{2}(x_2^2 + y_2^2) + \frac{c_2}{2}(x_1^2 + (l - y_1)^2)\\ \Pi = 2mgl\sin q^1 - mgl\cos q^2 + c_2 l^2(1 - \sin q^1) + c_1 l^2(1 - \sin(q^1 + q^2))\end{gathered} \tag{1.14}$$

Based on (1.14), generalized conservative forces can be determined for a specific case: $Q^{cs}_{q^1} = -\frac{\partial \Pi}{\partial q^1}$ and $Q^{cs}_{q^2} = -\frac{\partial \Pi}{\partial q^2}$.

The term for generalized damping forces is: $Q^w_\alpha = -\frac{\partial \Phi}{\partial \dot{q}^\alpha}$, where Φ can be expressed as a linear function of the square of the relative velocity and the damping coefficient of proportionality. Non-conservative forces are calculated as:

$$Q^{ncs}_{q^1} = -Fl\sin q^1 \quad \text{and} \quad Q^{ncs}_{q^2} = -Fl\cos q^2.$$

Finally, according to the (1.13), the total generalized force per generalized coordinate q^1 is: $Q_{q^1} = Q^{cs}_{q^1} + Q^w_{q^1} + Q^{ncs}_{q^1}$ and per generalized coordinate q^2: $Q_{q^2} = Q^{cs}_{q^2} + Q^w_{q^2} + Q^{ncs}_{q^2}$.

$$\begin{aligned} Q_{q^1} =& - 2mgl\cos q^1 + c_2 l^2 \cos q^1 + c_1 l^2 \cos(q^1 + q^2)\\ & - \beta l^2 \dot{q}^1 \sin^2 q^1 \left(1 + \frac{\cos q^1}{\sqrt{3 + \cos^2 q^1}}\right)^2 - Fl\sin q^1;\\ Q_{q^2} =& -mgl\sin q^2 + c_1 l^2 \cos(q^1 + q^2) - Fl\cos q^2\end{aligned} \tag{1.15}$$

1.5. Form Lagrangian equations of the second kind in covariant and contravariant formulation. Kinetic energy of the system can be expressed via the inertial coefficients of metric tensors $a_{(\nu)\alpha\beta}, \alpha, \beta = 1, 2, \ldots n$

$$T = \frac{1}{2}\dot{q}^\alpha \dot{q}^\beta \sum_{v=1}^{N} a_{(\nu)\alpha\beta} = \frac{1}{2} a_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta,$$

and how is the symmetry property of a covariant metric tensor valid, ie. $a_{12} = a_{21}$ for this task it follows:

$$T = \frac{1}{2}a_{11}\dot{q}^{1\,2} + a_{12}\dot{q}^1\dot{q}^2 + \frac{1}{2}a_{22}\dot{q}^{2\,2}. \tag{1.16}$$

By comparing the equations (1.12) and (1.16), the coefficients are obtained (see Appendix). Further, Christoffel symbols of the first kind are acquired as:

$$\begin{gathered}\Gamma_{11,1} = \frac{1}{2}\frac{\partial a_{11}}{\partial q^1}; \quad \Gamma_{12,1} = \Gamma_{21,1} = \frac{1}{2}\frac{\partial a_{11}}{\partial q^2};\\ \Gamma_{22,1} = \frac{\partial a_{21}}{\partial q^2} - \frac{1}{2}\frac{\partial a_{22}}{\partial q^1}; \quad \Gamma_{11,2} = \frac{\partial a_{21}}{\partial q^1} - \frac{1}{2}\frac{\partial a_{11}}{\partial q^2}; \quad \Gamma_{22,2} = \frac{1}{2}\frac{\partial a_{22}}{\partial q^2} = 0\end{gathered} \tag{1.17}$$

Finally, Lagrange's equations of the second kind in covariant formulation are:

$$a_{11}\ddot{q}^1 + a_{12}\ddot{q}^2 + \Gamma_{11,1}{\dot{q}^1}{}^2 + \Gamma_{22,1}{\dot{q}^2}{}^2 = Q_{q^1};$$
$$a_{21}\ddot{q}^1 + a_{22}\ddot{q}^2 + \Gamma_{11,2}{\dot{q}^1}{}^2 = Q_{q^2}.$$

With the substitution (4.1), (1.17) and (1.15) they become:

$$\Big(2ml^2 + ml^2\sin^2(q^1) + \frac{2ml^2\sin^2(q^1)\cos(q^1)}{\sqrt{3+\cos^2(q^1)}} + \frac{ml^2\sin^2(q^1)\cos^2(q^1)}{3+\cos(q^1)}\Big)\ddot{q}^1$$
$$+ (ml^2\sin(q^1+q^2))\ddot{q}^2 + ml^2\sin(2q^1){\dot{q}^1}{}^2$$
$$+ 2ml^2\frac{(\sin(2q^1)\cos(q^1) - \sin(q^1)^3)(3+\cos(q^1)^2)}{\sqrt{3+\cos^2(q^1)}}\dot{q}^{12}$$
$$+ ml^2\frac{\sin(2q^1)\cos(2q^1)(3+\cos(q^1)) + \sin(q^1)^3\cos(q^1)^2}{(3+\cos(q^1))^2}\dot{q}^{12}$$
$$+ 2ml^2\frac{\sin(q^1)^3\cos(q^1)^2}{\sqrt{3+\cos^2(q^1)}}\dot{q}^{12}$$
$$+ml^2\cos(q^1+q^2){\dot{q}^2}{}^2 = 2mgl\cos(q^1) + c_2l^2\cos(q^1) + c_1l^2\cos(q^1+q^2) - Fl\sin(q^1)$$
$$-\beta l^2\sin^2(q^1)\dot{q}^1\left(1 + \frac{\cos(q^1)}{\sqrt{3+\cos^2(q^1)}}\right)^2;$$

$$ml^2\sin(q^1+q^2)\ddot{q}^1 + ml^2\ddot{q}^2 + ml^2\cos(q^1+q^2)\dot{q}^{12} = -\mathrm{mgl}\sin(q^2)+$$
$$+c_1l^2\cos(q^1+q^2) - Fl\cos(q^2)$$

Similarly, $(*)$ represents Lagrange's equations of the second kind in contravariant form. The product of Kristofel's symbol of the first kind and the contravariant metric tensor represents Kristofel's symbol of the second kind.

$$(*)\qquad \begin{aligned} &\ddot{q}^1 + \Gamma^1_{11}\dot{q}^{12} + 2\Gamma^1_{12}\dot{q}^1\dot{q}^2 + \Gamma^1_{22}{\dot{q}^2}{}^2 = Q^{q^1}\\ &\ddot{q}^2 + \Gamma^2_{11}\dot{q}^{12} + 2\Gamma^2_{12}\dot{q}^1\dot{q}^2 + \Gamma^2_{22}{\dot{q}^2}{}^2 = Q^{q^2}\end{aligned}$$

where Q^{q^1} and Q^{q^2} are generalized forces in contravariant form (see Appendix).

The obtained results from both covariant and contravariant formulations are mutually and simultaneously the same as Lagrange equations of the first kind - Figure 3, so they wont be shown again. Also, two additional Figures representing q^1 and q^2 are given in Figure 4. On the whole, the positions of the points obtained using three different approaches are confirmed.

1.6. Lagrange function, generalized impulses and Hamiltonian mechanics. Instead of Lagrange variables, Hamilton proposed the variables t, q^α, p_α, $\alpha = 1, 2, \ldots, n$, where p_α are generalized impulses, defined as $p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}$ with Lagrange function or Lagrange kinetic potential, introduced as the difference between kinetic and potential energy $L = T - \Pi$. For the scleronomic system, which kinetic

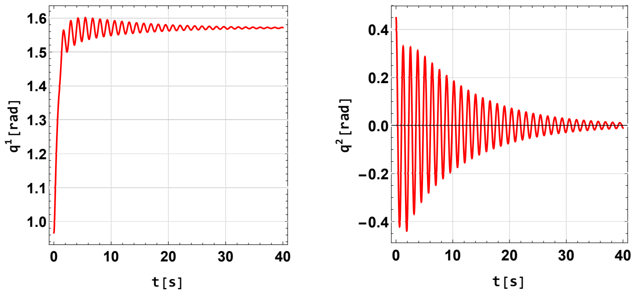


FIGURE 4. Generalized coordinates over time

energy does not explicitly depend on time, with two degrees of freedom and taking into account the components of the covariant metric tensor Hamilonian function H has the form (1.18):

$$H = \frac{1}{a}\left(\frac{1}{2}a_{22}p_{q^1}^2 - a_{12}p_{q^1}p_{q^2} + \frac{1}{2}a_{11}p_{q^2}^2\right) + \Pi, \tag{1.18}$$

where $a = a_{11}a_{22} - a_{12}a_{21}$. Generalized impulses are:

$$p_1 = \frac{\partial T}{\partial \dot{q}^1} = p_{q^1} = \frac{\partial T}{\partial \dot{q}^1} = a_{11}\dot{q}^1 + a_{12}\dot{q}^2 \text{ and } p_2 = \frac{\partial T}{\partial \dot{q}^2} = p_{q^2} = \frac{\partial T}{\partial \dot{q}^2} = a_{12}\dot{q}^1 + a_{22}\dot{q}^2$$

In this way, state of the system is completely described. By introducing Hamilton's variables, Lagrange equations of the second kind, which make system of n second order differential equations for determining n functions $q^\alpha = q^\alpha(t)$, can be replaced by equivalent system of $2n$ first order differential equations for determining $2n$ functions of $q^\alpha = q^\alpha(t)$, $p_\alpha = p_\alpha(t)$.

Graphs in generalized coordinates, as well as the positions of material points were calculated. Summarising the Hamiltonian equations also obtained the same solutions as by applying Lagrange equations of the first and second kind in the covariant and contravariant form. For the proposed system, result is given in the Appendix. After numerical calculation, graph of Cartesian and generalized coordinates over time obtained. It can be concluded that the Hamiltonian equations also obtained the same solutions as by applying all of the previous methods in Figures 3 and 4. Generalized momenta are presented in Figure 5.

2. Static equilibrium conditions and stability of the system

In the position of static equilibrium, the generalized forces and the generalized velocities of the points equal zero. Determining the position of static equilibrium comes down to determining the values of the generalized coordinates in which the system is at rest. The condition of static equilibrium is that the sum of all potential and non-potential forces is equal to zero:

$$-\frac{\partial \Pi}{\partial q^\alpha} + \tilde{Q}_\alpha = 0, \quad \alpha = 1, \dots, n \tag{2.1}$$

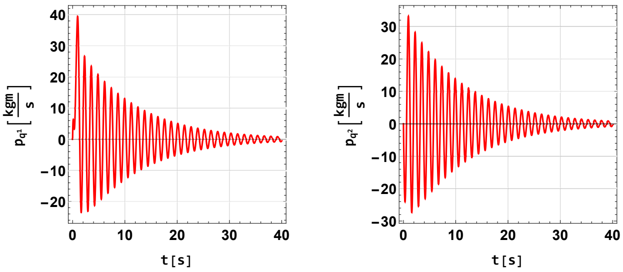


FIGURE 5. Generalized impulses over time

The conservative mechanical system is scleronomic and exposed exclusively to the action of conservative forces (potential forces whose potential energy does not depend explicitly on time). In this case, the required equilibrium conditions (2.1), due to the absence of nonconservative forces, take the form:

$$\frac{\partial \Pi}{\partial q^\alpha} = 0, \quad \alpha = 1, \dots, n,$$

so the examination of equilibrium stability comes down to considering the potential energy of a mechanical system. If the coordinates of the equilibrium position are known, they satisfy equations (2.1). Otherwise, if the equilibrium position is not known, solving algebraic equations (2.1), the obtained solutions determine the coordinates of possible equilibrium positions, the stability of which should be examined. According to the Lagrange–Dirichlet theorem, the equilibrium position of a conservative system, in which the potential energy has an isolated minimum, represents the position of the stable equilibrium of the system. This theorem gives only a sufficient condition for the stability of the equilibrium because it does not give the possibility to judge whether the equilibrium is stable or unstable if the potential energy has no minimum in the equilibrium position. For the application of the Lagrange–Dirichlet theorem, it is important that the potential energy is represented as an analytical function and that depends on all generalized coordinates of the system.

$$\Pi \approx \frac{1}{2} c_{\alpha\beta} q^\alpha q^\beta, \quad \text{where:} \quad c_{\alpha\beta} = \left(\frac{\partial^2 \Pi}{\partial q^\alpha \partial q^\beta}\right)_0, \quad \alpha, \beta = 1, \dots, n \tag{2.2}$$

The behaviour of the potential energy in the vicinity of the equilibrium position corresponds to the behaviour of a homogeneous square form with constant coefficients $c_{\alpha\beta}$. If the potential energy has a minimum in the equilibrium position and its approximation (2.2) has a minimum in the same position. The definiteness of the matrix is examined using the Sylvester criterion. If the material point M_2 is acted upon by a force of constant intensity $\boldsymbol{F_{st}} = 100\,\mathrm{N}$, the condition of static equilibrium (2.1) are obtained as (2.3):

$$\begin{aligned} l(c_2 l - 2mg)\cos q^1 + c_1 l \cos\big(q^1 + q^2\big) - F_{st}\sin q^1) = 0 \\ l(-F_{st}\cos q^2 + c_1 l \cos\big(q^1 + q^2\big) - mg\sin q^2) = 0 \end{aligned} \tag{2.3}$$

Force meets the condition: rot $\boldsymbol{F_{st}} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_{st} & 0 & 0 \end{vmatrix} = \frac{\partial F_{st}}{\partial z}\boldsymbol{j} - \frac{\partial F_{st}}{\partial y}\boldsymbol{k} = 0$, which is a necessary and sufficient condition for a stationary force field $\boldsymbol{F_{st}}$ to be potential. The results are given in Table 2, numerically, and graphical comparison is presented in Figure 6, which shows positions of equilibrium points in space and the change of the potential energy with change of angles for both cases. As it can be seen, from both cases, there are four equilibrium positions in Table 1. As previous examination was done in the case when $F = F_0e^{-\alpha t}$, here an additional Figure is provided: $F_{st} = 0\,\mathrm{N}$. Because the first and second case give the same results, only the second approach is depicted. Equations which describe this system are nonlinear and there is no point in checking system's stability, only the stability of the equilibriums [13]. There are many ways to examine the stability of the undisturbed motion of the specific system. For holonomic system from this task solutions were relatively easily obtained. An example of the non-holonomic system can be found in [14] and [15]. For example, previously mentioned Lagrange–Dirichlet theorem could be used. Also, in control theory, one of the most popular are Routh and Hurwitz criteria, i.e. via the basic principal minors of the Hurwitz matrix and with analysis of the characteristic polynomial.

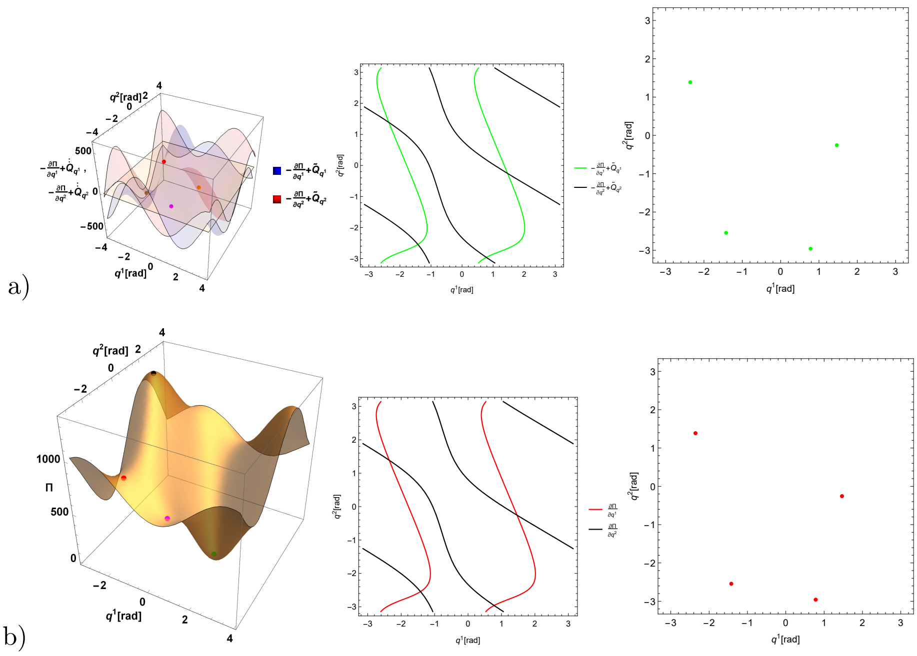


FIGURE 6. a) First approach: Spatial arrangement of equilibrium positions, Curves intersection, Static equilibriums; b) Second approach: Potential energy, Curves intersection, Static equilibriums

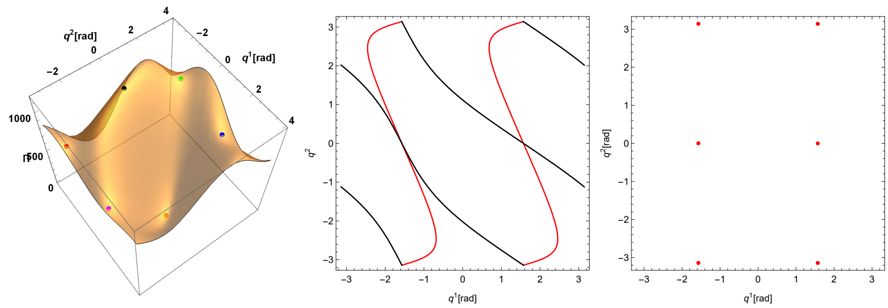


FIGURE 7. Second approach: Potential energy, Curves intersection, Static equilibriums when $F_{st} = 0\,\mathrm{N}$

TABLE 1. Static equilibrium positions

Case	q^1	q^2
eq_1	0.779777	-2.95918
eq_2	1.46544	-0.256309
eq_3	-1.42176	-2.54488
eq_4	-2.35647	1.3861

TABLE 2. The judgment on the stability of the position of static equilibrium is based on the Lagrange–Dirichlet theorem

Case	$\mathbf{C}_{\alpha,\beta}$	Δ_1	Δ_2	Stability
eq_1	$\begin{pmatrix} 111.015 & -208.317 \\ -208.317 & -306.19 \end{pmatrix}$	111.015	-77387.7	unstable
eq_2	$\begin{pmatrix} 576.559 & 237.481 \\ 237.481 & 387.748 \end{pmatrix}$	576.559	167163	stable
eq_3	$\begin{pmatrix} -119.042 & 186.514 \\ 186.514 & 155.356 \end{pmatrix}$	-119.042	-53281.4	unstable
eq_4	$\begin{pmatrix} -529.601 & -209.497 \\ -209.497 & -309.859 \end{pmatrix}$	-529.601	120212	unstable

2.0.1. *Lagrange–Dirichlet theorem.* If the quadratic form (2.2) can take negative values, then instability follows. This can be verified using Silvester's criterion. It is necessary and sufficient for all leading principal minors of the Hermit matrix to be positive in order to obtain positively definite quadratic form. As only the second condition fulfills the criterion, it is concluded that only the second equilibrium is stable.

TABLE 3. Characteristic polynomials

Case	Characteristic polynomial
eq_1	$\lambda^4 + 1.5464\lambda^3 - 65.7945\lambda^2 - 30.6665\lambda + 325.543$
eq_2	$\lambda^4 + 6.01551\lambda^3 + 38.3023\lambda^2 + 149.28\lambda + 305.992$
eq_3	$\lambda^4 + 5.33619\lambda^3 + 2.36749\lambda^2 + 53.0565\lambda - 83.5357$
eq_4	$\lambda^4 + 1.5464\lambda^3 - 65.7945\lambda^2 - 30.6665\lambda + 325.543$

TABLE 4. The judgment on the stability of the position of static equilibrium is based on the Routh criteria

Case	Routh table	Number of right hand side poles	Stability
eq_1	$\begin{pmatrix} 1.0 & -65.7945 & 325.543 \\ 1.5464 & -30.6665 & 0 \\ -45.9636 & 325.543 & 0 \\ -19.7139 & 0 & 0 \\ 325.543 & 0 & 0 \end{pmatrix}$	2	unstable
eq_2	$\begin{pmatrix} 1.0 & 38.3023 & 305.992 \\ 6.0155 & 149.28 & 0 \\ 13.4864 & 305.992 & 0 \\ 12.7950 & 0 & 0 \\ 305.992 & 0 & 0 \end{pmatrix}$	0	stable
eq_3	$\begin{pmatrix} 1.0 & 2.3675 & -83.5357 \\ 5.3362 & 53.0565 & 0 \\ -7.5753 & -83.5357 & 0 \\ -5.7879 & 0 & 0 \\ -83.5357 & 0 & 0 \end{pmatrix}$	1	unstable
eq_4	$\begin{pmatrix} 1.0 & -65.7945 & 325.543 \\ 1.5464 & -30.6665 & 0 \\ -45.9636 & 325.543 & 0 \\ -19.7139 & 0 & 0 \\ 325.543 & 0 & 0 \end{pmatrix}$	2	unstable

2.0.2. *Routh criterion.* The Rauth–Hurvitz criterion gives the conditions for asymptotic stability in the linear approximation. A necessary and sufficient condition for the system to be stable is that all the coefficients of the first Roth column are of the same sign. The system is unstable if there is a character change in the first Routh column. The number of sign changes corresponds to the number of unstable poles of the system.

TABLE 5. The judgment on the stability of the position of static equilibrium is based on the Hurwitz criteria

Case	Hurwitz matrix	Roots of characteristic polinomial	Stability
eq_1	$\begin{pmatrix} 30.6665 & -325.543 & 0 & 0 \\ -1.5464 & 65.7945 & 30.6665 & 0 \\ 0 & -1 & -1.564 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\lambda_1 = -8.3971$ $\lambda_2 = -2.5222$ $\lambda_3 = 2.1190$ $\lambda_4 = 7.2538$	unstable
eq_2	$\begin{pmatrix} -149.28 & -305.992 & 0 & 0 \\ -6.01551 & -38.3023 & -149.28 & 0 \\ 0 & -1 & -6.1551 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\lambda_1 = -2.92464 + 1.89756i$ $\lambda_2 = -2.9246 - 1.8976i$ $\lambda_3 = -0.0831 + 5.0169i$ $\lambda_4 = -0.0831 - 5.0169i$	stable
eq_3	$\begin{pmatrix} -53.0565 & 83.5357 & 0 & 0 \\ -5.3362 & -2.36749 & -53.0565 & 0 \\ 0 & -1 & -5.33619 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\lambda_1 = -6.5217$ $\lambda_2 = -0.035769 - 3.19186i$ $\lambda_3 = -0.035769 + 3.19186i$ $\lambda_4 = 1.25709$	unstable
eq_4	$\begin{pmatrix} 30.6665 & -325.543 & 0 & 0 \\ -1.5464 & 65.7945 & 30.6665 & 0 \\ 0 & -1 & -1.564 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\lambda_1 = -8.3971$ $\lambda_2 = -2.5222$ $\lambda_3 = 2.1190$ $\lambda_4 = 7.2538$	unstable

2.0.3. *Hurwitz criteria.* Hurwitz method is based on testsing well-known Hurwitz matrix. Stability is checked by three different criteria. All of them gave the same results-out of 4 equilibrium states, only the second is stable. Further, it can be seen from Table 3 that Case 1 and Case 4 have the same characteristic polynomial, so their Rauth tables (Table 4) and Hurwitz matrix (Table 5) are same.

2.1. Disturbed motion with non-linear and with linearized equations. Final solutions of differential equations do not provide a direct answer to the question of whether the state of the system is stable or unstable. Therefore, for stability testing, a disturbed state is considered. Stability of the equilibrium position is examinated in the sense of Lyapunov's definition. The disturbances ξ_α and η_α are differences between disturbed and non-disturbed motion. Initial disturbances are taken to be: $\xi_1 = 0.2$, $\xi_2 = 0$, $\eta_1 = 0.3$, $\eta_2 = 0$ and force of constant intensity $F_0 = 100\,\mathrm{N}$ acts on the material point M_2. The check was performed for all other equilibrium points and earlier conclusions were confirmed. If the system is brought to the position of the first static equilibrium, according to the previous results (Tables 2, 4 and 5), the undisturbed motion is determined. Based on the exact nonlinear differential equations of disturbed motion, the disturbed motion of the system in the vicinity of the static equilibrium position is determined. Linearized equations were calculated based on the linearization in the vicinity of the static equilibrium points and they differ greatly from the nonlinearized ones, Figures 8–10.

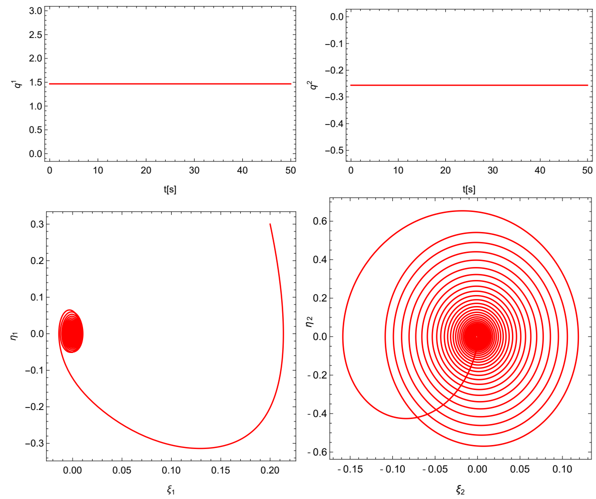
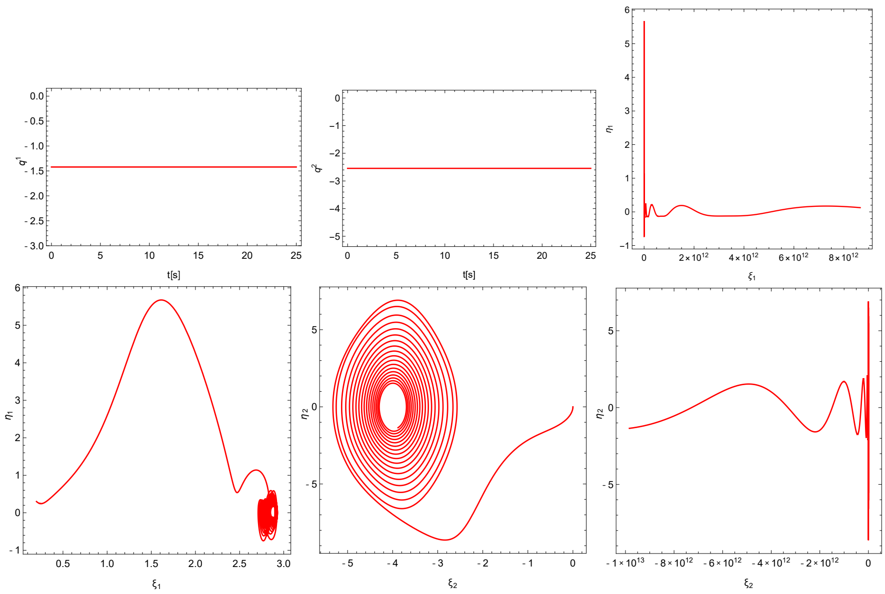


FIGURE 8. Second case with $F = 100\,\mathrm{N}$: Graphical confirmation of the position of static equilibrium, disturbed motion of equilibrium point with nonlinearized and linearized equations (they are the same)



a) b)

FIGURE 9. Third case with $F = 100\,\mathrm{N}$: Graphical confirmation of the position of static equilibrium, disturbed motion of equilibrium point with a) nonlinearized and b) linearized equations

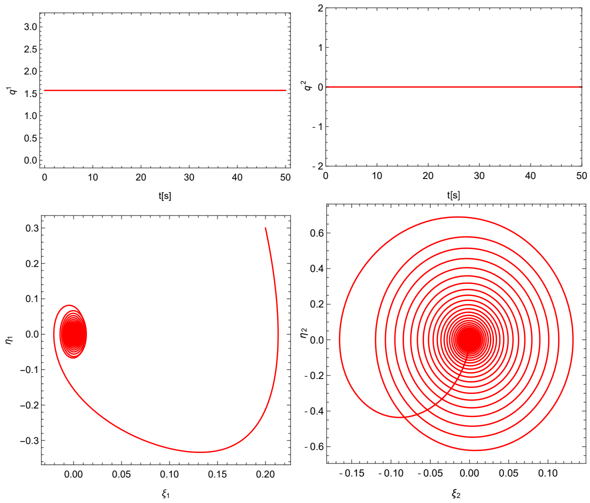


FIGURE 10. Stable case with $F = 0\,\mathrm{N}$ from Figre 7: Graphical confirmation of the position of static equilibrium, disturbed motion of equilibrium point with (non)linearized and linearized equations.

3. Conclusion

This paper provides mechanical construction which can be modelled as holonomic mechanical system with constrained motion. Particular initial conditions are given and the system has been analyzed with many different approaches: using Lagrange equations of the first and second kind and also with Hamiltonian mechanics. The system of the ten equations with ten variables was acquired by applying Lagrange's equations of the first kind. Further, it is determined that the analysis of the system in this way is complicated and redundant. More elegant approach for analyzing system is by applying Lagrange equations of the second kind. In order to obtain a less complicated system, generalized coordinates, as well as the Hamiltonians momenta were obtained; system's motion has been confirmed. Second order differential equations were replaced with Hamilton's equations, so solving the problem was simplified. At the minimum of the potential energy all of the equilibrium points are found. Their stability equilibriums was checked using three methods: Lagrange–Dirichlet, Routh and Hurwitz; system motion in disturbed and undisturbed case was investigated. Disturbed motion with nonlinear and linearized equations was presented for some cases. If the system is brought to the position of the second static equilibrium, according to the previous results the norm of disturbed motion strives to zero during time (since all of the earlier criteria have shown stability of this equilibrium). In exactly the same way as for the stable static equilibrium, it has been verified that positions of other static equilibriums are correctly determined and unstable. Same judgements were verified for different techniques. It can be concluded that linearized equations will not predict the behaviour of the system in the same way as nonlinear ones, so this system requires modelling with nonlinear equations with one of the presented methods.

4. Appendix

Parameters	Value	Name
$m_1(m)$	10 kg	Mass of material point M_1
$m_2(m)$	10 kg	Mass of material point M_2
$m_3(m)$	10 kg	Mass of slider-crank M_3
l	1.25 m	length
c_1	162.5 N/m	Spring stiffness
c_2	366.67 N/m	Spring stiffness
β	121 Ns/m	Coefficient of proportionality
F_0	100 N	Static force
φ_0	$4\pi/13$	Initial angle
θ_0	$\pi/7$	Initial angle
$\dot{\varphi}_0$	0	Initial velocity
$\dot{\theta}_0$	0	Initial velocity
α	0.7	Coefficient

Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} 2x_1 & 2y_1 & 0 & 0 & 0 & 0 \\ 2x_1 - 2x_2 & 2y_1 - 2y_2 & 2x_2 - 2x_1 & 2y_2 - 2y_1 & 0 & 0 \\ 2x_1 - 2x_3 & 2y_1 - 2y_3 & 0 & 0 & 2x_3 - 2x_1 & 2y_3 - 2y_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

The Jacobian matrix of transformation and the determinant has value:

$$\mathbf{J_1} = \begin{bmatrix} l\cos q^1 & 0 \\ -l\sin q^1 & -l\cos q^2 \end{bmatrix}; \quad |\mathbf{J_1}| = -l^2 \cos q^1 \cos q^2$$

Generalized conservative forces:

$$\begin{aligned} Q^{cs}_{q^1} &= -2mgl\cos q^1 + c_2 l^2 \cos q^1 + c_1 l^2 \cos\big(q^1 + q^2\big) \\ Q^{cs}_{q^2} &= -mgl\sin q^2 + c_1 l^2 \cos\big(q^1 + q^2\big) \end{aligned}$$

Generalized damping forces:

$$Q^{w}_{q^1} = -\beta l^2 \dot{q}^1 \sin^2 q^1 \left(1 + \frac{\cos q^1}{\sqrt{3+\cos^2 q^1}}\right)^2; \quad Q^{w}_{q^2} = 0$$

Coefficients of metric tensors:

$$\begin{aligned} a_{11} &= 2ml^2 + ml^2 \sin^2 q^1 \left(1 + \frac{\cos q^1}{\sqrt{3+\cos^2 q^1}}\right)^2; \\ a_{12} &= ml^2 \sin(\varphi + \theta); \qquad a_{22} = ml^2 \end{aligned} \tag{4.1}$$

Christoffel symbols of the first kind:

$$\Gamma_{11,1} = ml^2 \sin\left(2q^1\right) + 2ml^2 \frac{(\sin\left(2q^1\right)\cos\left(q^1\right) - \sin^3(q^1))(3+\cos^2(q^1))}{\sqrt{3+\cos^2(q^1)}} +$$

$$+ 2ml^2 \frac{\sin^3(q^1)\cos^2(q^1)}{\sqrt{3+\cos^2(q^1)}} + ml^2 \frac{\sin\left(2q^1\right)\cos\left(2q^1\right)(3+\cos\left(q^1\right)) + \sin^3(q^1)\cos^2(q^1)}{(3+\cos(q^1))^2}$$

$$\Gamma_{12,1} = 0; \quad \Gamma_{22,1} = ml^2\cos\left(q^1+q^2\right); \quad \Gamma_{11,2} = ml^2\cos\left(q^1+q^2\right); \quad \Gamma_{22,2} = 0$$

Kristofel's symbols of the second kind are determined by:

$$\Gamma^1_{11} = \frac{a_{22}\Gamma_{11,1} - a_{12}\Gamma_{11,2}}{a}; \quad \Gamma^1_{12} = \frac{a_{22}\Gamma_{12,1} - a_{12}\Gamma_{12,2}}{a} = 0 = \Gamma^1_{21}; \quad \Gamma^1_{22} = \frac{a_{22}\Gamma_{22,1}}{a};$$

$$\Gamma^2_{11} = \frac{-a_{21}\Gamma_{11,1} + a_{11}\Gamma_{11,2}}{a}; \quad \Gamma^2_{12} = 0 = \Gamma^2_{21}; \quad \Gamma^2_{22} = \frac{-a_{21}\Gamma_{22,1}}{a}; \quad a = a_{11}a_{22} - a_{12}a_{21}$$

Generalized forces in contravariant form are:

$$Q^{q^1} = \frac{a_{22}Q_{q^1} - a_{12}Q_{q^2}}{a}; \quad Q^{q^2} = \frac{-a_{21}Q_{q^1} + a_{11}Q_{q^2}}{a}.$$

Hamiltonian function:

$$H = (l(-gm\cos q^2 + (-1c_2l + 2gm)\sin q^1 + l(1c_1 + 1c_2 - 1c_1\sin\left(q^1+q^2\right))) + \frac{A}{B},$$

where $A = (l^2m(-49 + 1\cos 4q^1 - 16\cos\left(2(q^1+q^2)\right) + \sin^2 q^1(-16 + 8\sin^2 q^1 - 16\cos q^1\sqrt{4-\sin^2 q^1} - 8\sin^2(q^1+q^2))))$, and $B = ({p_{q^1}}^2(-16+4\sin^2 q^1) + {p_{q^2}}^2(-35 + 2\cos 2q^1 + 1\cos 4q^1 - 8\cos q^1 \sin^2 q^1\sqrt{4-\sin^2 q^1}) + p_{q^2}p_{q^1}(32 - 8\sin^2 q^1)\sin\left(q^1+q^2\right))$.

Acknowledgments. The research of the first author was supported by the Ministry of Education, Science and Technological Development of the The Republic of Serbia, under contract 451-03-9/2021-14/200105, from date 05/02/2021. This work of the second author was financially supported by the Ministry of Education, Science and Technological Development, Grant OI174001.

References

1. S. Black, D. Christie, N. Finlayson, *Spectral analysis of nonlinear systems*, Theor. Appl. Mech. **37**(2) (2010), 111–137.
2. L. Pars, *Treatise on Analytical Dynamics*, Heinemann, London, 1968.
3. J. Papastavridis, *Analytical Mechanics*, Oxford University Press, New York, 2002.
4. H. Josephs, R. Huston, *Dynamics of Mechanical Systems*, CRC Press, 2019.
5. S. Sosnitskii, *On the Lagrange stability of motion and final evolutions in the three-body problem*, Applied Mathematics **4**(2) (2013), 369–377.
6. L. Nivado, *Analytical Mechanics*, Universidade Federal Fluminense, Brazil, 2018.
7. V. Jurdjevic, J. P. Quinn, *Controllability and stability*, Journal of differential equations. **28**(3) (1978) 381–389.
8. Y. N. Fedorov, V. V. Kozlov, N. Finlayson, *Various aspects of n-dimensional rigid body dynamics*, Transl., Ser. 2, Am. Math. Soc. **168** (1995), 141–172.
9. R. Leine, N. van de Wouw, *Stability and convergence of mechanical systems with unilateral constraints*, Springer-Verlag, Berlin, Heidelberg, 2008.
10. M. Vesković, V. Čović, A. Obradović, *Instability of equilibrium of nonholonomic systems with dissipation and circulatory forces*, Appl. Math. Mech., Engl. Ed. **32**(2) (2011), 211–222.
11. A. I. Lurie, *Analytical Mechanics*, Springer-Verlag, Berlin, Heidelberg, New York, 2002.
12. M. Vesović, G. Petrović, R. Radulović, *Analysis of the motion and stability of the holonomic mechanical system in the arbitrary force field*, FME Transactions **49**(1) (2021), 195–205.

13. M. Vesović, R. Radulović, *Behaviour, examination and stability of the constrained mechanical system described with nonlinear equations*, 8$^{\text{th}}$ International Congress of the Serbian Society of Mechanics, Kragujevac, June 28–30, (2021), 54–63.
14. R. Radulović, D. Zeković, M. Lazarević, Š. Segl'a, B. Jeremić, *Analysis the brachistochronic motion of a mechanical system with nonlinear nonholonomic constraint*, FME Transactions **42**(4) (2014), 290–296.
15. R. Radulović, A. Obradović, B. Jeremić, *Analysis of the minimum required coefficient of sliding friction at brachistochronic motion of a nonholonomic mechanical system*, FME Transactions **42**(3) (2014), 201–206.

МОДЕЛОВАЊЕ И АНАЛИЗА СТАБИЛНОСТИ ДАТОГ НЕЛИНЕАРНОГ СИСТЕМА

Резиме. У производној индустрији инжењери се константно сусрећу са проблемом моделовања система. Успешно управљање система у великој мери зависи од тачности математичког модела који предвиђа његову динамику. У компликованим изазовима описивања нелинеарних машина користе се различите методе. У овом раду је приказано понашање и стабилност холономног механички система у произвољном пољу сила. За предложени систем су, на основу општих механичких теорема, добијене диференцијалне једначине понашања, кинетичка и потенцијална енергија система. Лагранжове једначине прве и друге врсте су поређене са генерализованим координатама и Хамилтоновим једначинама. Поред нумеричких поступака у раду, на одређеним местима дата су појашњења као и преглед теоријских основа на којима почивају дате методе. У наставку, положаји статичке равнотеже се проналазе коришћењем графичког и нумеричког приступа. Коначно, разматрана је стабилност равнотежних положаја коришћењем Лагранж-Дирихеове теореме и Рут-Хурвицовог критеријума. На крају је дато поређење линеаризованих и нелинеарних једначина и ранији закључци о стабилности равнотежних положаја су верификовани.

Department of Control Engineering
University of Belgrade
Belgrade
Serbia
mvesovic@mas.bg.ac.rs

(Received 01.11.2021.)
(Revised 18.03.2022.)
(Available online 15.04.2022.)

Department of Mechanical Engineering
University of Belgrade
Belgrade
Serbia
rradulovic@mas.bg.ac.rs