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CLASSICAL SOLUTIONS FOR A CLASS OF NONLINEAR WAVE EQUATIONS

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Abstract. We study a class of initial value problems subject to nonlinear partial differential equations of hyperbolic type. A new topological approach is applied to prove the existence of nontrivial nonnegative solutions. More precisely, we propose a new integral representation of the solutions for the considered initial value problems and using this integral representation we establish existence of classical solutions for the considered classes of nonlinear wave equations.

1. Introduction and auxiliary results

Wave equations arise in mathematical models which describe wave phenomena in fields like fluid dynamics and electromagnetics. Many authors such as H. Brésiz, J. Mawhin, and K. C. Chang have developed topological tools, index theory, and variational methods to get some existence results for the one-dimensional problem with various nonlinearities. The related results are [1, 2, 4–7] and the references therein. The fixed point theory for sums of operators has recently been developed (see [3, 6, 7, 13, 15, 17, 18] and the references therein). In [7], the authors have developed a new fixed point index for the sum of an expansive mapping and a k-set contraction defined in cones of Banach spaces. Nonlinear wave processes are usually modeled using nonlinear partial differential equations. For nonlinear analogs of the wave equation, let f be a nonlinear function, the structure of which is determined by the geometric and (or) physical features of the problem, where non-linear ripple effects are many and varied. A very important model of nonlinear waves is the nonlinear Klein-Gordon equation [16]

$$u_{tt} - u_{xx} = \phi(u),$$

where $\phi(u)$ is some smooth or discontinuous function that describes distributed nonlinear restoring forces. In the linear approximation $\phi(u) = -\kappa u (\kappa > 0)$, we have the well-known string model *on an elastic bed*.

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The main aim of this paper is to investigate the following initial value problem (IVP)

$$
\begin{aligned}
u_{tt} - \Delta u &= f(t,x,u), \quad t \geqslant 0,\ x \in \mathbb{R}^n,\\
u(0,x) &= u_0(x), \quad x \in \mathbb{R}^n,\\
u_t(0,x) &= u_1(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\tag{1.1}
$$

where $n \geqslant 3$, $u_0 \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$, $u_1 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_-)$ and $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R})$ satisfying a general growth condition. Here $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_- = (-\infty, 0]$.

When $f(t,x,u) = \mp u^5$, $n = 3$, we get the three-dimensional energy-critical nonlinear wave equation.

Throughout this work, by a solution we mean a nonnegative function $u \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}^n)$ with $u(t,x) \geqslant 0$ on $\mathbb{R}_+ \times \mathbb{R}^n$.

In [11], the problem (1.1) is investigated in the case when $n = 3$ and $f = f(t,x,y,z,u,u_t,u_x,u_y,u_z)$, where the authors give an integral representation of the solutions to the problem (1.1). In this paper, we propose a new integral representation of the solutions to the problem (1.1) for arbitrary $n \geqslant 3$, different than the integral representation of the solutions for the case $n = 3$. Note that the investigations in this paper can be applied to the case $n \geqslant 3$ and $f = f(t,x,u,u_t,u_x)$, where $u_x = (u_{x_1}, \dots, u_{x_n})$.

In [12] a Cauchy problem for a class of nonlinear wave equations in the case $n = 2$ is investigated. The authors give an integral representation of the solutions using the Green function for the wave equations in the case $n = 2$.

In [8], the authors considered the following IBVP subjected to a parabolic equation

$$
\begin{aligned}
u_t &= \Delta u + f(t,x,u), \quad (t,x) \in [0,T] \times \Omega,\\
u(0,x) &= \phi(x), \quad x \in \Omega,\\
\frac{\partial u}{\partial n} &= au + g \quad \text{on} \quad [0,T] \times \partial\Omega,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, $\partial\Omega$ is smoothed, $g \in \mathcal{C}([0,T] \times \partial\Omega, \mathbb{R}_+)$ and $f \in \mathcal{C}([0,T] \times \bar{\Omega} \times \mathbb{R}, \mathbb{R}_+)$ satisfying a general polynomial growth condition.

In [9], an IVP for the Burgers–Fisher equation has been studied

$$
\begin{aligned}
u_t - u_{xx} + \alpha(t) u u_x &= \beta(t) u(1-u), \quad t > 0,\ x \geqslant 0,\\
u(0,x) &= u_0(x), \quad x \geqslant 0,
\end{aligned}
$$

where $u_0 \in \mathcal{C}^2(\mathbb{R}_+)$, $\alpha, \beta \in \mathcal{C}(\mathbb{R}_+)$, $\alpha < 0$, $\beta \geqslant 0$ on $\mathbb{R}_+$.

Next, in [10], the following IVP subjected to a nonlinear PDE of hyperbolic type was investigated

$$
\begin{aligned}
u_{tt} - u_{xx} &= f(t,x,u_t), \quad t > 0,\ x > 0,\\
u(0,x) = u_0(x), &\quad u_t(0,x) = u_1(x), \qquad x \geqslant 0,
\end{aligned}
$$

where $u_0 \in \mathcal{C}^2(\mathbb{R}_+)$, $u_1 \in \mathcal{C}^1(\mathbb{R}_+)$ and $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R})$ satisfying a general polynomial growth condition.

Since this work is based upon a recent theoretical result, its novelty lies primarily in extending some of these contributions in the case of nonlinear PDEs. This work is concerned with the existence of bounded nontrivial nonnegative solutions for an IVP subject to an n-dimensional nonlinear hyperbolic equation, $n \geqslant 3$. An interesting point of the result is that the nonlinearity is of nonpositive sign. Moreover, it satisfies general polynomial growth conditions. New existence results of nontrivial nonnegative solutions are proved using a recent fixed point index theory on cones. We propose a new integral representation of the solutions to the considered initial value problems and using this integral representation we establish existence of classical solutions for the considered classes of nonlinear wave equations.

In what follows, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$. The following Proposition 1.1 will be used to prove our main result.

Proposition 1.1. [7,9] *Let Ω be a subset of $\mathcal{P}$ and U be a bounded open subset of $\mathcal{P}$ with $0 \in U$. Assume that the mapping $T\colon \Omega \subset \mathcal{P} \to E$ is such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $S\colon \bar{U} \to E$ is a k-set contraction with $0 \leqslant k < \gamma^{-1}$, and $S(\bar{U}) \subset (I-T)(\Omega)$. If*

$$Sx \neq (I-T)(\lambda x) \quad \text{for all } x \in \partial U \bigcap \Omega,\ \ \lambda \geqslant 1 \quad \text{and} \quad \lambda x \in \Omega,$$

then the fixed point index $i_*\left(T+S, U \bigcap \Omega, \mathcal{P}\right) = 1$.

The structure of the paper is as follows. In the next Section we give some preliminary results. In Section 3, we present and prove our main result. In the last section, Section 4, we give an example to illustrate our results.

2. Preliminary Result

Let

$$s = (s_1, \dots, s_n), \quad ds = ds_n \dots ds_1,$$

$$\prod_{j=1}^{n} \int_0^{x_j} (x_j - s_j)^2 (\cdot) ds = \int_0^{x_1} \cdots \int_0^{x_n} (x_1 - s_1)^2 \dots (x_n - s_n)^2 (\cdot) ds_n \dots ds_1,$$

$\alpha(n)$ be the volume of the unit ball in $\mathbb{R}^n$.

We will start with the following useful lemma. In it we propose a new integral representation of the solutions to the considered class of IVPs for the considered nonlinear wave equations. To the best of our knowledge, there is no such integral representation of the solutions for the IVP (1.1) in the existing references.

Lemma 2.1. *If $u \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}^n)$ is a solution to the integral equation*

$$\begin{aligned} 0 = {} & \frac{1}{2^{n+1} n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^{n} \int_0^{x_j} (x_j - s_j)^2 (t-t_1)^2 g(t_1, s) \int_{\mathbb{R}^n} \frac{1}{|s-y|^{n-2}} \\ & \times \left(-\int_0^{t_1} (t_1 - t_2) f(t_2, y, u(t_2, y)) dt_2 + u(t_1, y) - u_0(y) - t_1 u_1(y) \right) dy ds dt_1 \\ & + \frac{1}{2^{n+1}} \int_0^t \prod_{j=1}^{n} \int_0^{x_j} (x_j - s_j)^2 (t-t_1)^2 g(t_1, s) \int_0^{t_1} (t_1 - t_2) u(t_2, s) dt_2 ds dt_1, \end{aligned} \tag{2.1}$$

then it is a solution to the IVP (1.1).

Proof. Let $u \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}^n)$ be a solution to the integral equation (2.1). We differentiate the integral equation (2.1) three times in t, three times in x_1, three times in x_n and so on, and we find

$$\begin{aligned} 0 = \frac{1}{n(n-2)\alpha(n)} g(t,x) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \Big(- \int_0^t (t-t_1) f(t_1, y, u(t_1,y)) dt_1 \\ + u(t,y) - u_0(y) - t u_1(y) \Big) dy \\ + g(t,x) \int_0^t (t-t_1) u(t_1,x) dt_1, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \end{aligned}$$

whereupon

$$\begin{aligned} 0 = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \Big(- \int_0^t (t-t_1) f(t_1, y, u(t_1,y)) dt_1 \\ + u(t,y) - u_0(y) - t u_1(y) \Big) dy \\ + \int_0^t (t-t_1) u(t_1,x) dt_1, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \end{aligned} \tag{2.2}$$

Now we differentiate twice in t the last equation and we obtain

$$0 = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} (-f(t,y,u(t,y)) + u_{tt}(t,y)) dy + u(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

or

$$u(t,x) = -\frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} (-f(t,y,u(t,y)) + u_{tt}(t,y)) dy,$$

$(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Hence, using the main representation of the solutions to the Poisson equation, we arrive at

$$-\Delta u(t,x) = f(t,x,u(t,x)) - u_{tt}(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Now, we put $t = 0$ in the equation (2.2) and we find

$$0 = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} (u(0,y) - u_0(y)) dy, \quad x \in \mathbb{R}^n,$$

and by the Poisson equation, we obtain

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n.$$

We differentiate once in t the equation (2.2) and we get

$$\begin{aligned} 0 = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \Big(- \int_0^t f(t_1, y, u(t_1,y)) dt_1 + u_t(t,y) - u_1(y) \Big) dy \\ + \int_0^t u(t_1,x) dt_1, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \end{aligned}$$

We put $t = 0$ in the last equation and we find

$$0 = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} (u_t(0, y) - u_1(y))dy, \quad x \in \mathbb{R}^n.$$

From here and from the Poisson equation, we obtain

$$u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n.$$

This completes the proof. □

3. Main result

We first list the assumptions we will use in this paper:

(H1): $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R})$ is such that

$$0 \leqslant -f(t, x, u) \leqslant \sum_{j=1}^{l} c_j(t, x)|u|^{p_j}, \quad (t, x, u) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R},$$

$p_j > 0$ are given constants, $c_j \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n)$ are given functions, $j \in \{1, \dots, l\}$, $l \in \mathbb{N}$.

(H2): There exists an $r > 0$ such that $0 \leqslant u_0, -u_1 \leqslant \frac{r}{2}$ on $\mathbb{R}_+ \times \mathbb{R}^n$, and

$$(3 + 2n)\bigg(\sum_{j=1}^{l} r^{p_j} + 2r\bigg)l < 1,$$

for some given constant $l > 1$.

Let

$$g_k(t, x) = \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-2}} \int_0^t c_k(t_1, y)dt_1dy, \quad k \in \{1, \dots, l\}, \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Below, we suppose

(H3): there exists a nonnegative function $g \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n)$ that is positive almost everywhere on $\mathbb{R}_+ \times \mathbb{R}^n$ and a positive constant B such that

$$t^2(2 + t)(1 + t + t^2) \int_0^t \prod_{j=1, j\neq m}^{n} x_j^2(1 + x_m + x_m^2) \int_0^{x_j} \int_0^{x_m} g(t_1, s)(1 + g_k(t_1, s) + \log|s|)dsdt_1 \leqslant A, \quad k \in \{1, \dots, l\}, \ m \in \{1, \dots, n\}, \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

$$\frac{B}{2^{n+1}n(n-2)\alpha(n)} \int_1^{\frac{3}{2}} \prod_{j=1}^{n} \int_1^{\frac{3}{2}} (2 - s_j)^2(2 - t_1)^2 g(t_1, s) \int_{n \leqslant |y| \leqslant 2n} \frac{1}{|s - y|^{n-2}} dydsdt_1 \geqslant \frac{A}{l}.$$

Our main result is as follows.

THEOREM 3.1. *Suppose* (H1)–(H3) *hold. Then the IVP* (1.1) *has at least one nontrivial nonnegative solution* $u \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}^n)$ *satisfying* $\|u\| < \min(r, R)$ *and*

$$u(t, x) \geqslant u_0(x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad u(t, x) \geqslant u_0(x) + B, \quad t \in [1, 2], \ x_j \in [1, 2],$$

$j \in \{1, \dots, n\}$, $x = (x_1, \dots, x_n)$.

Let $E = \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}^n)$ be endowed with the norm

$$\|u\| = \|u\|_\infty + \|u_t\|_\infty + \|u_{tt}\|_\infty + \sum_{j=1}^n \|u_{x_j}\|_\infty + \sum_{j=1}^n \|u_{x_j x_j}\|_\infty,$$

provided it exists, where

$$\|v\|_\infty = \sup_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n} |v(t,x)|.$$

For $u \in E$, define the operator

$$\begin{aligned}
Fu(t,x) &= \frac{1}{2^{n+1}n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^n \int_0^{x_j} (x_j - s_j)^2 (t-t_1)^2 g(t_1,s) \int_{\mathbb{R}^n} \frac{1}{|s-y|^{n-2}} \\
&\quad \times \Big(-\int_0^{t_1} (t_1 - t_2) f(t_2, y, u(t_2,y)) dt_2 + u(t_1,y) - u_0(y) - t_1 u_1(y) \Big) dy ds dt_1 \\
&\quad + \frac{1}{2^{n+1}} \int_0^t \prod_{j=1}^n \int_0^{x_j} (x_j - s_j)^2 (t-t_1)^2 g(t_1,s) \int_0^{t_1} (t_1 - t_2) u(t_2,s) dt_2 ds dt_1,
\end{aligned}$$

$(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Lemma 3.1. *Suppose* (H1), (H2) *and* (H3) *hold. For* $u \in E$, $\|u\| \leqslant r$, *we have*

$$\|Fu\| \leqslant (3+2n)\Big(\sum_{j=1}^l r^{p_j} + 2r\Big) A.$$

Proof. We have

$$\begin{aligned}
|Fu(t,x)| &\leqslant \frac{1}{2^{n+1}n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^n \int_0^{x_j} (x_j - s_j)^2 (t-t_1)^2 g(t_1,s) \int_{\mathbb{R}^n} \frac{1}{|s-y|^{n-2}} \\
&\qquad \times \Big(\int_0^{t_1} (t_1 - t_2) \sum_{k=1}^l c_k(t_2,y) |u(t_2,y)|^{p_k} dt_2 \\
&\qquad + u(t_1,y) + u_0(y) - t_1 u_1(y) \Big) dy ds dt_1 \\
&\quad + \frac{1}{2^{n+1}} \int_0^t \prod_{j=1}^n \int_0^{x_j} (x_j - s_j)^2 (t-t_1)^2 g(t_1,s) \int_0^{t_1} (t_1 - t_2) |u(t_2,s)| dt_2 ds dt_1 \\
&\leqslant \frac{1}{2^{n+1}n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^n \int_0^{x_j} (x_j - s_j)^2 (t-t_1)^2 g(t_1,s) \int_{\mathbb{R}^n} \frac{1}{|s-y|^{n-2}} \\
&\qquad \times \Big(\sum_{k=1}^l r^{p_k} \int_0^{t_1} (t_1 - t_2) c_k(t_2,y) dt_2 + (2+t_1) r \Big) dy ds dt_1 \\
&\qquad + \frac{1}{2^{n+1}} r \int_0^t \prod_{j=1}^n t_1^2 \int_0^{x_j} (x_j - s_j)^2 (t-t_1)^2 g(t_1,s) ds dt_1
\end{aligned}$$

$$\leqslant \sum_{k=1}^{l} r^{p_k} \frac{t^4}{2^{n+1} n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^{n} x_j^2 \int_0^{x_j} g(t_1,s) g_k(t_1,s) ds dt_1$$

$$+ r \frac{t^3(2+t)}{2^{n+1} n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^{n} x_j^2 \int_0^{x_j} g(t_1,s) \int_{\mathbb{R}^n} \frac{1}{|s-y|^{n-2}} ds$$

$$+ r \frac{1}{2^{n+1}} t^4 \int_0^t \prod_{j=1}^{n} x_j^2 \int_0^{x_j} g(t_1,s) ds dt_1$$

$$\leqslant \left(\sum_{k=1}^{l} r^{p_k} + 2r \right) A, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

and

$$\left| \frac{\partial}{\partial t} Fu(t,x) \right| \leqslant \frac{1}{2^n n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^{n} \int_0^{x_j} (x_j - s_j)^2 (t-t_1) g(t_1,s) \int_{\mathbb{R}^n} \frac{1}{|s-y|^{n-2}}$$

$$\times \left(\int_0^{t_1} (t_1 - t_2) \sum_{k=1}^{l} c_k(t_2,y) |u(t_2,y)|^{p_k} dt_2 \right.$$

$$\left. + u(t_1,y) + u_0(y) - t_1 u_1(y) \right) dy ds dt_1$$

$$+ \frac{1}{2^n} \int_0^t \prod_{j=1}^{n} \int_0^{x_j} (x_j - s_j)^2 (t-t_1) g(t_1,s) \int_0^{t_1} (t_1 - t_2) |u(t_2,s)| dt_2 ds dt_1$$

$$\leqslant \frac{1}{2^n n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^{n} \int_0^{x_j} (x_j - s_j)^2 (t-t_1) g(t_1,s) \int_{\mathbb{R}^n} \frac{1}{|s-y|^{n-2}}$$

$$\times \left(\sum_{k=1}^{l} r^{p_k} \int_0^{t_1} (t_1 - t_2) c_k(t_2,y) dt_2 + (2+t_1) r \right) dy ds dt_1$$

$$+ \frac{1}{2^n} r \int_0^t \prod_{j=1}^{n} t_1^2 \int_0^{x_j} (x_j - s_j)^2 (t-t_1) g(t_1,s) ds dt_1$$

$$\leqslant \sum_{k=1}^{l} r^{p_k} \frac{t^3}{2^n n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^{n} x_j^2 \int_0^{x_j} g(t_1,s) g_k(t_1,s) ds dt_1$$

$$+ r \frac{t^2(2+t)}{2^n n(n-2)\alpha(n)} \int_0^t \prod_{j=1}^{n} x_j^2 \int_0^{x_j} g(t_1,s) \int_{\mathbb{R}^n} \frac{1}{|s-y|^{n-2}} ds$$

$$+ r \frac{1}{2^n} t^3 \int_0^t \prod_{j=1}^{n} x_j^2 \int_0^{x_j} g(t_1,s) ds dt_1$$

$$\leqslant \left(\sum_{k=1}^{l} r^{p_k} + 2r \right) A, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

and

$$\begin{aligned}
\Big|\frac{\partial^2}{\partial t^2}Fu(t,x)\Big| &\leqslant \frac{1}{2^n n(n-2)\alpha(n)}\int_0^t \prod_{j=1}^n \int_0^{x_j}(x_j-s_j)^2 g(t_1,s)\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}\\
&\quad\times\Big(\int_0^{t_1}(t_1-t_2)\sum_{k=1}^l c_k(t_2,y)|u(t_2,y)|^{p_k}dt_2\\
&\quad + u(t_1,y)+u_0(y)-t_1u_1(y)\Big)dydsdt_1\\
&\quad+\frac{1}{2^n}\int_0^t\prod_{j=1}^n\int_0^{x_j}(x_j-s_j)^2g(t_1,s)\int_0^{t_1}(t_1-t_2)|u(t_2,s)|dt_2dsdt_1\\
&\leqslant \frac{1}{2^n n(n-2)\alpha(n)}\int_0^t\prod_{j=1}^n\int_0^{x_j}(x_j-s_j)^2g(t_1,s)\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}\\
&\quad\times\Big(\sum_{k=1}^l r^{p_k}\int_0^{t_1}(t_1-t_2)c_k(t_2,y)dt_2+(2+t_1)r\Big)dydsdt_1\\
&\quad+\frac{1}{2^n}r\int_0^t\prod_{j=1}^n t_1^2\int_0^{x_j}(x_j-s_j)^2g(t_1,s)dsdt_1\\
&\leqslant \sum_{k=1}^l r^{p_k}\frac{t^2}{2^n n(n-2)\alpha(n)}\int_0^t\prod_{j=1}^n x_j^2\int_0^{x_j}g(t_1,s)g_k(t_1,s)dsdt_1\\
&\quad+r\frac{t(2+t)}{2^n n(n-2)\alpha(n)}\int_0^t\prod_{j=1}^n x_j^2\int_0^{x_j}g(t_1,s)\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}ds\\
&\quad+r\frac{1}{2^n}t^2\int_0^t\prod_{j=1}^n x_j^2\int_0^{x_j}g(t_1,s)dsdt_1\\
&\leqslant\Big(\sum_{k=1}^l r^{p_k}+2r\Big)A,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\frac{\partial}{\partial x_m}Fu(t,x)\Big|\\
&\leqslant\frac{1}{2^n n(n-2)\alpha(n)}\int_0^t\prod_{j=1,j\neq m}^n\int_0^{x_j}(x_j-s_j)^2\int_0^{x_m}(x_m-s_m)(t-t_1)^2g(t_1,s)\\
&\quad\times\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}\Big(\int_0^{t_1}(t_1-t_2)\sum_{k=1}^l c_k(t_2,y)|u(t_2,y)|^{p_k}dt_2\\
&\quad+u(t_1,y)+u_0(y)-t_1u_1(y)\Big)dydsdt_1
\end{aligned}$$

$$+\frac{1}{2^n}\int_0^t\prod_{j=1,j\neq m}^n\int_0^{x_j}(x_j-s_j)^2$$

$$\times\int_0^{x_m}(x_m-s_m)(t-t_1)^2g(t_1,s)\int_0^{t_1}(t_1-t_2)|u(t_2,s)|dt_2dsdt_1$$

$$\leqslant\frac{1}{2^nn(n-2)\alpha(n)}\int_0^t\prod_{j=1,j\neq m}^n$$

$$\times\int_0^{x_j}(x_j-s_j)^2\int_0^{x_m}(x_m-s_m)(t-t_1)^2g(t_1,s)$$

$$\times\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}\bigg(\sum_{k=1}^l r^{p_k}\int_0^{t_1}(t_1-t_2)c_k(t_2,y)dt_2+(2+t_1)r\bigg)dydsdt_1$$

$$+\frac{1}{2^n}r\int_0^t\prod_{j=1,j\neq m}^n t_1^2\int_0^{x_j}(x_j-s_j)^2\int_0^{x_m}(x_m-s_m)(t-t_1)^2g(t_1,s)dsdt_1$$

$$\leqslant\sum_{k=1}^l r^{p_k}\frac{t^4}{2^nn(n-2)\alpha(n)}\int_0^t\prod_{j=1,j\neq m}^n x_j^2x_m\int_0^{x_j}\int_0^{x_m}g(t_1,s)g_k(t_1,s)dsdt_1$$

$$+r\frac{t^3(2+t)}{2^nn(n-2)\alpha(n)}\int_0^t\prod_{j=1,j\neq m}^n x_j^2x_m\int_0^{x_j}\int_0^{x_m}g(t_1,s)\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}ds$$

$$+r\frac{1}{2^n}t^4\int_0^t\prod_{j=1,j\neq m}^n x_j^2x_m\int_0^{x_j}\int_0^{x_m}g(t_1,s)dsdt_1$$

$$\leqslant\bigg(\sum_{k=1}^l r^{p_k}+2r\bigg)A,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n,\quad m\in\{1,\dots,n\},$$

and

$$\left|\frac{\partial^2}{\partial x_m^2}Fu(t,x)\right|$$

$$\leqslant\frac{1}{2^nn(n-2)\alpha(n)}\int_0^t\prod_{j=1,j\neq m}^n\times\int_0^{x_j}(x_j-s_j)^2\int_0^{x_m}(t-t_1)^2g(t_1,s)\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}$$

$$\times\bigg(\int_0^{t_1}(t_1-t_2)\sum_{k=1}^l c_k(t_2,y)|u(t_2,y)|^{p_k}dt_2+u(t_1,y)+u_0(y)-t_1u_1(y)\bigg)dydsdt_1$$

$$+\frac{1}{2^n}\int_0^t\prod_{j=1,j\neq m}^n\times\int_0^{x_j}(x_j-s_j)^2\int_0^{x_m}(t-t_1)^2g(t_1,s)\int_0^{t_1}(t_1-t_2)|u(t_2,s)|dt_2dsdt_1$$

$$\leqslant\frac{1}{2^nn(n-2)\alpha(n)}\int_0^t\prod_{j=1,j\neq m}^n\times\int_0^{x_j}(x_j-s_j)^2\int_0^{x_m}(t-t_1)^2g(t_1,s)\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}$$

$$
\begin{aligned}
&\times\left(\sum_{k=1}^{l} r^{p_k}\int_0^{t_1}(t_1-t_2)c_k(t_2,y)dt_2+(2+t_1)r\right)dydsdt_1\\
&+\frac{1}{2^n}r\int_0^t\prod_{j=1,j\neq m}^{n}t_1^2\int_0^{x_j}(x_j-s_j)^2\int_0^{x_m}(t-t_1)^2 g(t_1,s)dsdt_1\\
&\leqslant\sum_{k=1}^{l} r^{p_k}\frac{t^4}{2^n n(n-2)\alpha(n)}\int_0^t\prod_{j=1,j\neq m}^{n}x_j^2\int_0^{x_j}\int_0^{x_m}g(t_1,s)g_k(t_1,s)dsdt_1\\
&+r\frac{t^3(2+t)}{2^n n(n-2)\alpha(n)}\int_0^t\prod_{j=1,j\neq m}^{n}x_j^2 x_m\int_0^{x_j}\int_0^{x_m}g(t_1,s)\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}ds\\
&+r\frac{1}{2^n}t^4\int_0^t\prod_{j=1,j\neq m}^{n}x_j^2\int_0^{x_j}\int_0^{x_m}g(t_1,s)dsdt_1\\
&\leqslant\left(\sum_{k=1}^{l} r^{p_k}+2r\right)A,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n,\quad m\in\{1,\dots,n\}.
\end{aligned}
$$

Hence,

$$
\|Fu\|\leqslant(3+2n)\left(\sum_{j=1}^{l} r^{p_j}+2r\right)A.
$$

This completes the proof. □

3.1. Proof of the main result. For $u\in E$, define the operators

$$
\begin{aligned}
Tu(t,x)&=(1-\epsilon)u(t,x),\\
Su(t,x)&=\epsilon u(t,x)+Fu(t,x)\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n.
\end{aligned}
$$

Note that if $u\in E$ is a fixed point of the operator $T+S$, then it is a solution to the IVP (1.1). Really, let $u\in E$ be a fixed point of $T+S$. Then

$$
\begin{aligned}
u(t,x)&=(1-\epsilon)u(t,x)+\epsilon u(t,x)+Fu(t,x)\\
&=u(t,x)+Fu(t,x),\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n
\end{aligned}
$$

or

$$
Fu(t,x)=0,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n.
$$

Hence, by Lemma 2.1, we conclude that u is a solution to the IVP (1.1). Define, for $R>1$,

$$
\widetilde{\mathcal{P}}=\{u\in E: u(t,x)\geqslant 0,\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n\},
$$

and $\mathcal{P}$ is the set of all equi-continuous families in $\widetilde{\mathcal{P}}$ (an example for an equi-continuous family is the family $\{(3+\sin(t+m))\prod_{i=1}^n(4+\sin(x_i+m)) : t\in\mathbb{R}_+,\quad x_i\in\mathbb{R}, i=1,\dots,n\}_{m\in\mathbb{N}}$),

$$
\begin{aligned}
\Omega&=\{u\in\mathcal{P}:\|u\|\leqslant R\},\\
U&=\{u\in\mathcal{P}: u(t,x)\geqslant u_0(x),\quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n,\\
&\quad u(t,x)\geqslant u_0(x)+B,\quad t\in[1,2],\quad x_j\in[1,2],
\end{aligned}
$$

$$j \in \{1, \dots, n\}, \quad x = (x_1, \dots, x_n), \quad \|u\| < r\}.$$

For $u \in U$, we have that $Fu(t,x) \geqslant 0$.

(1) For $u \in \Omega$, we have

$$(I - T)u(t,x) = \epsilon u(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

and

$$\|(I - T)u\| = \epsilon \|u\| \geqslant \frac{\epsilon}{l}\|u\|.$$

Thus, $I - T\colon \Omega \to E$ is Lipschitz invertible with a constant $\gamma \leqslant \frac{l}{\epsilon}$.

(2) By Lemma 3.1, for $u \in \bar{U}$, we find

$$\|Su\| \leqslant \epsilon\|u\| + \|Fu\| \leqslant \epsilon r + (3+2n)\biggl(\sum_{j=1}^{l} r^{p_j} + 2r\biggr)A.$$

Therefore, $S\colon \bar{U} \to E$ is uniformly bounded. Since $S\colon \bar{U} \to E$ is continuous, we obtain that $S(\bar{U})$ is equi-continuous and then $S\colon \bar{U} \to E$ is relatively compact. Consequently, $\colon \bar{U} \to E$ is a 0-set contraction.

(3) Let $u \in \bar{U}$. For $z \in \Omega$, define the operator

$$Lz = Tz + Su.$$

For $z \in \Omega$, we get

$$\begin{aligned}
\|Lz\| &= \|Tz + Su\| \\
&\leqslant \|Tz\| + \|Su\| \\
&\leqslant (1-\epsilon)R + \epsilon r + (3+2n)\biggl(\sum_{j=1}^{l} r^{p_j} + 2r\biggr)A \\
&\leqslant (1-\epsilon)R + \frac{\epsilon}{2l} \\
&< (1-\epsilon)R + \epsilon R \\
&= R.
\end{aligned}$$

Therefore, $L\colon \Omega \to \Omega$. Next, for $z_1, z_2 \in \Omega$, we have

$$\begin{aligned}
\|Lz_1 - Lz_2\| &= \|Tz_1 - Tz_2\| \\
&= (1-\epsilon)\|z_1 - z_2\|,
\end{aligned}$$

i.e., $L\colon \Omega \to \Omega$ is a contraction mapping. Hence, there exists a unique $z \in \Omega$ such that

$$\begin{aligned}
z &= Lz \\
&= Tz + Su
\end{aligned}$$

or

$$(I - T)z = Su.$$

Consequently, $S(\bar{U}) \subset (I - T)(\Omega)$.

(4) Assume that there are $u \in \partial U$ and $\lambda \geqslant 1$ such that

$$Su = (I - T)(\lambda u), \quad \lambda u \in \Omega.$$

We have

$$\epsilon u + Fu = \epsilon \lambda u$$

or

$$\epsilon(\lambda - 1)u = Fu.$$

Hence,

$$\begin{aligned} \epsilon(\lambda - 1)r &= \epsilon(\lambda - 1)\|u\| \\ &= \|Fu\| \\ &\leqslant (3 + 2n)\bigg(\sum_{j=1}^{l} r^{p_j} + 2r\bigg)A. \end{aligned}$$

Thus,

$$\begin{aligned} (3 + 2n)\bigg(\sum_{j=1}^{l} r^{p_j} + 2r\bigg)A &\geqslant \epsilon(\lambda - 1)\|u\| = \|Fu\| \\ &\geqslant \frac{B}{2^{n+1} n(n-2)\alpha(n)} \int_1^{\frac{3}{2}} \prod_{j=1}^{n} \int_1^{\frac{3}{2}} (2 - s_j)^2 (2 - t_1)^2 g(t_1, s) \\ &\qquad \times \int_{n \leqslant |y| \leqslant 2n} \frac{1}{|s - y|^{n-2}} dy ds dt_1 \geqslant \frac{A}{l}, \end{aligned}$$

i.e.,

$$(3 + 2n)\bigg(\sum_{j=1}^{l} r^{p_j} + 2r\bigg) \geqslant \frac{1}{l},$$

which is a contradiction.

By (1), (2), (3), (4) and Proposition 1.1, we conclude that the operator $T + S$ has a fixed point in $U \bigcap \Omega$. This completes the proof.

4. Example

Let

$$h(x) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \qquad s \in \mathbb{R},\ s \neq \pm 1.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}. \end{aligned}$$

Therefore,

$$-\infty < \lim_{t\to\infty} t^2(2+t)(1+t+t^2)h(t) < \infty,$$
$$-\infty < \lim_{t\to\infty} t^2(2+t)(1+t+t^2)l(t) < \infty,$$
$$-\infty < \lim_{s\to\pm\infty} s^2(1+s+s^2)h(s) < \infty,$$
$$-\infty < \lim_{s\to\pm\infty} s^2(1+s+s^2)l(s) < \infty.$$

Hence, there exists a positive constant C_1 such that

$$t^2(2+t)(1+t+t^2)\left(\frac{1}{44\sqrt{2}}\log\frac{1+t^{11}\sqrt{2}+t^{22}}{1-t^{11}\sqrt{2}+t^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{t^{11}\sqrt{2}}{1-t^{22}}\right)\leqslant C_1,$$
$$s^2(1+s+s^2)\left(\frac{1}{44\sqrt{2}}\log\frac{1+s^{11}\sqrt{2}+s^{22}}{1-s^{11}\sqrt{2}+s^{22}}+\frac{1}{22\sqrt{2}}\arctan\frac{s^{11}\sqrt{2}}{1-s^{22}}\right)\leqslant C_1,$$

$t\in\mathbb{R}_+$, $s\in\mathbb{R}$. Note that by [14, p. 707, Integral 79], we have

$$\int\frac{dz}{1+z^4}=\frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

Let

$$g_1(t,x)=\frac{t^{10}}{1+t^{44}}\prod_{j=1}^{n}\frac{x_j^{10}}{1+x_j^{44}}\frac{1}{(1+g_k(t,s)+\log|s|)^4},$$

where

$$g_k(t,s)=\int_{\mathbb{R}^n}\frac{1}{|s-y|^{n-2}}\int_0^t\frac{1}{(1+|y|)^{3n}(1+t_1^2)^{10}}dt_1dy,$$

$(t,s)\in\mathbb{R}_+\times\mathbb{R}^n$, $k\in\{1,\dots,n\}$, $n\geqslant 3$. Moreover, there exists a positive constant C_2 such that

$$\frac{1}{2^{n+1}n(n-2)\alpha(n)}\int_1^{\frac{3}{2}}\prod_{j=1}^{n}\int_1^{\frac{3}{2}}(2-s_j)^2(2-t_1)^2g_1(t,s)\int_{n\leqslant|y|\leqslant 2n}\frac{1}{|s-y|^{n-2}}dydsdt_1\geqslant C_2.$$

|Take

$$g(t,s)=\frac{g_1(t,x)}{C_1^{n+1}},\quad A=2,\quad l=1,\quad r=\frac{1}{10000(3+2n)},\quad \epsilon=\frac{1}{2},\quad B=\frac{3}{C_2}.$$

Hence,

$$2\left(\epsilon r+(3+2n)\left(\sum_{j=1}^{l}r^{p_j}+2r\right)A\right)<\frac{\epsilon}{l},$$

i.e., $(H3)$ holds. Next,

$$t^2(2+t)(1+t+t^2)\int_0^t\prod_{j=1,j\neq m}^{n}x_j^2(1+x_m+x_m^2)$$
$$\times\int_0^{x_j}\int_0^{x_m}g(t_1,s)(1+g_k(t_1,s)+\log|s|)dsdt_1\leqslant 1\leqslant A,$$

$k \in \{1, \dots, l\}$, $m \in \{1, \dots, n\}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, and

$$\frac{B}{2^{n+1} n(n-2)\alpha(n)} \int_1^{\frac{3}{2}} \prod_{j=1}^{n} \int_1^{\frac{3}{2}} (2-s_j)^2 (2-t_1)^2 g(t_1, s) \int_{n \leqslant |y| \leqslant 2n} \frac{1}{|s-y|^{n-2}} dy ds dt_1 \geqslant 3 \geqslant \frac{A}{l}.$$

Therefore, $(H4)$ holds. Consider the IVP

$$\begin{aligned} u_{tt} - \Delta u &= -\frac{1}{(1+|x|)^{30}(1+t^2)^{10}} |u|^p, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, x) &= \frac{1}{30000(3+2n)(1+|x|^2)^{4n}}, \\ u_t(0, x) &= -\frac{1}{30000(3+2n)(1+|x|^2)^{5n}}, \quad x \in \mathbb{R}^n, \end{aligned} \tag{4.1}$$

where $n \geqslant 3$, $p > 1$. Here

$$\begin{aligned} u_0(x) &= \frac{1}{30000(3+2n)(1+|x|^2)^{4n}}, \\ u_1(x) &= -\frac{1}{30000(3+2n)(1+|x|^2)^{5n}}, \quad x \in \mathbb{R}^n. \end{aligned}$$

We have $0 \leqslant u_0, -u_1 < \frac{r}{2}$ and

$$(3+2n)(r^p + 2r)l < 1.$$

Hence, by using Theorem 3.1, it follows that the IVP (4.1) has at least one nontrivial nonnegative solution $u \in \mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}^n)$.

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КЛАСИЧНА РЕШЕЊА ЈЕДНЕ КЛАСЕ НЕЛИНЕАРНИХ ТАЛАСНИХ ЈЕДНАЧИНА

Резиме. Проучавамо класу парцијалних једначина хиперболичког типа са задатим почетним граничним условима. Примењен је нови тополошки приступ да се докаже постојање нетривијалних ненегативних решења. Тачније, предлажемо нову интегралну репрезентацију решења и утврђујемо постојање класичних решења за разматране класе нелинеарних таласних једначина.

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