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TULCZYJEW'S TRIPLET FOR LIE GROUPS III: HIGHER ORDER DYNAMICS AND REDUCTIONS FOR ITERATED BUNDLES

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Abstract. Given a Lie group G, we elaborate the dynamics on T^*T^*G and T^*TG, which is given by a Hamiltonian, as well as the dynamics on the Tulczyjew symplectic space TT^*G, which may be defined by a Lagrangian or a Hamiltonian function. As the trivializations we adapted respect the group structures of the iterated bundles, we exploit all possible subgroup reductions (Poisson, symplectic or both) of higher order dynamics.

1. Introduction

The tangent and the cotangent bundles of a Lie group admit global trivializations, as well as the Lie group structures, induced from the underlying Lie group itself. These structures may further be carried over the iterated bundles T^*TG, TT^*G, and T^*T^*G. These iterated bundles constitute Tulczyjew's triplet, introduced for a geometric description of the Legendre transformation from the Lagrangian description on TG to the Hamiltonian description on T^*G for a mechanical system having G as the configuration space. Such a system admits G as kinematical symmetries, and the reduction of the Lagrangian dynamics results in the Euler-Poincré equations on the Lie algebra $\mathfrak{g}$ of G. Similarly, the reduction of the Hamiltonian dynamics to $\mathfrak{g}^*$ is described by the Lie–Poisson equations.

The present note is intended as a sequel to [19, 20]. In the first part [19], we gave a detailed description of the possible trivializations of the iterated bundles T^*TG, TT^*G, and T^*T^*G, which are Lie group isomorphisms. Moreover, we described the group structures up to the second iterated bundles, as well as the canonical involutions on them. Having explicit descriptions of the cotangent and the Tulczyjew symplectic structures, we performed the Marsden-Weinstein reduction by kinematical symmetries to obtain the reduced Tulczyjew triplet for the Legendre transformation from Euler–Poincaré to Lie–Poisson equations. Then, in the second part [20], we studied the Lagrangian and the Hamiltonian dynamical

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equations at each stage of the Tulczyjew construction under the trivializations respecting the Lie group structures. The dynamics we considered is defined either by a Lagrangian on TG, or by a Hamiltonian on T^*G, which, in the framework of Tulczyjew construction, corresponds to Lagrangian submanifolds of T^*TG or T^*T^*G, respectively. In other words, first order dynamics considered in [20] restricts to the fiber coordinates of the second iterated bundles.

In this work, we aim to give a complete description of the higher order dynamics and its reductions by considering the Lagrangian and/or Hamiltonian functions on the second iterated bundles, taking full advantage of the trivializations at our disposal. Obviously, releasing the condition that the dynamics on iterated bundles is described by Lagrangian submanifolds opens up the possibility to obtain higher order forms of Euler–Poincaré and Lie–Poisson equations. The underlying structure will, indeed, offer more than this generalization.

The geometric/algebraic constructions as well as dynamical equations we offer in this study admit several applications in Hamiltonian dynamics. Let us list some of these applications that particularly lie in our sphere of interest.

(1) Consider a Lagrangian theory determined by a second order Lagrangian function on the second order tangent bundle T^2G [11,22,24]. This higher order geometry is proper for investigations of variational aspects of higher order dynamical systems [14]. In this case, after considering the inclusion $T^2G \hookrightarrow TTG$, one defines the Legendre transformation from T^2G to a subbundle of the iterated cotangent bundle T^*TG. Due to the dimensional argument, independent of the degeneracy level of the Lagrangian function, any mapping $T^2G \mapsto T^*TG$ fails to be an isomorphism. Consequently, not only the dynamics on T^*TG but also all of its reductions are particularly important for Hamiltonian analysis of such a Lagrangian system. We deal with the geometric/algebraic analysis of this transformation in an upcoming work.

(2) As another application, consider the motion of a single particle whose momentum phase space is the cotangent group T^*G. Its complete tangent (or cotangent) lift is necessarily a Hamiltonian system on the tangent bundle TT^*G (resp. T^*T^*G). Then taking the vertical (evolutionary) representative of the associated Hamiltonian vector field, one arrives at the kinetic theory governing the collective motion of a bunch of such particles, each obeying individually dynamics on the base level T^*G. That is a geometric pathway from the individual motion of particles to the kinetic equation of the medium consisting of such particles [16–18]. The relation between all possible reductions of the cotangent bundle T^*G and the iterated bundles TT^*G and T^*T^*G are, therefore, particularly interesting from this point of view.

(3) As a third comment, we may add that these geometries serve well to concrete realizations of Tulczyjew–Legendre transformations even for singular theories. The Lie group approach to Tulczyjew construction is particularly important for fluid and plasma theories where the configuration spaces are diffeomorphism groups [41, 44]. As a final comment, we would like to note that formalisms we address in the present work contain very interesting geometric and algebraic features even in their own right. Because of the beautiful geometry they have, we think these systems deserve detailed analysis, even if they don't have any field of application.

An immediate generalization of the results of the papers [19, 20], and the present work is to apply them to fibered spaces admitting local trivializations, or Ehresmann connections. In a recent work [21], we have addressed this issue in the particular case of the principal G-bundles, and their associated vector bundles.

1.1. Trivializations. One observes that the form of equations governing dynamics on Lie groups depends on the kind of trivializations adapted on iterated bundles [11, 12, 24, 38]. Additional terms in these equations may or may not appear depending on whether trivialization preserves semidirect product and group structures or not. If one preserves the group structures, canonical embeddings of factors involving trivialization define subgroups of iterated bundles and reductions of dynamics with these subgroups become possible.

Based on exhaustive investigation of trivializations in our previous work [19], we shall present all reductions of dynamics on iterated bundles of a Lie group with the convenient trivialization of the first kind. In trivialization of the first kind, we identify tangent TG and cotangent T^*G bundles with their semidirect product trivializations $G \circledS \mathfrak{g}$ and $G \circledS \mathfrak{g}^*$, respectively. Then, we trivialize the iterated bundles $T(G \circledS \mathfrak{g})$, $T(G \circledS \mathfrak{g}^*)$, $T^*(G \circledS \mathfrak{g})$ and $T^*(G \circledS \mathfrak{g}^*)$ by considering them as tangent and cotangent groups again. As an example, we obtain

$$TT^*G \simeq T(G \circledS \mathfrak{g}^*) \simeq (G \circledS \mathfrak{g}^*) \circledS \operatorname{Lie}(G \circledS \mathfrak{g}^*) \simeq (G \circledS \mathfrak{g}^*) \circledS (\mathfrak{g} \circledS \mathfrak{g}^*)$$

for which the trivialization maps preserve lifted group structures, thereby making various reductions of dynamics possible. On the other hand, in trivialization of the second kind, one distributes functors T and T^* to $G \circledS \mathfrak{g}$ and $G \circledS \mathfrak{g}^*$, obtains products of first order bundles and then trivializes each factor involving the products. This results in, for example,

$$^{2}TT^*G \simeq T(G \circledS \mathfrak{g}^*) \to TG \circledS T\mathfrak{g}^* \simeq (G \circledS \mathfrak{g}) \circledS (\mathfrak{g}^* \times \mathfrak{g}^*) \tag{1.1}$$

for which distributions of functors mix up orders of fibrations, and do not preserve group structures [19]. Throughout this work we shall use trivialization of the first kind unless otherwise stated. A subscript of $\mathfrak{g}$ and $\mathfrak{g}^*$ will show its position in the original trivialization of an iterated bundle.

1.2. Content of the work. Here is a brief description of what we present in each section.

Section 2. This section is intended as a reference section of the present work. Notations and conventions are fixed. Trivializations of all spaces TG, T^*G, T^*TG, TT^*G, T^*T^*G and their induced group structures are defined. Subgroups are listed. Subgroups with symplectic actions are identified. The trivialized form of the symplectic two-forms, as well as the associated one-forms and the invariant vector fields on the cotangent bundles and Tulczyjew's symplectic space TT^*G are given.

Section 3. The dynamics on the first order (both tangent and cotangent) bundles are considered. The first order Lagrangian and Hamiltonian dynamics on TG

and T^*G are described by Euler-Lagrange and Hamilton's equations

$$\frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi} = T_e^* R_g \frac{\delta \bar{L}}{\delta g} + \operatorname{ad}_\xi^* \frac{\delta \bar{L}}{\delta \xi}, \tag{1.2}$$

$$\frac{dg}{dt} = T_e R_g \Big(\frac{\delta \bar{H}}{\delta \mu}\Big), \quad \frac{d\mu}{dt} = \operatorname{ad}^*_{\frac{\delta \bar{H}}{\delta \mu}} \mu - T_e^* R_g \frac{\delta \bar{H}}{\delta g},$$

respectively. Reduction of (1.2) by G gives the Euler–Poincaré equations. Poisson and Marsden–Weinstein reductions on T^*G are performed to obtain the Lie–Poisson equations.

Section 4. Hamiltonian dynamics on T^*TG is given by the equations

$$\Big(\frac{d}{dt} - \operatorname{ad}^*_{\frac{\delta H}{\delta \mu}}\Big)(\operatorname{ad}^*_\xi \nu - \mu) = T_e^* R_g \frac{\delta H}{\delta g}, \quad \frac{dg}{dt} = T_e R_g \frac{\delta H}{\delta \mu}$$

equivalent to four-component Hamilton's equations. There are remarkable differences arising from the use of different trivializations. Reductions by G, $\mathfrak{g}$ and $G \circledS \mathfrak{g}$ are performed. Structures of the reduced spaces are studied in detail.

Section 5. Hamiltonian dynamics on T^*T^*G is generated by the vector fields with components of the form

$$\frac{dg}{dt} = T_e R_g \Big(\frac{\delta H}{\delta \nu}\Big), \quad \frac{d\mu}{dt} = \frac{\delta H}{\delta \xi} + \operatorname{ad}^*_{\frac{\delta H}{\delta \nu}} \mu,$$

$$\frac{d\nu}{dt} = \operatorname{ad}^*_{\frac{\delta H}{\delta \mu}} \mu + \operatorname{ad}^*_{\frac{\delta H}{\delta \nu}} \nu - T_e^* R_g \Big(\frac{\delta H}{\delta g}\Big) - \operatorname{ad}^*_\xi \frac{\delta H}{\delta \xi}, \quad \frac{d\xi}{dt} = -\frac{\delta H}{\delta \mu} + \Big[\xi, \frac{\delta H}{\delta \nu}\Big].$$

Reductions by G, $\mathfrak{g}^*$ and $G \circledS \mathfrak{g}^*$ are performed. Structures of the reduced spaces are exhibited in detail. The correspondence between the dynamics on T^*T^*G and on T^*TG is established by symplectic diffeomorphisms and Poisson maps.

Section 6. On TT^*G, there are both Lagrangian and Hamiltonian formalisms. If a function E on TT^*G is regarded as a Hamiltonian, then Hamilton's equations with the Tulczyjew symplectic structure are

$$\frac{dg}{dt} = TR_g \Big(\frac{\delta E}{\delta \nu}\Big), \quad \frac{d\mu}{dt} = -\frac{\delta E}{\delta \xi}, \quad \frac{d\xi}{dt} = \frac{\delta E}{\delta \mu}, \quad \frac{d\nu}{dt} = \operatorname{ad}^*_{\frac{\delta E}{\delta \nu}} \nu - T^* R_g \Big(\frac{\delta E}{\delta g}\Big).$$

Reduction by G results in a reduced Tulczyjew triplet considered in [20] before. Reductions of the Tulczyjew structure by $\mathfrak{g}$, by a symplectic action of $\mathfrak{g}^*$ that may be connected with a symplectic diffeomorphism from TT^*G to T^*T^*G, by $G \circledS \mathfrak{g}$ and by $G \circledS \mathfrak{g}^*$ are studied in detail.

If the function E on TT^*G is regarded as a Lagrangian function, it then gives the Euler–Lagrange dynamics

$$\frac{d}{dt}\Big(\frac{\delta E}{\delta \xi}\Big) = T_e^* R_g \Big(\frac{\delta E}{\delta g}\Big) - \operatorname{ad}^*_{\frac{\delta E}{\delta \mu}} \mu + \operatorname{ad}^*_\xi \Big(\frac{\delta E}{\delta \xi}\Big) - \operatorname{ad}^*_{\frac{\delta E}{\delta \nu}} \nu$$

$$\frac{d}{dt}\Big(\frac{\delta E}{\delta \nu}\Big) = \frac{\delta E}{\delta \mu} - \operatorname{ad}_\xi \frac{\delta E}{\delta \nu}.$$

Reductions of these equations by G, $\mathfrak{g}^*$ and $G \circledS \mathfrak{g}^*$ are described. The latter gives the Euler–Poincaré equations on $\mathfrak{g} \circledS \mathfrak{g}^*$.

2. Geometry of iterated bundles

Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G) \simeq T_eG$ be its Lie algebra, and $\mathfrak{g}^* = \mathrm{Lie}^*(G)$ be the dual of $\mathfrak{g}$. We shall adapt the letters

$$g, h \in G, \quad \xi, \eta, \zeta \in \mathfrak{g}, \quad \mu, \nu, \lambda \in \mathfrak{g}^*$$

as elements of the spaces shown. For a tensor field which is either right or left invariant, we shall use $V_g \in T_gG$, $\alpha_g \in T_g^*G$, etc... We shall denote left and right multiplications on G by L_g and R_g, respectively. The right inner automorphism $I_g = L_{g^{-1}} \circ R_g$ is a right representation of G on G satisfying $I_g \circ I_h = I_{hg}$. The right adjoint action $\mathrm{Ad}_g = T_eI_g$ of G on $\mathfrak{g}$ is defined as the tangent map of I_g at the identity $e \in G$. The infinitesimal right adjoint representation $\mathrm{ad}_\xi \eta$ is $[\xi, \eta]$ and is defined as a derivative of Ad_g over the identity. A right invariant vector field X_ξ^G generated by $\xi \in \mathfrak{g}$ is of the form $X_\xi^G(g) = T_eR_g\xi$. The identity $[\xi, \eta] = [X_\xi^G, X_\eta^G]_{JL}$ defines the isomorphism between $\mathfrak{g}$ and the space $\mathfrak{X}^R(G)$ of right invariant vector fields endowed with the Jacobi–Lie bracket. The coadjoint action Ad_g^* of G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is a right representation and is the linear algebraic dual of $\mathrm{Ad}_{g^{-1}}$, namely,

$$\langle \mathrm{Ad}_g^* \mu, \xi \rangle = \langle \mu, \mathrm{Ad}_{g^{-1}} \xi \rangle \tag{2.1}$$

holds for all $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. The inverse element g^{-1} appears in the definition (2.1) in order to make Ad_g^* a right action. The infinitesimal coadjoint action ad_ξ^* of $\mathfrak{g}$ on $\mathfrak{g}^*$ is the linear algebraic dual of ad_ξ. Note that the infinitesimal generator of the coadjoint action Ad_g^* is minus the infinitesimal coadjoint action ad_ξ^*, that is, if $g^t \subset G$ is a curve passing through the identity in the direction of $\xi \in \mathfrak{g}$, then

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{g^t}^* \mu = -\,\mathrm{ad}_\xi^* \mu.$$

In the diagrams of this work, EL and EP will stand for Euler–Lagrange and Euler–Poincaré equations, respectively, and PR, SR, LR, EPR and EPR will denote Poisson, symplectic, Lagrangian, and Euler–Poincaré reductions, respectively.

2.1. The first order tangent group *TG*.

The trivialization

$$\mathrm{triv}_{TG} \colon TG \longrightarrow G \circledS \mathfrak{g}_1, \quad V_g \mapsto (g, T_gR_{g^{-1}}V_g) =: (g, \xi) \tag{2.2}$$

enables us to endow TG with the semi-direct product group structure on $G \circledS \mathfrak{g}_1$ given by

$$(g, \xi^{(1)})(\tilde{g}, \tilde{\xi}^{(1)}) = (g\tilde{g}, \xi^{(1)} + \mathrm{Ad}_{g^{-1}} \tilde{\xi}^{(1)}), \tag{2.3}$$

for any $\xi^{(1)}, \tilde{\xi}^{(1)} \in \mathfrak{g}_1 = \mathfrak{g}$. Accordingly, the Lie algebra of $TG \cong G \circledS \mathfrak{g}$ is the semi-direct sum Lie algebra $\mathfrak{g}_2 \circledS \mathfrak{g}_3 := \mathfrak{g} \circledS \mathfrak{g}$ with the Lie bracket

$$[(\xi^{(2)}, \xi^{(3)}), (\tilde{\xi}^{(2)}, \tilde{\xi}^{(3)})] = ([\xi^{(2)}, \tilde{\xi}^{(2)}], \mathrm{ad}_{\xi^{(2)}} \tilde{\xi}^{(3)} - \mathrm{ad}_{\tilde{\xi}^{(2)}} \xi^{(3)})$$

for any $\xi^{(2)}, \tilde{\xi}^{(2)} \in \mathfrak{g}_2 = \mathfrak{g}$, and any $\xi^{(3)}, \tilde{\xi}^{(3)} \in \mathfrak{g}_3 = \mathfrak{g}$. For further details on the tangent group, see [26, 31, 38, 45, 46].

REMARK 2.1. Here, the subindices on the Lie algebra $\mathfrak{g}$ serve to distinguish the copies of $\mathfrak{g}$. For the tangent bundle TG, we denote the trivialization as $TG \cong G \circledS \mathfrak{g}_1$. The Lie algebra of this space is denoted by $\mathfrak{g}_2 \circledS \mathfrak{g}_3$, that is,

$$\mathrm{Lie}(TG) \cong \mathrm{Lie}(G \circledS \mathfrak{g}_1) \cong \mathfrak{g}_2 \circledS \mathfrak{g}_3,$$

where $\mathrm{Lie}(H)$ stands for the Lie algebra of the group H. We have that the dual space of the Lie algebra $\mathfrak{g}_2 \circledS \mathfrak{g}_3$ is the Cartesian space $\mathfrak{g}_2^* \times \mathfrak{g}_3^*$, that is,

$$(\mathfrak{g}_2 \circledS \mathfrak{g}_3)^* \cong \mathfrak{g}_2^* \times \mathfrak{g}_3^*,$$

where $\mathfrak{g}_2^*$ and $\mathfrak{g}_3^*$ are the dual spaces of the constitutive Lie algebras $\mathfrak{g}_2$ and $\mathfrak{g}_3$, respectively.

2.2. The first order cotangent group T^*G. The cotangent bundle T^*G can also be endowed with a group structure borrowed from the semi-direct product group $G \circledS \mathfrak{g}^*$ via the right trivialization

$$\mathrm{triv}_{T^*G} \colon T^*G \to G \circledS \mathfrak{g}^*, \quad \alpha_g \mapsto (g,\, T_e^* R_g \alpha_g).$$

The group operation on $G \circledS \mathfrak{g}^*$ is

$$(g, \mu_1)(\tilde{g}, \mu_2) := \Big(g\tilde{g},\, \mu_1 + \mathrm{Ad}^*_{g^{-1}} \mu_2\Big). \tag{2.4}$$

On the other hand, we can pull back the canonical 1-form and the symplectic 2-form on the cotangent bundle T^*G with the structure of an exact symplectic manifold symplectic 2-form Ω_{T^*G} and a potential 1-form θ_{T^*G}.

A right invariant vector field $X^{T^*G}_{(\xi,\nu)}$ on the trivialization $T^*G \cong G \circledS \mathfrak{g}^*$ corresponding to $(\xi, \nu) \in \mathfrak{g} \circledS \mathfrak{g}^*$ at a point (g, μ) is given by [23, App. B. (B.9)]

$$X^{T^*G}_{(\xi,\nu)}(g, \mu) = \Big(T_e R_g \xi, \nu + \mathrm{ad}^*_\xi \mu\Big).$$

Accordingly, the values of the canonical 1-form θ_{T^*G} and the symplectic 2-form Ω_{T^*G} on a right invariant vector field are [1, 3, 23, 34]

$$\langle \theta_{T^*G}, X^{T^*G}_{(\xi,\nu)} \rangle (g, \mu) = \langle \mu, \xi \rangle, \tag{2.5}$$

$$\langle \Omega_{T^*G}; (X^{T^*G}_{(\xi_1,\nu_1)}, X^{T^*G}_{(\xi_2,\nu_2)}) \rangle (g, \mu) = \langle \nu_1, \xi_2 \rangle - \langle \nu_2, \xi_1 \rangle + \langle \mu, [\xi_1, \xi_2] \rangle. \tag{2.6}$$

REMARK 2.2. The symplectic 2-form Ω_{T^*G} is not conserved under the group operation (2.4), as such, $G \circledS \mathfrak{g}^*$ is not a symplectic Lie group as defined in [33].

2.3. The cotangent group of the tangent group T^*TG.

2.3.1. *Trivialization.* The global trivialization of $T^*TG \simeq T^*(G \circledS \mathfrak{g})$ can be achieved by trivializing $T^*(G \circledS \mathfrak{g})$ into the semidirect product group $G \circledS \mathfrak{g}_1$ and the dual $\mathfrak{g}_2^* \times \mathfrak{g}_3^*$ of its Lie algebra $\mathfrak{g}_2 \circledS \mathfrak{g}_3$

$$\begin{aligned} \mathrm{triv}_{T^*(G \circledS \mathfrak{g})} &\colon T^*(G \circledS \mathfrak{g}_1) \to (G \circledS \mathfrak{g}_1) \circledS (\mathfrak{g}_2^* \times \mathfrak{g}_3^*) \\ &\colon (\alpha_g, \alpha_\xi) \to (g, \xi, T_e^* R_g(\alpha_g) + \mathrm{ad}^*_\xi \alpha_\xi, \alpha_\xi), \end{aligned} \tag{2.7}$$

which preserves the group multiplication rule

$$\begin{aligned} &(g, \xi, \mu_1, \mu_2)(h, \eta, \nu_1, \nu_2) \\ &\quad = (gh, \xi + \mathrm{Ad}_{g^{-1}} \eta, \mu_1 + \mathrm{Ad}^*_{g^{-1}}(\nu_1 + \mathrm{ad}^*_{\mathrm{Ad}_g \xi} \nu_2), \mu_2 + \mathrm{Ad}^*_{g^{-1}} \nu_2) \end{aligned} \tag{2.8}$$

on T^*TG and results in the following subgroups. Here, the Cartesian space $\mathfrak{g}_2^* \times \mathfrak{g}_3^*$ is the one defined in Remark 2.1.

PROPOSITION 2.1. *The canonical immersions of the following submanifolds*

$$G, \quad \mathfrak{g}_1, \quad \mathfrak{g}_2^*, \quad \mathfrak{g}_3^*, \quad G \circledS \mathfrak{g}_1, \quad G \circledS \mathfrak{g}_2^*, \quad G \circledS \mathfrak{g}_3^*, \quad \mathfrak{g}_2^* \times \mathfrak{g}_3^*, \\ \mathfrak{g}_1 \circledS (\mathfrak{g}_2^* \times \mathfrak{g}_3^*), \quad (G \circledS \mathfrak{g}_1) \circledS \mathfrak{g}_2^*, \quad G \circledS (\mathfrak{g}_2^* \times \mathfrak{g}_3^*) \tag{2.9}$$

define subgroups of T^*TG and hence they act on T^*TG by actions induced from the multiplication in Eq. (2.8).

Here, the group structure on $G \circledS \mathfrak{g}_1$ is the one given in (2.3), whereas the group structure on $G \circledS g^*$ is in the form of Eq. (2.4) and we obtain the multiplications

$$(g, \xi, \mu)(h, \eta, \nu) = (gh, \xi + \mathrm{Ad}_{g^{-1}} \eta, \mu + \mathrm{Ad}^*_{g^{-1}} \nu) \tag{2.10}$$

$$(g, \mu_1, \mu_2)(h, \nu_1, \nu_2) = (gh, \mu_1 + \mathrm{Ad}^*_{g^{-1}} \nu_1, \mu_2 + \mathrm{Ad}^*_{g^{-1}} \nu_2) \tag{2.11}$$

$$(\xi, \mu_1, \mu_2)(\eta, \nu_1, \nu_2) = (\xi + \eta, \mu_1 + \nu_1 + \mathrm{ad}^*_{\xi} \nu_2, \mu_2 + \nu_2) \tag{2.12}$$

defining the group structures on $(G \circledS \mathfrak{g}_1) \circledS \mathfrak{g}_2^*$, $G \circledS (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ and $\mathfrak{g}_1 \circledS (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$, respectively.

2.3.2. *Symplectic structure.* By requiring that the trivialization $\mathrm{triv}_{T^*(G \circledS \mathfrak{g})}$ be a symplectic map, we define a canonical one-form θ_{T^*TG} and a symplectic two-form Ω_{T^*TG} on the trivialized cotangent bundle T^*TG. To this end, we recall that a right invariant vector field $X^{T^*TG}_{(\eta_1, \eta_2, \nu_1, \nu_2)}$ on T^*TG is generated by an element $(\eta_1, \eta_2, \nu_1, \nu_2)$ in the Lie algebra $(\mathfrak{g} \circledS \mathfrak{g}) \circledS (\mathfrak{g}^* \times \mathfrak{g}^*)$ of T^*TG by means of the tangent lift of right translation on T^*TG. At a point (g, ξ, μ_1, μ_2) in T^*TG, the value of such a right invariant vector field at the point (g, ξ, μ_1, μ_2) reads

$$X^{T^*TG}_{(\eta_1, \eta_2, \nu_1, \nu_2)} = (T_e R_g \eta_1, \eta_2 + \mathrm{ad}_{\xi} \eta_1, \nu_1 + \mathrm{ad}^*_{\eta_1} \mu_1 + \mathrm{ad}^*_{\eta_2} \mu_2, \nu_2 + \mathrm{ad}^*_{\eta_1} \mu_2) \tag{2.13}$$

and is an element of the fiber $T_{(g, \xi, \mu_1, \mu_2)}(T^*TG)$. The values of canonical forms θ_{T^*TG} and Ω_{T^*TG} on right invariant vector fields can then be computed at the point (g, ξ, μ_1, μ_2) as

$$\langle \theta_{T^*TG}; X^{T^*TG}_{(\xi_1, \xi_2, \nu_1, \nu_2)} \rangle = \langle \mu_1, \xi_1 \rangle + \langle \mu_2, \xi_2 \rangle$$

$$\begin{aligned} \langle \Omega_{T^*TG}; (X^{T^*TG}_{(\xi_1, \xi_2, \lambda_1, \lambda_2)}, X^{T^*TG}_{(\eta_1, \eta_2, \nu_1, \nu_2)}) \rangle &= \langle \lambda_1, \eta_1 \rangle + \langle \lambda_2, \eta_2 \rangle \\ &- \langle \nu_1, \xi_1 \rangle - \langle \nu_2, \xi_2 \rangle + \langle \mu_1, [\xi_1, \eta_1] \rangle + \langle \mu_2, [\xi_1, \eta_2] - [\xi_2, \eta_1] \rangle. \end{aligned}$$

The musical isomorphism $\Omega^{\flat}_{T^*TG}$, induced from the symplectic two-form Ω_{T^*TG}, maps the tangent bundle $T(T^*TG)$ to the cotangent bundle $T^*(T^*TG)$. It takes the right invariant vector field in Eq. (2.13) to an element of the cotangent bundle $T^*_{(g, \xi, \mu_1, \mu_2)}(T^*TG)$ with coordinates at the point $(g, \xi, \mu_1, \mu_2))$ as

$$\Omega^{\flat}_{T^*TG}(X^{T^*TG}_{(\eta_1, \eta_2, \lambda_1, \lambda_2)}) = (T^*_g R_{g^{-1}}(\lambda_1 - \mathrm{ad}^*_{\xi} \lambda_2), \lambda_2, -\eta_1, -\eta_2).$$

REMARK 2.3. The actions of the subgroups $\mathfrak{g}_2^*$ and $\mathfrak{g}_3^*$ are not symplectic, nor is any subgroup in the list of Eq. (2.9) containing $\mathfrak{g}_2^*$ and $\mathfrak{g}_3^*$. There remains only the action of the group $G \circledS \mathfrak{g}$ to perform symplectic reduction on T^*TG.

2.4. The cotangent group of the cotangent group T^*T^*G. The global trivialization of the iterated cotangent bundle can be achieved by semidirect product of the group $G Ⓢ \mathfrak{g}_1^*$ and the dual $\mathfrak{g}_2^* \times \mathfrak{g}_3$ of its Lie algebra [19]. The trivialization map

$$\begin{aligned}\mathrm{triv}_{T^*T^*G} \colon T^*(G Ⓢ \mathfrak{g}_1^*) &\to (G Ⓢ \mathfrak{g}_1^*) Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3)\\ : (\alpha_g, \alpha_\mu) &\to (g, \mu, T_e^* R_g(\alpha_g) - \mathrm{ad}^*_{\alpha_\mu} \mu, \alpha_\mu)\end{aligned}$$

implies on T^*T^*G the group multiplication rule

$$\begin{aligned}(2.14)\quad &(g, \mu_1, \mu_2, \xi)(h, \nu_1, \nu_2, \eta)\\ &\quad = (gh, \mu_1 + \mathrm{Ad}^*_{g^{-1}} \nu_1, \mu_2 + \mathrm{Ad}^*_{g^{-1}} \nu_2 - \mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}} \eta} \mu_1, \xi + \mathrm{Ad}_{g^{-1}} \eta).\end{aligned}$$

PROPOSITION 2.2. *Embeddings of the following subspaces*

$$\begin{aligned}(2.15)\quad G,\quad \mathfrak{g}_1^*,\quad \mathfrak{g}_2^*,\quad \mathfrak{g}_3,\quad G Ⓢ \mathfrak{g}_1^*,\quad G Ⓢ \mathfrak{g}_2^*,\quad G Ⓢ \mathfrak{g}_3,\quad \mathfrak{g}_1^* Ⓢ \mathfrak{g}_2^*,\quad \mathfrak{g}_2^* \times \mathfrak{g}_3,\\ (G Ⓢ \mathfrak{g}_1^*) Ⓢ \mathfrak{g}_2^*,\quad G Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3),\quad \mathfrak{g}_1^* Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3)\end{aligned}$$

define subgroups of T^*T^*G and hence they act on T^*T^*G by actions induced from the multiplication in Eq. (2.14).

The group structures on $G Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3)$, $G Ⓢ \mathfrak{g}_1^* Ⓢ \mathfrak{g}_2^*$, $\mathfrak{g}_1^* Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3)$ are (up to some reordering) given by Eqs. (2.10), (2.11) and (2.12), respectively.

2.4.1. *Symplectic structure on T^*T^*G.* The canonical one-form and the symplectic two-form on T^*T^*G can be mapped by $\mathrm{triv}_{T^*T^*G}$ to T^*T^*G based on the fact that the trivialization map is a symplectic diffeomorphism. Consider a right invariant vector field $X^{T^*T^*G}_{(\eta_1,\nu_1,\nu_2,\eta_2)}$ generated by an element $(\eta_1, \nu_1, \nu_2, \eta_2)$ in the Lie algebra $(\mathfrak{g} Ⓢ \mathfrak{g}^*) Ⓢ (\mathfrak{g}^* \times \mathfrak{g})$ of T^*T^*G. At the point (g, μ_1, μ_2, ξ), the right invariant vector

$$(2.16)\quad X^{T^*T^*G}_{(\eta_1,\nu_1,\nu_2,\eta_2)} = (TR_g \eta_1, \nu_1 + \mathrm{ad}^*_{\eta_1} \mu_1, \nu_2 + \mathrm{ad}^*_{\eta_1} \mu_2 - \mathrm{ad}^*_\xi \nu_1, \eta_2 + \mathrm{ad}_\xi \eta_1)$$

is an element of $T_{(g,\mu_1,\mu_2,\xi)}(T^*T^*G)$. At the point (g, μ_1, μ_2, ξ), the values of canonical forms $\theta_{T^*T^*G}$ and $\Omega_{T^*T^*G}$ at right invariant vector fields can now be evaluated to be

$$\langle \theta_{T^*T^*G}, X^{T^*T^*G}_{(\xi_1,\lambda_1,\lambda_2,\xi_2)} \rangle = \langle \xi_1, \mu_2 \rangle + \langle \lambda_1, \xi \rangle$$

$$\begin{aligned}(2.17)\quad \langle \Omega_{T^*T^*G}; (X^{T^*T^*G}_{(\xi_1,\lambda_1,\lambda_2,\xi_2)}, X^{T^*T^*G}_{(\eta_1,\nu_1,\nu_2,\eta_2)}) \rangle &= \langle \lambda_2, \eta_1 \rangle + \langle \nu_1, \xi_2 \rangle\\ &- \langle \nu_2, \xi_1 \rangle - \langle \lambda_1, \eta_2 \rangle + \langle \mu_2, [\xi_1, \xi_2] \rangle - \langle \nu_1, [\xi_1, \xi] \rangle + \langle \lambda_1, [\eta_1, \xi] \rangle.\end{aligned}$$

The musical isomorphism $\Omega^\flat_{T^*T^*G}$, induced from the symplectic two-form $\Omega_{T^*T^*G}$ in Eq. (2.17), maps $T(T^*T^*G)$ to $T^*(T^*T^*G)$. At the point (g, μ_1, μ_2, ξ), $\Omega^\flat_{T^*T^*G}$ takes the vector in Eq. (2.16) to the element

$$\Omega^\flat_{T^*T^*G}(X^{T^*T^*G}_{(\eta_1,\nu_1,\nu_2,\eta_2)}) = (T_g^* R_{g^{-1}}(\nu_2 + \mathrm{ad}^*_{\eta_2} \mu_1), \eta_2, -\eta_1, -\nu_1)$$

in $T^*_{(g,\mu_1,\mu_2,\xi)}(T^*T^*G)$.

REMARK 2.4. Actions of subgroups $\mathfrak{g}_2^*$ and $\mathfrak{g}_3$, and hence any subgroup in the list (2.15) containing $\mathfrak{g}_2^*$ and $\mathfrak{g}_3$, are not symplectic. Thus, there remains only the action of the group $G Ⓢ \mathfrak{g}_1^*$ to perform symplectic reduction on T^*T^*G.

2.5. The tangent group of the cotangent group TT^*G. $TT^*G \simeq T(G\circledS\mathfrak{g}^*)$ can be trivialized as a semidirect product of the group $G\circledS\mathfrak{g}^*$ and its Lie algebra $\mathfrak{g}\circledS\mathfrak{g}^*$ by

$$\begin{aligned} \operatorname{triv}_{TT^*G}\colon T(G\circledS\mathfrak{g}^*) &\to (G\circledS\mathfrak{g}_1^*)\circledS(\mathfrak{g}_2\circledS\mathfrak{g}_3^*) \\ : (V_g, V_\mu) &\to (g, \mu, TR_{g^{-1}}V_g, V_\mu - \operatorname{ad}^*_{TR_{g^{-1}}V_g}\mu), \end{aligned} \tag{2.18}$$

where $(V_g, V_\mu) \in T_{(g,\mu)}(G\circledS\mathfrak{g}^*)$ [19]. The group multiplication on TT^*G is

$$\begin{aligned} &(g, \mu_1, \xi, \mu_2)(h, \nu_1, \eta, \nu_2) \\ &\qquad = (gh, \mu_1 + \operatorname{Ad}^*_{g^{-1}}\nu_1, \xi + \operatorname{Ad}_{g^{-1}}\eta, \mu_2 + \operatorname{Ad}^*_{g^{-1}}\nu_2 - \operatorname{ad}^*_{\operatorname{Ad}_{g^{-1}}\eta}\mu_1) \end{aligned}$$

and embedded subgroups of TT^*G follow.

Remark 2.5. As in the case of the tangent group, the subindices appear on the Lie algebra and the dual space serves to distinguish the copies of $\mathfrak{g}$ and the dual $\mathfrak{g}^*$. For the cotangent bundle T^*G, we denote the trivialization as $TG \cong G\circledS\mathfrak{g}_1^*$. The Lie algebra of this space is denoted by $\mathfrak{g}_2\circledS\mathfrak{g}_3^*$, that is,

$$\operatorname{Lie}(T^*G) \cong \operatorname{Lie}(G\circledS\mathfrak{g}_1^*) \cong \mathfrak{g}_2\circledS\mathfrak{g}_3^*,$$

where $\operatorname{Lie}(H)$ stands for the Lie algebra of the group H. The dual space of the Lie algebra $\mathfrak{g}_2\circledS\mathfrak{g}_3^*$ is

$$(\mathfrak{g}_2\circledS\mathfrak{g}_3^*)^* \cong \mathfrak{g}_2^* \times \mathfrak{g}_3,$$

where $\mathfrak{g}_2^*$ and $\mathfrak{g}_3^{**} \cong \mathfrak{g}_3$ are the dual spaces of the constitutive Lie algebras $\mathfrak{g}_2$ and $\mathfrak{g}_3^*$, respectively.

Proposition 2.3. *The embeddings of the subspaces*

$$\begin{gathered} G, \quad \mathfrak{g}_1^*, \quad \mathfrak{g}_2, \quad \mathfrak{g}_3^* \quad G\circledS\mathfrak{g}_1^*, \quad G\circledS\mathfrak{g}_2, \quad G\circledS\mathfrak{g}_3^*, \quad \mathfrak{g}_1^*\circledS\mathfrak{g}_3^*, \quad \mathfrak{g}_2\circledS\mathfrak{g}_3^*, \\ (G\circledS\mathfrak{g}_1^*)\circledS\mathfrak{g}_3^*, \quad G\circledS(\mathfrak{g}_2\circledS\mathfrak{g}_3^*), \quad \mathfrak{g}_1^*\circledS(\mathfrak{g}_2\circledS\mathfrak{g}_3^*) \end{gathered}$$

of TT^*G define its subgroups. The group structures on $G\circledS\mathfrak{g}$, $G\circledS\mathfrak{g}^*$ are defined by Eqs. (2.3) *and* (2.4)*, respectively. The group structures on the product spaces $(G\circledS\mathfrak{g}_1^*)\circledS\mathfrak{g}_3^*$, $G\circledS(\mathfrak{g}_2\circledS\mathfrak{g}_3^*)$ and $\mathfrak{g}_1^*\circledS(\mathfrak{g}_2\circledS\mathfrak{g}_3^*)$ are defined* (*up to some reordering*) *by Eqs.* (2.11), (2.10) *and* (2.12)*, respectively. The group multiplications on $\mathfrak{g}_1^*$, $\mathfrak{g}_2$, $\mathfrak{g}_3^*$, $\mathfrak{g}_1^* \times \mathfrak{g}_3^*$ and $\mathfrak{g}_2 \times \mathfrak{g}_3^*$ are vector additions.*

2.5.1. *Tulczyjew symplectic strcuture on TT^*G.* TT^*G is central in Tulczyjew's triplet and carries a two-sided symplectic two-form. An element $(\eta_1, \nu_1, \eta_2, \nu_2)$ in the semidirect product Lie algebra $(\mathfrak{g}\circledS\mathfrak{g}^*)\circledS(\mathfrak{g}\circledS\mathfrak{g}^*)$ defines a right invariant vector field on TT^*G by the tangent lift of right translation in TT^*G. At a point (g, μ_1, ξ, μ_2), a right invariant vector is given by

$$X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} = (TR_g\xi_2, \nu_2 + \operatorname{ad}^*_{\xi_2}\mu_1, \xi_3 + \operatorname{ad}_\xi \xi_2, \nu_3 + \operatorname{ad}^*_{\xi_2}\mu_2 - \operatorname{ad}^*_\xi \nu_2). \tag{2.19}$$

The bundle $T(G\circledS\mathfrak{g}^*)$ carries Tulczyjew's symplectic two-form $\Omega_{T(G\circledS\mathfrak{g}^*)}$ with two potential one-forms. The one-forms θ_1 and θ_2 are obtained by taking derivations of the symplectic two-form Ω_{T^*G} and the canonical one-form θ_{T^*G} respectively in Eq. (2.5) [19]. By requiring that the trivialization $\operatorname{triv}_{TT^*G}$ in Eq. (2.18) be a symplectic mapping, we obtain an exact symplectic structure Ω_{TT^*G} with two potential one-forms θ_1 and θ_2 taking the values

$$\langle \Omega_{TT^*G}; (X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}, X^{TT^*G}_{(\bar\xi_2,\bar\nu_2,\bar\xi_3,\bar\nu_3)})\rangle(g,\mu_1,\xi,\mu_2) = \langle \nu_3,\bar\xi_2\rangle + \langle \nu_2,\bar\xi_3\rangle \\ - \langle \bar\nu_2,\xi_3\rangle - \langle \bar\nu_3,\xi_2\rangle + \langle \mu_2,[\xi_2,\bar\xi_2]\rangle + \langle \mu_1,[\xi_3,\bar\xi_2]+[\xi_2,\bar\xi_3]+[\xi,[\xi_2,\bar\xi_2]]\rangle, \tag{2.20}$$

$$\langle \theta_1, X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\rangle = \langle \mu_2,\xi_2\rangle - \langle \nu_2,\xi\rangle + \langle \mu_1,[\xi,\xi_2]\rangle, \tag{2.21}$$

$$\langle \theta_2, X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\rangle = \langle \mu_1,\xi_3\rangle + \langle \mu_2,\xi_2\rangle + \langle \mu_1,[\xi,\xi_2]\rangle \tag{2.22}$$

on right invariant vector fields of the form of Eq. (2.19). At a point $(g,\mu_1,\xi,\mu_2)\in TT^*G$, the musical isomorphism $\Omega^\flat_{TT^*G}$, induced from Ω_{TT^*G}, maps the image of a right invariant vector field $X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ to an element

$$\Omega^\flat_{TT^*G}(X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}) = (T^*_gR_{g^{-1}}(\nu_3 - \operatorname{ad}^*_\xi \nu_2), -(\xi_3+[\xi,\xi_2]), \nu_2 + \operatorname{ad}^*_{\xi_2}\mu_1, -\xi_2)$$

of $T^*_{(g,\mu,\xi,\nu)}(TT^*G)$.

3. Dynamics on the first order bundles

3.1. Lagrangian dynamics on the tangent group TG. Given a Lagrangian function $L\colon TG\to\mathbb{R}$, let $\bar L\colon G\circledS\mathfrak{g}\to\mathbb{R}$ be the corresponding function determined by $\bar L\circ \operatorname{triv}_{TG} = L$. The variation of the action integral of the latter is computed as

$$\delta\int_a^b \bar L(\xi,g)dt = \int_a^b \Big(\Big\langle \frac{\delta \bar L}{\delta\xi},\delta\xi\Big\rangle_e + \Big\langle \frac{\delta\bar L}{\delta g},\delta g\Big\rangle_g\Big)dt,$$

applying Hamilton's principle to the variations of the group (base) component, and the reduced variational principle

$$\delta\xi = \dot\eta + [\xi,\eta]$$

to the variations of the Lie algebra (fiber) component. For the reduced variational principle we refer to [9,20,27,30,39] and for the Lagrangian dynamics on semidirect products to [4,8,28,37,40,47,48]. For the following result, see [5,12,13,15,20].

Proposition 3.1. *The trivialized Euler–Lagrange dynamics generated by a Lagrangian function $\bar L\colon G\circledS\mathfrak{g}\to\mathbb{R}$ is given by*

$$\frac{d}{dt}\frac{\delta\bar L}{\delta\xi} = T^*_eR_g\frac{\delta\bar L}{\delta g} + \operatorname{ad}^*_\xi\frac{\delta\bar L}{\delta\xi}. \tag{3.1}$$

If, in addition, the Lagrangian function $\bar L\colon G\circledS\mathfrak{g}\to\mathbb{R}$ is right invariant (namely, it is independent of the group variable, that is, $\bar L(g,\xi)=\ell(\xi)$), then Eq. (3.1) reduces to the Euler–Poincaré equations on $\mathfrak{g}=(G\circledS\mathfrak{g})/G$

$$\frac{d}{dt}\frac{\delta l}{\delta\xi} = \operatorname{ad}^*_\xi\frac{\delta l}{\delta\xi}. \tag{3.2}$$

Along the motion, for any Lagrangian $\bar L=\bar L(g,\xi)$, we compute that

$$\frac{dL}{dt} = \Big\langle\frac{\delta\bar L}{\delta g},\dot g\Big\rangle + \Big\langle\frac{\delta\bar L}{\delta\xi},\dot\xi\Big\rangle = \Big\langle\frac{\delta\bar L}{\delta g},T_eR_g\xi\Big\rangle + \Big\langle\frac{\delta\bar L}{\delta\xi},\dot\xi\Big\rangle \tag{3.3}$$

$$
\begin{aligned}
&= \Big\langle T_e^* R_g \frac{\delta \bar{L}}{\delta g}, \xi \Big\rangle + \Big\langle \frac{\delta \bar{L}}{\delta \xi}, \dot{\xi} \Big\rangle \\
&= \Big\langle \frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi} - \mathrm{ad}_\xi^* \frac{\delta \bar{L}}{\delta \xi}, \xi \Big\rangle + \Big\langle \frac{\delta \bar{L}}{\delta \xi}, \dot{\xi} \Big\rangle \\
&= \Big\langle \frac{d}{dt}\frac{\delta \bar{L}}{\delta \xi}, \xi \Big\rangle + \Big\langle \frac{\delta \bar{L}}{\delta \xi}, \dot{\xi} \Big\rangle = \frac{d}{dt}\Big\langle \frac{\delta \bar{L}}{\delta \xi}, \xi \Big\rangle,
\end{aligned}
$$

where, according to the trivialization (2.2), we have employed the identification $\dot{g} = T_e R_g \xi$ in the first line, whereas we substitute the Euler–Lagrange equations (3.1) in the third line. The calculation (3.3) reads that the quantity $\langle \delta\bar{L}/\delta\xi, \xi\rangle - L$ is a constant of the motion.

3.2. Hamiltonian dynamics on the cotangent group T^*G. Given $\bar{H}\colon T^*G \cong G Ⓢ \mathfrak{g}^* \to \mathbb{R}$, one obtains Hamilton's equations

$$
\frac{dg}{dt} = T_e R_g \Big(\frac{\delta \bar{H}}{\delta \mu}\Big), \quad \frac{d\mu}{dt} = \mathrm{ad}^*_{\frac{\delta \bar{H}}{\delta \mu}} \mu - T_e^* R_g \frac{\delta \bar{H}}{\delta g} \tag{3.4}
$$

on the semidirect product $T^*G \cong G Ⓢ \mathfrak{g}^*$ from the very definition

$$
i_{X_{\bar{H}}^{T^*G}} \Omega_{T^*G} = -d\bar{H},
$$

where the right invariant vector field

$$
X_{\bar{H}}^{T^*G}(g,\mu) := \Big(T_e R_g \frac{\delta \bar{H}}{\delta \mu}, \mathrm{ad}^*_{\frac{\delta \bar{H}}{\delta \mu}} \mu - T_e^* R_g \frac{\delta \bar{H}}{\delta g}\Big)
$$

is the Hamiltonian vector field associated to

$$
\Big(\frac{\delta \bar{H}}{\delta \mu}, -T_e^* R_g \frac{\delta \bar{H}}{\delta g}\Big) \in \mathfrak{g} Ⓢ \mathfrak{g}^*.
$$

For further details of Hamiltonian dynamics on semi-direct products we refer the reader to [4, 7, 11, 20, 29, 35–38, 40, 42, 46].

The canonical Poisson bracket of two functionals $\bar{F}, \bar{K} : G Ⓢ \mathfrak{g}^* \to \mathbb{R}$ at a point $(g,\mu) \in G Ⓢ \mathfrak{g}^*$ is given by

$$
\{\bar{F}, \bar{K}\}_{T^*G} = \Big\langle T_e^* R_g \frac{\delta \bar{F}}{\delta g}, \frac{\delta \bar{K}}{\delta \mu} \Big\rangle - \Big\langle T_e^* R_g \frac{\delta \bar{K}}{\delta g}, \frac{\delta \bar{F}}{\delta \mu} \Big\rangle + \Big\langle \mu, \Big[\frac{\delta \bar{F}}{\delta \mu}, \frac{\delta \bar{K}}{\delta \mu}\Big]\Big\rangle. \tag{3.5}
$$

3.2.1. *Reduction of T^*G by G.* The right action of G on $G Ⓢ \mathfrak{g}^*$ is

$$
(G Ⓢ \mathfrak{g}^*) \times G \to G Ⓢ \mathfrak{g}^* : ((g,\mu); h) \to (gh, \mu) \tag{3.6}
$$

with the infinitesimal generator $X_{(\xi,0)}^{T^*G}$. If $\bar{H}$, defined on $G Ⓢ \mathfrak{g}^*$, is independent of g, it becomes right invariant under G. In this case, dropping the terms involving $\delta\bar{H}/\delta g$ in Poisson bracket (3.5) is the Poisson reduction $G Ⓢ \mathfrak{g}^* \to (G Ⓢ \mathfrak{g}^*)/G \simeq \mathfrak{g}^*$. When $\bar{F}$ and $\bar{K}$ are independent of the group variable $g \in G$, that is, when $\bar{F} = f(\mu)$ and $\bar{K} = k(\mu)$, we have the Lie–Poisson bracket

$$
\{f, k\}_{\mathfrak{g}^*}(\mu) = \Big\langle \mu, \Big[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\Big]\Big\rangle \tag{3.7}
$$

from which the Lie–Poisson equation

$$\dot{\mu} = \operatorname{ad}^*_{\frac{\delta h}{\delta \mu}} \mu \tag{3.8}$$

on the dual space $\mathfrak{g}^*$ follows. The Lie–Poisson bracket given in Eq. (3.7) can also be obtained by pulling back the non-degenerate Poisson bracket in Eq. (3.5) with the embedding $\mathfrak{g}^* \to G \circledS \mathfrak{g}^*$.

For the symplectic leaves of this Poisson structure [50], we apply the Marsden–Weinstein symplectic reduction theorem [43] to $G \circledS \mathfrak{g}^*$ with the action of G. The action (3.6) is symplectic and it induces the momentum mapping

$$\mathbf{J}_{G \circledS \mathfrak{g}^*} \colon G \circledS \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \colon (g, \mu) \to \mu,$$

which is also Poisson, and hence it projects trivialized Hamiltonian dynamics in (3.4) to the Lie–Poisson dynamics in (3.8).

The inverse image $\mathbf{J}^{-1}_{G \circledS \mathfrak{g}^*}(\mu) \subset G \circledS \mathfrak{g}^*$ of a regular value $\mu \in \mathfrak{g}^*$ consists of two-tuples (g, μ) for $g \in G$ and fixed $\mu \in \mathfrak{g}^*$. We may identify $\mathbf{J}^{-1}_{G \circledS \mathfrak{g}^*}(\mu)$ with the group G. Let G_μ be the isotropy group of the coadjoint action Ad^*, defined in (2.1), preserving the momenta μ. Then, we have the isomorphism

$$\mathbf{J}^{-1}_{G \circledS \mathfrak{g}^*}(\mu)/G_\mu \simeq G/G_\mu \simeq \mathcal{O}_\mu$$

identifying the equivalence class $[g]$ of g in G/G_μ with the coadjoint orbit

$$\mathcal{O}_\mu = \{\operatorname{Ad}^*_g \mu \colon g \in G\}$$

through the point μ in $\mathfrak{g}^*$ [41]. We denote the reduced symplectic two-form on $\mathcal{O}_\mu$ by $\Omega^{/G}_{T^*G}(\mu)$, which is the Kostant–Kirillov–Souriau two-form [30, 41, 44]. The value of $\Omega^{/G}_{T^*G}(\mu)$ on two vector fields $\operatorname{ad}^*_\xi \mu$, $\operatorname{ad}^*_\eta \mu$ in $T_\mu \mathcal{O}_\mu$ is

$$\langle \Omega^{/G}_{T^*G}; (\operatorname{ad}^*_\xi \mu, \operatorname{ad}^*_\eta \mu) \rangle = -\langle \mu, [\xi, \eta]_{\mathfrak{g}} \rangle. \tag{3.9}$$

3.2.2. *Reduction of T^*G by G_μ.* The isotropy subgroup G_μ acts on $G \circledS \mathfrak{g}^*$ as described by Eq. (3.6). Then, Poisson and symplectic reductions of dynamics are possible. The Poisson reduction of the symplectic manifold $G \circledS \mathfrak{g}^*$ under the action of the isotropy group G_μ results in

$$(G \circledS \mathfrak{g}^*)/G_\mu \simeq \mathcal{O}_\mu \times \mathfrak{g}^*,$$

with a Poisson bracket

$$\{H, K\}_{\mathcal{O}_\mu \times \mathfrak{g}^*}(\mu, \nu) = \left\langle \mu, \left[\frac{\delta H}{\delta \mu}, \frac{\delta K}{\delta \mu}\right] \right\rangle + \left\langle \nu, \left[\frac{\delta H}{\delta \mu}, \frac{\delta K}{\delta \nu}\right] - \left[\frac{\delta K}{\delta \mu}, \frac{\delta H}{\delta \nu}\right] \right\rangle$$

which is not a direct product of Lie–Poisson structures on $\mathcal{O}_\mu$ and $\mathfrak{g}^*$.

The coadjoint action of $G \circledS \mathfrak{g}$ on the dual $\mathfrak{g}^* \times \mathfrak{g}^*$ of its Lie algebra is

$$\operatorname{Ad}^*_{(g,\xi)} \colon \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^* \times \mathfrak{g}^* \colon (\mu, \nu) \mapsto (\operatorname{Ad}^*_g \mu - \operatorname{ad}^*_\xi \operatorname{Ad}^*_g \nu, \operatorname{Ad}^*_g \nu). \tag{3.10}$$

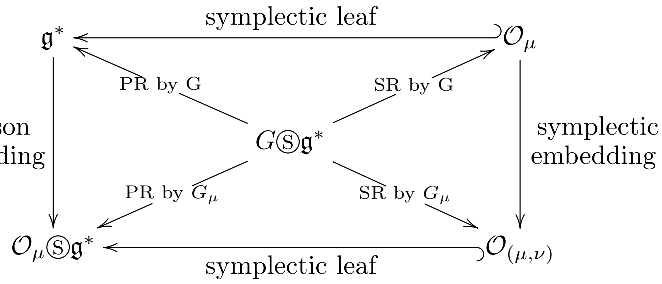
The symplectic reduction of $G \circledS \mathfrak{g}^*$ under the action of the isotropy subgroup G_μ results in the coadjoint orbit $\mathcal{O}_{(\mu,\nu)}$ in $\mathfrak{g}^* \times \mathfrak{g}^*$ through the point (μ, ν) under the action in Eq. (3.10). The reduced symplectic two-form $\Omega_{\mathcal{O}_{(\mu,\nu)}}$ takes the value

$$\langle \Omega_{\mathcal{O}_{(\mu,\nu)}}; (\eta, \zeta), (\bar{\eta}, \bar{\zeta}) \rangle (\mu, \nu) = \langle \mu, [\bar{\eta}, \eta] \rangle + \langle \nu, [\bar{\eta}, \zeta] - [\eta, \bar{\zeta}] \rangle$$

on two vectors (η,ζ) and $(\bar\eta,\bar\zeta)$ in $T_{(\mu,\nu)}\mathcal{O}_{(\mu,\nu)}$.

We summarize reductions of the symplectic space $G\circledS\mathfrak{g}^*$ in the following diagram.

(3.11)



Reductions of $T^*G = G\circledS\mathfrak{g}^*$

3.3. The Legendre transformation. For a (hyper)regular Lagrangian $\bar L$ on $TG = G\circledS\mathfrak{g}$, the Legendre transformation is

$$G\circledS\mathfrak{g} \to G\circledS\mathfrak{g}^* \colon (g,\xi) \to \left(g, \frac{\delta \bar L}{\delta\xi} = \mu\right),$$

which identifies $\delta\bar L/\delta\xi$ with the fiber variable μ of $G\circledS\mathfrak{g}^*$. Define a Hamiltonian function

$$H(g,\mu) = \langle \mu,\xi\rangle - L(g,\xi)$$

for which the Hamiltonian dynamics in Eq. (3.4) gives Euler-Lagrange equations (3.1). When $\bar L$ is independent of the group variable, we have Euler–Poincare equations (3.2) and the Legendre transformation $\mu = \delta l/\delta\xi$ maps these to Lie–Poisson equations (3.8) with the Hamiltonian function

$$h(\mu) = \langle\mu,\xi\rangle - l(\xi).$$

When the Lagrangian function is degenerate, the fiber derivative is not invertible and hence a direct passage from the Lagrangian dynamics to the Hamiltonian one is not possible. One possible way to define a general Legendre transformation, including the degenerate cases, is possible in Tulczyjew's approach [49]. We refer to [19, 20, 25], where Tulczyjew's triplet is constructed for Lie groups.

4. Hamiltonian dynamics on the cotangent of the tangent bundle

For a Hamiltonian function(al) $H = H(g,\xi,\mu,\nu)$ on the symplectic manifold (T^*TG, Ω_{T^*TG}), Hamilton's equations read

$$i_{X_H^{T^*TG}}\Omega_{T^*TG} = -dH,$$

where the right invariant Hamiltonian vector field $X_H^{T^*TG}$ is generated by [2]

$$\left(\frac{\delta H}{\delta\mu}, \frac{\delta H}{\delta\nu}, -T_e^*R_g\left(\frac{\delta H}{\delta g}\right) - \mathrm{ad}_\xi^*\left(\frac{\delta H}{\delta\xi}\right), -\frac{\delta H}{\delta\xi}\right) \in (\mathfrak{g}\circledS\mathfrak{g})\circledS(\mathfrak{g}^*\times\mathfrak{g}^*).$$

PROPOSITION 4.1. *Components of* $X_H^{T^*TG}$ *are trivialized Hamilton's equations on* (T^*TG, Ω_{T^*TG})

$$\frac{dg}{dt} = T_eR_g \frac{\delta H}{\delta \mu}, \tag{4.1}$$

$$\frac{d\xi}{dt} = \frac{\delta H}{\delta \nu} + \mathrm{ad}_\xi \frac{\delta H}{\delta \mu}, \tag{4.2}$$

$$\frac{d\mu}{dt} = -T_e^*R_g \frac{\delta H}{\delta g} - \mathrm{ad}_\xi^* \frac{\delta H}{\delta \xi} + \mathrm{ad}^*_{\frac{\delta H}{\delta \mu}} \mu + \mathrm{ad}^*_{\frac{\delta H}{\delta \nu}} \nu, \tag{4.3}$$

$$\frac{d\nu}{dt} = -\frac{\delta H}{\delta \xi} + \mathrm{ad}^*_{\frac{\delta H}{\delta \mu}} \nu. \tag{4.4}$$

From the equations (4.2) and (4.4), we single out $\delta H/\delta \nu$ and $\delta H/\delta \xi$, respectively. By substituting these into Eq. (4.3), we obtain the system

$$\left(\frac{d}{dt} - \mathrm{ad}^*_{\frac{\delta H}{\delta \mu}}\right)(\mathrm{ad}_\xi^* \nu - \mu) = T_e^*R_g \frac{\delta H}{\delta g}, \quad \frac{dg}{dt} = T_eR_g \frac{\delta H}{\delta \mu}$$

equivalent to Eqs. (4.1)–(4.4).

REMARK 4.1. Hamilton's equations (4.1)–(4.4) have extra terms, compared to the ones, for example, in [**11**,**24**], coming from the choice of trivialization preserving group structure. The trivialization of [**11**] is of the second kind given by Eq. (1.1), whereas Eqs. (4.1)–(4.4) result from trivializations of the first kind. Reference [**6**] studies geometric integrators of this Hamiltonian dynamics.

4.1. Reduction of T^*TG by G. We shall first perform Poisson reduction of the Hamiltonian system on T^*TG under the action of G given by

$$((g, \xi, \mu, \nu); \tilde{g}) \to (g\tilde{g}, \xi, \mu, \nu)$$

for a right invariant Hamiltonian $H = H(\xi, \mu, \nu)$.

PROPOSITION 4.2. *The Poisson reduced manifold* $\mathfrak{g}_1 Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ *carries the Poisson bracket*

$$\begin{aligned} \{H, K\}_{\mathfrak{g}_1 Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)}(\xi, \mu, \nu) = &-\left\langle \frac{\delta H}{\delta \xi}, \frac{\delta K}{\delta \nu} \right\rangle + \left\langle \frac{\delta K}{\delta \xi}, \frac{\delta H}{\delta \nu} \right\rangle + \left\langle \mu, \left[\frac{\delta H}{\delta \mu}, \frac{\delta K}{\delta \mu}\right] \right\rangle \\ &+ \left\langle \mathrm{ad}_\xi^* \frac{\delta K}{\delta \xi}, \frac{\delta H}{\delta \mu} \right\rangle - \left\langle \mathrm{ad}_\xi^* \frac{\delta H}{\delta \xi}, \frac{\delta K}{\delta \mu} \right\rangle + \left\langle \nu, \left[\frac{\delta H}{\delta \mu}, \frac{\delta K}{\delta \nu}\right] - \left[\frac{\delta K}{\delta \mu}, \frac{\delta H}{\delta \nu}\right] \right\rangle, \end{aligned} \tag{4.5}$$

for two right invariant functionals H and K on T^*TG.

REMARK 4.2. $T^*\mathfrak{g}_1 = \mathfrak{g}_1 \times \mathfrak{g}_3^*$ carries a canonical Poisson bracket, and $\mathfrak{g}_2^*$ carries a Lie–Poisson bracket. The immersions $\mathfrak{g}_1 \times \mathfrak{g}_3^* \to \mathfrak{g}_1 Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ and $\mathfrak{g}_2^* \to \mathfrak{g}_1 Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ are Poisson maps. However, the Poisson structure described by Eq. (4.5) on $\mathfrak{g}_1 Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ is not a direct product of these. In fact, a direct product structure on $\mathfrak{g}_1 \times (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ whereas the group structure on $G Ⓢ g^*$ is in the form of in [**11**] and [**24**]. In these cases the second line of (4.5) disappears. Here, we are employing the notations presented in Remark 2.1.

Proposition 4.3. *The Marsden–Weinstein symplectic reduction by the action of G on T^*TG with the momentum mapping*

$$\mathbf{J}^G_{T^*TG}\colon T^*TG \to \mathfrak{g}^*\colon (g,\xi,\mu,\nu) \to \mu$$

results in the reduced symplectic two-form $\Omega^{/G}_{T^*TG}$ on the reduced space $\mathcal{O}_\mu \times \mathfrak{g}_1 \times \mathfrak{g}_3^*$. The value of $\Omega^{/G}_{T^*TG}$ on two vectors $(\eta_{\mathfrak{g}^*}(\mu),\zeta,\lambda)$ and $(\bar\eta_{\mathfrak{g}^*}(\mu),\bar\zeta,\bar\lambda)$ is

$$\Omega^{/G}_{T^*TG}((\eta_{\mathfrak{g}^*}(\mu),\zeta,\lambda),(\bar\eta_{\mathfrak{g}^*}(\mu),\bar\zeta,\bar\lambda)) = \langle\lambda,\bar\zeta\rangle - \langle\bar\lambda,\zeta\rangle - \langle\mu,[\eta,\bar\eta]\rangle \tag{4.6}$$

and reduced Hamilton's equations for a right invariant Hamiltonian H are

$$\frac{d\zeta}{dt} = \frac{\delta H}{\delta\lambda}, \qquad \frac{d\lambda}{dt} = -\frac{\delta H}{\delta\zeta}, \qquad \frac{d\mu}{dt} = \mathrm{ad}^*_{\frac{\delta H}{\delta\mu}}\,\mu.$$

Remark 4.3. The reduced space $\mathcal{O}_\mu \times \mathfrak{g}_1 \times \mathfrak{g}_3^*$ is a symplectic leaf [50] for the Poisson manifold $\mathfrak{g}_1 Ⓢ (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ of Proposition 4.2 as well as for $\mathfrak{g}_1 \times \mathfrak{g}_2^* \times \mathfrak{g}_3^*$ with the direct product Poisson structure described in Remark 4.2 above. Here, we are employing the notations presented in Remark 2.1.

The symplectic two-form $\Omega^{/G}_{T^*TG}$ given in Eq. (4.6) on $\mathcal{O}_\mu \times \mathfrak{g}_1 \times \mathfrak{g}_3^*$ is in a direct product form. Hence a reduction is possible by the additive action of $\mathfrak{g}_1$ to the second factor in $\mathcal{O}_\mu \times \mathfrak{g}_1 \times \mathfrak{g}_3^*$.

Proposition 4.4. *The momentum map of additive action of $\mathfrak{g}_1$ on the symplectic manifold $(\mathcal{O}_\mu \times \mathfrak{g}_1 \times \mathfrak{g}_3^*, \Omega^{/G}_{T^*TG})$ is*

$$\mathbf{J}^{\mathfrak{g}}_{\mathcal{O}_\mu\times\mathfrak{g}_1\times\mathfrak{g}_3^*}\colon \mathcal{O}_\mu \times \mathfrak{g}_1 \times \mathfrak{g}_3^* \to \mathfrak{g}_3^*\colon (\mu,\xi,\nu) \to \nu$$

and the symplectic reduction results in the orbit $\mathcal{O}_\mu$ with a Kostant–Kirillov–Souriau two-form (3.9).

4.2. Reduction of T^*TG by $\mathfrak{g}$. The vector space structure of $\mathfrak{g}$ makes it an Abelian group, and according to the immersion in Eq. (2.9), $\mathfrak{g}$ is an Abelian subgroup of T^*TG. It acts on the total space T^*TG by

$$((g,\xi,\mu,\nu);\eta) \to (g,\xi + Ad_{g^{-1}}\eta,\mu,\nu).$$

Since the action of $GⓈ\mathfrak{g}$ on its cotangent bundle T^*TG is symplectic, the subgroup $\mathfrak{g}$ of $GⓈ\mathfrak{g}$ also acts on T^*TG symplectically. The following results describe Poisson and symplectic reductions of T^*TG by $\mathfrak{g}$ assuming that functions $K = K(g,\mu,\nu)$ and $H = H(g,\mu,\nu)$ defined on $GⓈ(\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ are right invariant under the above action of $\mathfrak{g}$.

Proposition 4.5. *Poisson reduction of T^*TG by the Abelian subgroup $\mathfrak{g}$ gives the Poisson manifold $GⓈ(\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ endowed with the Poisson bracket*

$$\begin{aligned}\{H,K\}_{GⓈ(\mathfrak{g}_2^*\times\mathfrak{g}_3^*)}(g,\mu,\nu) = &-\left\langle T_e^*R_g\frac{\delta H}{\delta g},\frac{\delta K}{\delta\mu}\right\rangle + \left\langle T_e^*R_g\frac{\delta K}{\delta g},\frac{\delta H}{\delta\mu}\right\rangle \\ &+ \left\langle\mu,\left[\frac{\delta H}{\delta\mu},\frac{\delta K}{\delta\mu}\right]\right\rangle + \left\langle\nu,\left[\frac{\delta H}{\delta\mu},\frac{\delta K}{\delta\nu}\right] - \left[\frac{\delta K}{\delta\mu},\frac{\delta H}{\delta\nu}\right]\right\rangle.\end{aligned} \tag{4.7}$$

Remark 4.4. Recall that $G \times \mathfrak{g}_2^*$ is canonically symplectic with the Poisson bracket in Eq. (3.5) and the immersion $G \times \mathfrak{g}_2^* \to G \circledS (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ is a Poisson map. On the other hand, $\mathfrak{g}_3^*$ is naturally Lie–Poisson and $\mathfrak{g}_3^* \to \mathfrak{g}_1 \circledS (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ is also a Poisson map. The Poisson bracket in Eq. (4.7) is, however, not a direct product of these structures. Here, we are employing the notations presented in Remark 2.1.

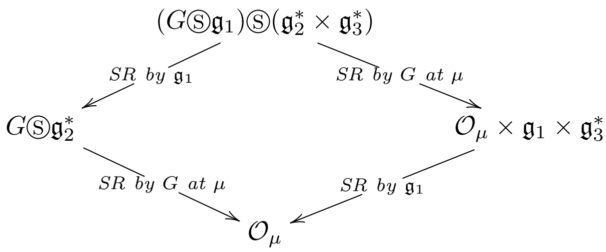
Proposition 4.6. *The Marsden–Weinstein symplectic reduction by the action of $\mathfrak{g}$ on T^*TG with the momentum mapping*

$$\mathbf{J}^{\mathfrak{g}}_{T^*TG} : T^*TG \to \mathfrak{g}_3^* : (g, \xi, \mu, \nu) \to \nu$$

results in the reduced symplectic space $(\mathbf{J}^{\mathfrak{g}}_{T^*TG})^{-1}/\mathfrak{g}$ isomorphic to $G \circledS \mathfrak{g}_2^*$ and with the canonical symplectic two-form $\Omega_{G \circledS \mathfrak{g}_2^*}$ in Eq. (2.6).

It follows that the immersion $G \circledS \mathfrak{g}_2^* \to G \circledS (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$ defines symplectic leaves of the Poisson manifold $G \circledS (\mathfrak{g}_2^* \times \mathfrak{g}_3^*)$. The symplectic reduction of $G \circledS \mathfrak{g}_2^*$ under the action of G results in the total space $\mathcal{O}_\mu$ with the Kostant–Kirillov–Souriau two-form (3.9). We arrive at the following proposition.

Proposition 4.7. *Reductions by actions of $\mathfrak{g}$ and G make the following diagram commutative*



Symplectic Reductions of T^*TG.

Note that the symplectic reduction of T^*TG by the total action of the group $G \circledS \mathfrak{g}_1$ does not result in $\mathcal{O}_\mu$ as reduced space. This is a matter of Hamiltonian reduction by stages theorem [35]. In the following subsection, we will discuss the reduction of T^*TG under the action of $G \circledS \mathfrak{g}_1$ as well as the implications of the Hamiltonian reduction by stages theorem for this case.

4.3. Reduction of T^*TG by $G \circledS \mathfrak{g}$. The Lie algebra of $G \circledS \mathfrak{g}$ is the space $\mathfrak{g} \circledS \mathfrak{g}$ endowed with the semidirect product Lie algebra bracket

$$[(\xi_1, \xi_2), (\eta_1, \eta_2)]_{\mathfrak{g} \circledS \mathfrak{g}} = ([\xi_1, \eta_1], [\xi_1, \eta_2] - [\eta_1, \xi_2])$$

for (ξ_1, ξ_2) and (η_1, η_2) in $\mathfrak{g} \circledS \mathfrak{g}$. Accordingly, the dual space $\mathfrak{g}_2^* \times \mathfrak{g}_3^*$ has the Lie–Poisson bracket

$$\{F, E\}_{\mathfrak{g}_2^* \times \mathfrak{g}_3^*}(\mu, \nu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta E}{\delta \mu}\right] \right\rangle + \left\langle \nu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta E}{\delta \nu}\right] - \left[\frac{\delta E}{\delta \mu}, \frac{\delta F}{\delta \nu}\right] \right\rangle \tag{4.8}$$

for two functionals F and E on $\mathfrak{g}_2^* \times \mathfrak{g}_3^*$.

Proposition 4.8. *The Lie–Poisson structure on* $\mathfrak{g}_2^*\times\mathfrak{g}_3^*$ *is given by the bracket in Eq. (4.8) and the Lie–Poisson equations for a function* $H(\mu,\nu)$ *on* $\mathfrak{g}_2^*\times\mathfrak{g}_3^*$ *read*

$$\frac{d\mu}{dt}=\operatorname{ad}^*_{\frac{\delta H}{\delta\mu}}\mu+\operatorname{ad}^*_{\frac{\delta H}{\delta\nu}}\nu,\qquad \frac{d\nu}{dt}=\operatorname{ad}^*_{\frac{\delta H}{\delta\mu}}\nu. \tag{4.9}$$

Alternatively, the Lie–Poisson equations (4.9) can be obtained by Poisson reduction of T^*TG with the action of $G\circledS\mathfrak{g}$ given by

$$((g,\xi,\mu,\nu);(\tilde g,\eta))\mapsto(g\tilde g,\xi+\operatorname{Ad}_{g^{-1}}\eta,\mu,\nu) \tag{4.10}$$

and restricting the Hamiltonian function H to the fiber variables (μ,ν). In this case, the Lie–Poisson dynamics of g and ξ remains the same but the dynamics governing μ and ν has the reduced form given by Eq. (4.9). This is a manifestation of the fact that the projections to the last two factors in the trivialization (2.7) are a momentum map under the left Hamiltonian action of the group $G\circledS\mathfrak{g}_1$ to its trivialized cotangent bundle T^*TG. Yet another way is to reduce the bracket (4.5) on $\mathfrak{g}_1\circledS(\mathfrak{g}_2^*\times\mathfrak{g}_3^*)$ by assuming that functionals depend on elements of the dual spaces. That is, to consider the Abelian group action of $\mathfrak{g}_1$ on $\mathfrak{g}_1\circledS(\mathfrak{g}_2^*\times\mathfrak{g}_3^*)$ given by

$$((\xi,\mu,\nu);\eta)\mapsto(\xi+\eta,\mu,\nu)$$

and then apply Poisson reduction. Note finally that the immersion $\mathfrak{g}_2^*\times\mathfrak{g}_3^*\to\mathfrak{g}_1\circledS(\mathfrak{g}_2^*\times\mathfrak{g}_3^*)$ is a Poisson map. Here, we are employing the notations presented in Remark 2.1.

Application of the Marsden–Weinstein reduction to the symplectic manifold T^*TG results in the symplectic leaves of the Poisson structure on $\mathfrak{g}_2^*\times\mathfrak{g}_3^*$. The action in Eq. (4.10) has the momentum mapping

$$\mathbf{J}^{G\circledS\mathfrak{g}}_{T^*TG}\colon T^*TG\to\mathfrak{g}_2^*\times\mathfrak{g}_3^*\colon(g,\xi,\mu,\nu)\to(\mu,\nu).$$

The pre-image $(\mathbf{J}^{G\circledS\mathfrak{g}}_{T^*TG})^{-1}(\mu,\nu)$ of an element $(\mu,\nu)\in\mathfrak{g}_2^*\times\mathfrak{g}_3^*$ is diffeomorphic to $G\circledS\mathfrak{g}$. The isotropy group $(G\circledS\mathfrak{g})_{(\mu,\nu)}$ of (μ,ν) consists of pairs (g,ξ) in $G\circledS\mathfrak{g}$ satisfying

$$\operatorname{Ad}^*_{(g,\xi)}(\mu,\nu)=(\operatorname{Ad}^*_g(\mu+\operatorname{ad}^*_\xi\nu),\operatorname{Ad}^*_g\nu)=(\mu,\nu), \tag{4.11}$$

which means that $g\in G_\nu\cap G_\mu$ and the representation of $\operatorname{Ad}_g\xi$ on ν is null, that is, $\operatorname{ad}^*_{\operatorname{Ad}_g\xi}\nu=0$. From the general theory, we deduce that the quotient space

$$(\mathbf{J}^{G\circledS\mathfrak{g}}_{T^*TG})^{-1}(\mu,\nu)/(G\circledS\mathfrak{g})_{(\mu,\nu)}\simeq\mathcal{O}_{(\mu,\nu)}$$

is diffeomorphic to the coadjoint orbit $\mathcal{O}_{(\mu,\nu)}$ in $\mathfrak{g}_2^*\times\mathfrak{g}_3^*$ through the point (μ,ν) under the action $\operatorname{Ad}^*_{(g,\xi)}$ in Eq. (4.11), that is,

$$\mathcal{O}_{(\mu,\nu)}=\left\{(\mu,\nu)\in\mathfrak{g}_2^*\times\mathfrak{g}_3^*:\operatorname{Ad}^*_{(g,\xi)}(\mu,\nu)=(\mu,\nu)\right\}. \tag{4.12}$$

Proposition 4.9. *The symplectic reduction of* T^*TG *results in the coadjoint orbit* $\mathcal{O}_{(\mu,\nu)}$ *in* $\mathfrak{g}_2^*\times\mathfrak{g}_3^*$ *through the point* (μ,ν)*. The reduced symplectic two-form* $\Omega^{G\circledS\mathfrak{g}\backslash}_{T^*TG}$ *(denoted simply by* $\Omega_{\mathcal{O}_{(\mu,\nu)}}$*) takes the value*

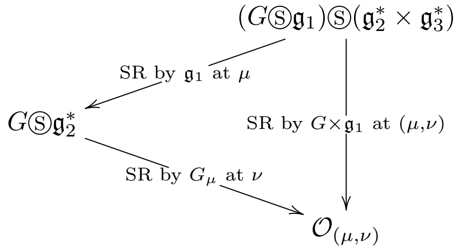
$$\langle\Omega_{\mathcal{O}_{(\mu,\nu)}};(\eta,\zeta),(\bar\eta,\bar\zeta)\rangle(\mu,\nu)=\langle\mu,[\bar\eta,\eta]\rangle+\langle\nu,[\bar\eta,\zeta]-[\eta,\bar\zeta]\rangle \tag{4.13}$$

on two vectors (η,ζ) *and* $(\bar\eta,\bar\zeta)$ *in* $T_{(\mu,\nu)}\mathcal{O}_{(\mu,\nu)}$.

This reduction can also be achieved by stages as described in [28,35]. That is, dynamics is first trivialized by the action of Lie algebra $\mathfrak{g}$ on T^*TG, which results in the Poisson structure on the product $G\circledS(\mathfrak{g}_2^*\times\mathfrak{g}_3^*)$ given by Eq. (4.7). Symplectic leaves of this Poisson structure are spaces diffeomorphic to $G\circledS\mathfrak{g}_2^*$ with a symplectic two-form given in Eq. (2.6). The isotropy group G_μ of an element $\mu\in\mathfrak{g}^*$ acts on $G\circledS\mathfrak{g}_2^*$ with a the same way as assigned in Eq. (3.6), that is,

$$(G\circledS\mathfrak{g}_2^*)\times G_\mu\to G\circledS\mathfrak{g}_2^*\colon ((h,\lambda);g)\to(hg,\lambda).$$

Then, the Hamiltonian reduction by stages theorem states that the symplectic reduction of $G\circledS\mathfrak{g}_2^*$ under the action of G_μ will result in $\mathcal{O}_{(\mu,\nu)}$ as the reduced space endowed with the symplectic two-form $\Omega_{\mathcal{O}_{(\mu,\nu)}}$ in Eq. (4.13). The following diagram summarizes the Hamiltonian reduction by stages theorem for the case of T^*TG under consideration



Hamiltonian reduction by stages for T^*TG.

There exists a momentum mapping $\mathbf{J}^{G_\mu}_{G\circledS\mathfrak{g}_2^*}$ from $G\circledS\mathfrak{g}_2^*$ to the dual space $\mathfrak{g}_\mu^*$ of the isotropy subalgebra $\mathfrak{g}_\mu$ of G_μ. The isotropy subgroup $G_{\mu,\nu}$ of the coadjoint action is

$$G_{\mu,\nu}=\{g\in G_\mu:\operatorname{Ad}^*_{g^{-1}}\nu=\nu\}.$$

The quotient symplectic space

$$(\mathbf{J}^{G_\mu}_{G\circledS\mathfrak{g}_2^*})^{-1}(\nu)/G_{\mu,\nu}\simeq\mathcal{O}_{(\mu,\nu)}$$

is diffeomorphic to the coadjoint orbit $\mathcal{O}_{(\mu,\nu)}$ defined in (4.12).

It is also possible to establish the Poisson reduction of the symplectic manifold $G\circledS\mathfrak{g}_2^*$ under the action of the isotropy group G_μ. This results in

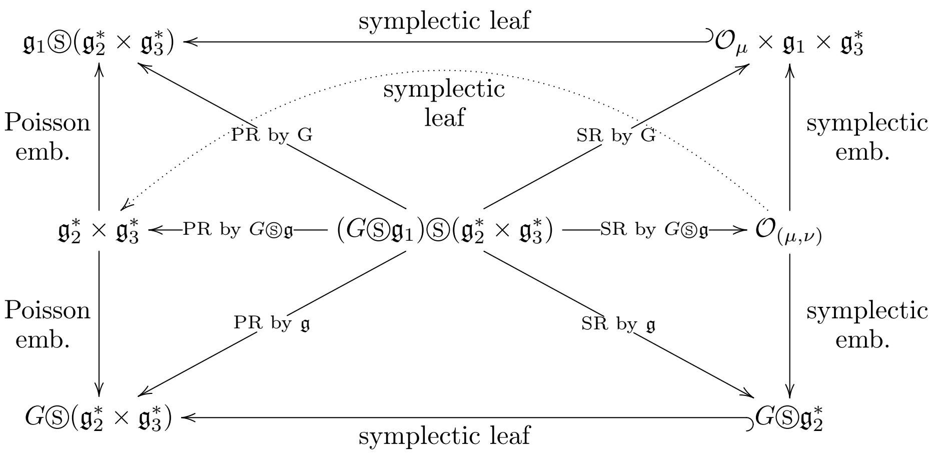
$$G_\mu\backslash(G\circledS\mathfrak{g}_2^*)\simeq\mathcal{O}_\mu\times\mathfrak{g}_2^*$$

with the Poisson bracket

$$\{H,K\}_{\mathcal{O}_\mu\times\mathfrak{g}_2^*}(\mu,\nu)=\Big\langle\mu,\Big[\frac{\delta H}{\delta\mu},\frac{\delta K}{\delta\mu}\Big]\Big\rangle+\Big\langle\nu,\Big[\frac{\delta H}{\delta\mu},\frac{\delta K}{\delta\nu}\Big]-\Big[\frac{\delta K}{\delta\mu},\frac{\delta H}{\delta\nu}\Big]\Big\rangle.$$

We note again that the Poisson structure on $\mathcal{O}_\mu\times\mathfrak{g}^*$ is not a direct product of the Lie–Poisson structures on $\mathcal{O}_\mu$ and $\mathfrak{g}_2^*$. The following diagram illustrates various reductions of T^*TG under the actions of G, $\mathfrak{g}$ and $G\circledS\mathfrak{g}$. Diagram (3.11), describing reductions of $G\circledS\mathfrak{g}^*$, can be attached to the lower right corner of this diagram to have a complete picture of reductions. In the light of the notation in Remark 2.1 we have the following.

(4.14)



Reductions of T^*TG

5. Hamiltonian dynamics on the cotangent of the cotangent bundle

PROPOSITION 5.1. *A Hamiltonian function $H = H(g, \mu, \nu, \xi)$ on the iterated cotangent bundle T^*T^*G determines Hamilton's equations*

$$i_{X_H^{T^*T^*G}} \Omega_{T^*T^*G} = -dH$$

by uniquely defining the Hamiltonian vector field $X_H^{T^*T^*G}$. The Hamiltonian vector field is a right invariant vector field generated by a 4-tuple Lie algebra element

$$\Big(\frac{\delta H}{\delta \nu}, \frac{\delta H}{\delta \xi}, \mathrm{ad}^*_{\frac{\delta H}{\delta \mu}} \mu - T_e^* R_g \Big(\frac{\delta H}{\delta g}\Big), -\frac{\delta H}{\delta \mu}\Big)$$

in $(\mathfrak{g} \circledS \mathfrak{g}^*) \circledS (\mathfrak{g}^* \times \mathfrak{g})$. At the point (g, μ, ν, ξ), Hamilton's equations are

$$\frac{dg}{dt} = T_e R_g \Big(\frac{\delta H}{\delta \nu}\Big), \tag{5.1}$$

$$\frac{d\mu}{dt} = \frac{\delta H}{\delta \xi} + \mathrm{ad}^*_{\frac{\delta H}{\delta \nu}} \mu$$

$$\frac{d\nu}{dt} = \mathrm{ad}^*_{\frac{\delta H}{\delta \mu}} \mu + \mathrm{ad}^*_{\frac{\delta H}{\delta \nu}} \nu - T_e^* R_g \Big(\frac{\delta H}{\delta g}\Big) - \mathrm{ad}^*_\xi \frac{\delta H}{\delta \xi}$$

$$\frac{d\xi}{dt} = -\frac{\delta H}{\delta \mu} - [\frac{\delta H}{\delta \nu}, \xi]. \tag{5.2}$$

5.1. Reduction of T^*T^*G by G.

It follows from Eq. (2.14) that the right action of G on T^*T^*G is

$$((g, \mu, \nu, \xi); \tilde{g}) \to (g\tilde{g}, \mu, \nu, \xi)$$

with the infinitesimal generator $X_{(\eta,0,0,0)}^{T^*T^*G}$ being a right invariant vector field as in Eq. (2.16) generated by $(\eta, 0, 0, 0)$ for $\eta \in \mathfrak{g}$.

PROPOSITION 5.2. *Poisson reduction of T^*T^*G under the action of G results in $\mathfrak{g}_1^*Ⓢ(\mathfrak{g}_2^* \times \mathfrak{g}_3)$ endowed with the Poisson bracket*

$$\{H,K\}_{\mathfrak{g}_1^*Ⓢ(\mathfrak{g}_2^*\times\mathfrak{g}_3)}(\mu,\nu,\xi) = -\Big\langle \frac{\delta H}{\delta\mu}, \frac{\delta K}{\delta\xi}\Big\rangle + \Big\langle \frac{\delta K}{\delta\mu}, \frac{\delta H}{\delta\xi}\Big\rangle + \Big\langle \nu, \Big[\frac{\delta H}{\delta\nu}, \frac{\delta K}{\delta\nu}\Big]\Big\rangle + \Big\langle \xi, -\operatorname{ad}^*_{\frac{\delta H}{\delta\nu}} \frac{\delta K}{\delta\xi} + \operatorname{ad}^*_{\frac{\delta K}{\delta\nu}} \frac{\delta H}{\delta\xi}\Big\rangle + \Big\langle \mu, \Big[\frac{\delta H}{\delta\mu}, \frac{\delta K}{\delta\nu}\Big] - \Big[\frac{\delta K}{\delta\mu}, \frac{\delta H}{\delta\nu}\Big]\Big\rangle,$$

and symplectic reduction gives $\mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^*$ with the symplectic two-form defined by

$$\Omega^{/G}_{T^*T^*G}((\eta_{\mathfrak{g}^*}(\mu),\lambda,\zeta),(\bar\eta_{\mathfrak{g}^*}(\mu),\bar\lambda,\bar\zeta)) = \langle\zeta,\bar\lambda\rangle - \langle\bar\zeta,\lambda\rangle - \langle\mu,[\eta,\bar\eta]\rangle$$

on two elements $(\eta_{\mathfrak{g}^*}(\mu),\lambda,\zeta)$ and $(\bar\eta_{\mathfrak{g}^*}(\mu),\bar\lambda,\bar\zeta)$ of $T_\mu\mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^*$.

Recall that, in the previous section, the Poisson and symplectic reductions of T^*TG result in reduced spaces $\mathfrak{g}Ⓢ(\mathfrak{g}^* \times \mathfrak{g}^*)$ and $\mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^*$, respectively. The reduced Poisson bracket on $\mathfrak{g}Ⓢ(\mathfrak{g}^* \times \mathfrak{g}^*)$ is given by Eq. (4.5) and the reduced symplectic two-form $\Omega^{/G}_{T^*TG}$ on $\mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}$ is in Eq. (4.6). We have the following proposition from [19] relating the reductions of cotangent bundles T^*T^*G and T^*TG. We refer to [32] for a detailed study on the canonical maps between semidirect products.

5.2. Reduction of T^*T^*G by $\mathfrak{g}^*$. The action of $\mathfrak{g}^*$ on T^*T^*G, given by

$$((g,\mu,\nu,\xi);\lambda) \to (g,\mu + \operatorname{Ad}^*_g\lambda,\nu,\xi). \tag{5.3}$$

is generated by $X^{T^*T^*G}_{(0,\lambda,0,0)} = (0,\lambda,-\operatorname{ad}^*_\xi\lambda,0)$, as the action of $GⓈ\mathfrak{g}^*$ on its cotangent bundle T^*T^*G is symplectic, and $\mathfrak{g}^*$ is a subgroup. The action in Eq. (5.3) is symplectic, hence we can perform Poisson and symplectic reductions of T^*T^*G.

PROPOSITION 5.3. *The Poisson reduction of T^*T^*G with the action of $\mathfrak{g}^*$ results in $GⓈ(\mathfrak{g}_2^* \times \mathfrak{g}_3)$ endowed with the bracket*

$$\{H,K\}_{GⓈ(\mathfrak{g}_2^*\times\mathfrak{g}_3)}(g,\nu,\xi) = -\Big\langle T_e^*R_g\frac{\delta H}{\delta g}, \frac{\delta K}{\delta\nu}\Big\rangle + \Big\langle T_e^*R_g\frac{\delta K}{\delta g}, \frac{\delta H}{\delta\nu}\Big\rangle + \Big\langle \xi, -\operatorname{ad}^*_{\frac{\delta H}{\delta\nu}} \frac{\delta K}{\delta\xi} + \operatorname{ad}^*_{\frac{\delta K}{\delta\nu}} \frac{\delta H}{\delta\xi}\Big\rangle + \Big\langle \nu, \Big[\frac{\delta H}{\delta\nu}, \frac{\delta K}{\delta\nu}\Big]\Big\rangle. \tag{5.4}$$

The application of the Marsden–Weinstein symplectic reduction with the action of $\mathfrak{g}^*$ on T^*T^*G having the momentum mapping

$$\mathbf{J}^{\mathfrak{g}^*}_{T^*T^*G}\colon T^*T^*G \to \mathfrak{g}_2\colon (g,\mu,\nu,\xi) \to \xi$$

results in the reduced symplectic space $(\mathbf{J}^{\mathfrak{g}^*}_{T^*T^*G})^{-1}(\xi)/\mathfrak{g}^*$ isomorphic to $GⓈ\mathfrak{g}_3^*$ with the canonical symplectic two-form $\Omega_{GⓈ\mathfrak{g}_2^*}$ in Eq. (2.6).

5.3. Reduction of T^*T^*G by $GⓈ\mathfrak{g}^*$. The Lie algebra of the group $GⓈ\mathfrak{g}_1^*$ is the space $\mathfrak{g}Ⓢ\mathfrak{g}^*$ carrying the bracket

$$[(\xi,\mu),(\eta,\nu)]_{\mathfrak{g}Ⓢ\mathfrak{g}^*} = ([\xi,\eta],-\operatorname{ad}^*_\xi\nu + \operatorname{ad}^*_\eta\mu). \tag{5.5}$$

The dual space $\mathfrak{g}_2^* \times \mathfrak{g}_3$ carries the Lie–Poisson bracket

$$\{F,E\}_{\mathfrak{g}_2^*\times\mathfrak{g}_3}(\nu,\xi) = \Big\langle \nu, \Big[\frac{\delta F}{\delta\nu}, \frac{\delta E}{\delta\nu}\Big]\Big\rangle + \Big\langle \xi, \operatorname{ad}^*_{\frac{\delta E}{\delta\nu}} \frac{\delta F}{\delta\xi} - \operatorname{ad}^*_{\frac{\delta F}{\delta\nu}} \frac{\delta E}{\delta\xi}\Big\rangle, \tag{5.6}$$

which follows from the Lie algebra bracket in Eq. (5.5).

PROPOSITION 5.4. *The Lie–Poisson bracket, in Eq.* (5.6)*, on* $\mathfrak{g}_2^* \times \mathfrak{g}_3$ *defines the Hamiltonian vector field* $X_E^{\mathfrak{g}^*\times\mathfrak{g}}$ *by*

$$\{F, E\}_{\mathfrak{g}_2^*\times\mathfrak{g}_3} = -\left\langle dF, X_E^{\mathfrak{g}_2^*\times\mathfrak{g}_3}\right\rangle,$$

whose components are the Lie–Poisson equations

$$\frac{d\nu}{dt} = \operatorname{ad}^*_{\frac{\delta H}{\delta \nu}} \nu - \operatorname{ad}^*_\xi \frac{\delta H}{\delta \xi}, \qquad \frac{d\xi}{dt} = \left[\frac{\delta H}{\delta \nu}, \xi\right]. \tag{5.7}$$

Although these equations result from Eq. (5.6), it is possible to obtain them starting from Hamilton's equations (5.1)–(5.2) on T^*T^*G and applying a Poisson reduction with the action of $G \circledS \mathfrak{g}^*$ given by

$$\begin{aligned} T^*T^*G \times (G \circledS \mathfrak{g}^*) \to T^*T^*G &\colon ((h, \nu, \lambda, \xi); (g, \mu)) \\ &\mapsto (hg, \nu + \operatorname{Ad}^*_h \mu, \lambda, \xi). \end{aligned} \tag{5.8}$$

In other words, choosing the Hamiltonian function H in Eqs. (5.1)–(5.2) depending on fiber variables only, that is, $H = H(\nu, \xi)$, Eq. (5.7) follows.

To reduce Hamilton's equations (5.1)–(5.2) on T^*T^*G symplectically, we first compute the momentum mapping

$$\mathbf{J}^{G_\xi}_{G \circledS \mathfrak{g}_3^*} \colon T^*T^*G \to \mathfrak{g}^* \times \mathfrak{g} \colon (g, \mu, \nu, \xi) \to (\nu, \xi),$$

associated with the action of $G \circledS \mathfrak{g}^*$ in Eq. (5.8) and the quotient space

$$(\mathbf{J}^{G_\xi}_{G \circledS \mathfrak{g}_3^*})^{-1}(\nu, \xi)/G_{(\nu,\xi)} \simeq \mathcal{O}_{(\nu,\xi)}. \tag{5.9}$$

Here, $G_{(\nu,\xi)}$ is the isotropy subgroup of $G \circledS \mathfrak{g}^*$ consisting of elements preserved under the coadjoint action $G \circledS \mathfrak{g}^*$ on the dual space $\mathfrak{g}^* \times \mathfrak{g}$ of its Lie algebra

$$\begin{aligned} \operatorname{Ad}^* \colon &(G \circledS \mathfrak{g}^*) \times (\mathfrak{g}^* \times \mathfrak{g}) \to \mathfrak{g}^* \times \mathfrak{g} \\ &\colon ((g, \mu), (\nu, \xi)) \to (\operatorname{Ad}^*_g(\nu + \operatorname{ad}^*_\xi \mu), \operatorname{Ad}_g \xi) \end{aligned} \tag{5.10}$$

and the space $\mathcal{O}_{(\nu,\xi)}$ is the coadjoint orbit passing through the point (ν, ξ) under this coadjoint action.

PROPOSITION 5.5. *The symplectic reduction of* T^*T^*G *results in the coadjoint orbit* $\mathcal{O}_{(\nu,\xi)}$ *in* $\mathfrak{g}_2^* \times \mathfrak{g}$ *through the point* (ν, ξ)*. The reduced symplectic two-form* $\Omega^{/(G \circledS \mathfrak{g}^*)}_{T^*T^*G}$ *(denoted simply by* $\Omega_{\mathcal{O}_{(\nu,\xi)}}$*) takes the value*

$$\langle \Omega_{\mathcal{O}_{(\nu,\xi)}}; (\lambda, \eta), (\bar{\lambda}, \bar{\eta})\rangle(\nu, \xi) = \langle \nu, [\bar{\eta}, \eta]\rangle + \langle \xi, \operatorname{ad}^*_\eta \bar{\lambda} - \operatorname{ad}^*_{\bar{\eta}} \lambda]\rangle \tag{5.11}$$

on two vectors (λ, η) *and* $(\bar{\lambda}, \bar{\eta})$ *in* $T_{(\nu,\xi)}\mathcal{O}_{(\nu,\xi)}$*.*

Alternatively, this reduction can be performed in two steps by applying the Hamiltonian reduction by stages theorem [35]. The first step consists of the symplectic reduction of T^*T^*G with the action of $\mathfrak{g}^*$ which has already been established in the previous subsection and resulted in the reduced symplectic space $(\mathbf{J}^{\mathfrak{g}^*}_{T^*T^*G})^{-1}(\xi)/\mathfrak{g}^*$, isomorphic to $G \circledS \mathfrak{g}_3^*$, with the canonical symplectic two-form

$\Omega_{GⓈ\mathfrak{g}_3^*}$ in Eq. (2.6). For the second step, we recall the adjoint group action $\mathrm{Ad}_{g^{-1}}$ of G on $\mathfrak{g}$ and define the isotropy subgroup

$$G_\xi = \{g \in G : \mathrm{Ad}_{g^{-1}} \xi = \xi\} \tag{5.12}$$

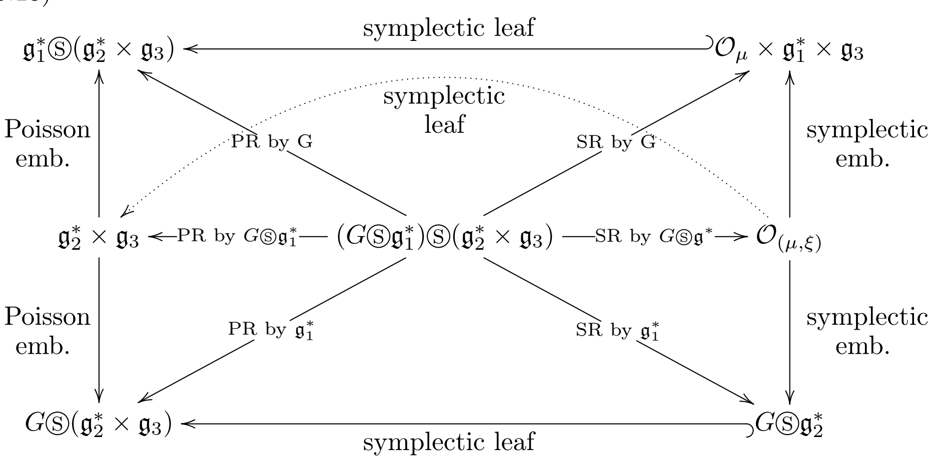
for an element
$xi \in \mathfrak{g}$ under the adjoint action. The Lie algebra $\mathfrak{g}_\xi$ of G_ξ consists of vectors $\eta \in \mathfrak{g}$ satisfying $[\eta, \xi] = 0$. The isotropy subgroup G_ξ acts on $GⓈ\mathfrak{g}_3^*$ in the same way as described in Eq. (3.6). This action is Hamiltonian and has the momentum mapping

$$\mathbf{J}^{G_\xi}_{GⓈ\mathfrak{g}_3^*} : GⓈ\mathfrak{g}_3^* \to \mathfrak{g}_\xi^*,$$

where $\mathfrak{g}_\xi^*$ is the dual space of $\mathfrak{g}_\xi$. The quotient space

$$(\mathbf{J}^{G_\xi}_{GⓈ\mathfrak{g}_3^*})^{-1}(\nu)/G_{\xi,\nu} \simeq \mathcal{O}_{(\nu,\xi)}$$

is diffeomorphic to the coadjoint orbit $\mathcal{O}_{(\nu,\xi)}$ in Eq. (5.9). Referring to the notations presented in Remark 2.5 we draw the following graph.
(5.13)



Reduction of $T^*T^*G = (GⓈ\mathfrak{g}_1^*)Ⓢ(\mathfrak{g}_2^* \times \mathfrak{g}_3)$

6. Hamiltonian and Lagrangian dynamics on Tulczyjew symplectic space

6.1. Hamiltonian dynamics on TT^*G.

Proposition 6.1. *Given a Hamiltonian function $E = E(g, \mu, \xi, \nu)$ on TT^*G, Hamilton's equation*

$$i_{X_E^{TT^*G}} \Omega_{TT^*G} = -dE$$

defines a Hamiltonian right invariant vector field $X_E^{TT^*G}$ generated by the element

$$\Big(\frac{\delta E}{\delta \nu}, -\Big(\frac{\delta E}{\delta \xi} + \mathrm{ad}^*_{\frac{\delta E}{\delta \nu}} \mu\Big), \frac{\delta E}{\delta \mu} - \mathrm{ad}_\xi \frac{\delta E}{\delta \nu}, -\Big(T^* R_g \frac{\delta E}{\delta g} + \mathrm{ad}^*_\xi \frac{\delta E}{\delta \xi} + \mathrm{ad}^*_\xi \mathrm{ad}^*_{\frac{\delta E}{\delta \nu}} \mu\Big)\Big)$$

of the Lie algebra $(\mathfrak{g}Ⓢ\mathfrak{g}^*)Ⓢ(\mathfrak{g}Ⓢ\mathfrak{g}^*)$. *Components of* $X_E^{TT^*G}$ *define Hamilton's equations*

$$\dot{g} = TR_g\Big(\frac{\delta E}{\delta \nu}\Big), \quad \dot{\mu} = -\frac{\delta E}{\delta \xi}, \quad \dot{\xi} = \frac{\delta E}{\delta \mu}, \quad \dot{\nu} = \operatorname{ad}^*_{\frac{\delta E}{\delta \nu}} \nu - T^*R_g\Big(\frac{\delta E}{\delta g}\Big) \tag{6.1}$$

in the adapted trivialization of TT^*G.

6.1.1. *Reduction of* TT^*G *by* G.

PROPOSITION 6.2. *The Poisson reduction of* TT^*G *under the action of* G *results in the total space* $\mathfrak{g}_1^*Ⓢ(\mathfrak{g}_2Ⓢ\mathfrak{g}_3^*)$ *endowed with the Poisson bracket*

$$\{E, F\}_{\mathfrak{g}_1^*Ⓢ(\mathfrak{g}_2Ⓢ\mathfrak{g}_3^*)}(\mu, \xi, \nu) = \Big\langle \frac{\delta F}{\delta \xi}, \frac{\delta E}{\delta \mu} \Big\rangle - \Big\langle \frac{\delta E}{\delta \xi}, \frac{\delta F}{\delta \mu} \Big\rangle + \Big\langle \nu, \Big[\frac{\delta E}{\delta \nu}, \frac{\delta F}{\delta \nu}\Big] \Big\rangle.$$

REMARK 6.1. Here, the Poisson bracket on $\mathfrak{g}_1^*Ⓢ(\mathfrak{g}_2Ⓢ\mathfrak{g}_3^*)$ is the direct product of the canonical Poisson bracket on $\mathfrak{g}_1^* \times \mathfrak{g}_2$ and Lie–Poisson bracket on $\mathfrak{g}_3^*$ whereas in Eq. (4.5) we obtained a Poisson bracket, on the isomorphic space $\mathfrak{g}Ⓢ(\mathfrak{g}^* \times \mathfrak{g}^*)$, which is not in the form of a direct product. Here, we are employing the notations presented in Remark 2.5.

The action of G is Hamiltonian with the momentum mapping

$$\mathbf{J}^G_{TT^*G} \colon TT^*G \to \mathfrak{g}^* \colon (g, \mu, \xi, \nu) \to \nu + \operatorname{ad}^*_\xi \mu. \tag{6.2}$$

The quotient space of the preimage $\mathbf{J}^{-1}_{TT^*G}(\lambda)$ of an element $\lambda \in \mathfrak{g}^*$ under the action of the isotropy subgroup G_λ is

$$\mathbf{J}^{-1}_{TT^*G}(\lambda)/G_\lambda \simeq \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}.$$

Pushing forward a right invariant vector field $X^{TT^*G}_{(\eta,\nu,\zeta,\bar{\nu})}$ in the form of Eq. (2.19) by the symplectic projection $TT^*G \to \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}$, we obtain the vector field

$$X^{\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}}_{(\eta,\nu,\zeta)}(\operatorname{Ad}^*_{g^{-1}} \lambda, \mu, \xi) = (\operatorname{ad}^*_\eta \circ \operatorname{Ad}^*_{g^{-1}} \lambda, \nu + \operatorname{ad}^*_\eta \mu, \zeta + [\xi, \eta])$$

on the quotient space $\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}$. We refer to [20] for the proof of the following proposition.

PROPOSITION 6.3. *Reduced Tulczyjew's space* $\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}$ *has an exact symplectic two-form* $\Omega_{\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}}$ *with two potential one-forms* χ_1 *and* χ_2 *whose values on vector fields of the form of Eq.* (2.19) *at the point* $(\operatorname{Ad}^*_{g^{-1}} \lambda, \mu, \xi)$ *are*

$$\langle \Omega_{\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}}, (X^{\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}}_{(\eta,\nu,\zeta)}, X^{\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}}_{(\bar{\eta},\bar{\nu},\bar{\zeta})})\rangle = \langle \nu, \bar{\zeta} \rangle - \langle \bar{\nu}, \zeta \rangle - \langle \lambda, [\eta, \bar{\eta}] \rangle,$$
$$\langle \chi_1, X^{\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}}_{(\eta,\nu,\zeta)} \rangle(\operatorname{Ad}^*_{g^{-1}} \lambda, \mu, \xi) = \langle \lambda, \eta \rangle - \langle \nu, \xi \rangle,$$
$$\langle \chi_2, X^{\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}}_{(\eta,\nu,\zeta)} \rangle(\operatorname{Ad}^*_{g^{-1}} \lambda, \mu, \xi) = \langle \lambda, \eta \rangle + \langle \mu, \zeta \rangle.$$

6.1.2. *Reduction of* TT^*G *by* $\mathfrak{g}$.

PROPOSITION 6.4. *The action of* $\mathfrak{g}_2$ *on* TT^*G *is given, for* $\eta \in \mathfrak{g}_2$, *by*

$$\varphi_\eta \colon TT^*G \to TT^*G \colon ((g, \mu, \xi, \nu); \eta) \to (g, \mu, \xi + \eta, \nu) \tag{6.3}$$

and it is symplectic.

PROOF. Push forward of a vector field $X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ in the form of Eq. (2.19) by the transformation φ_η is also a right invariant vector field

$$(\varphi_\eta)_* X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} = X^{TT^*G}_{(\xi_2,\nu_2,\xi_3-[\eta,\xi_2],\nu_3+\mathrm{ad}^*_\eta \nu_2)}.$$

By direct calculation, one establishes the identity

$$\varphi^*_\eta \Omega_{TT^*G}(X,Y)(g,\mu,\xi,\nu) = \Omega_{TT^*G}((\varphi_\eta)_* X, (\varphi_\eta)_* Y)(g,\mu,\xi+\eta,\nu) \tag{6.4}$$

which gives the desired result. In Eq. (6.4) X and Y are right invariant vector fields as in Eq. (2.19) and Ω_{TT^*G} is the symplectic two-form given in Eq. (2.20). □

PROPOSITION 6.5. *The Poisson reduction of TT^*G under the action in Eq. (6.3) of $\mathfrak{g}_2$ results in $(G \circledS \mathfrak{g}_1^*) \circledS \mathfrak{g}_3^*$ endowed with the bracket at the point (g,μ,ν) as*

$$\{E,F\}_{(G\circledS\mathfrak{g}_1^*)\circledS\mathfrak{g}_3^*} = -\Big\langle T^*_e R_g \frac{\delta E}{\delta g}, \frac{\delta F}{\delta \nu} \Big\rangle + \Big\langle T^*_e R_g \frac{\delta F}{\delta g}, \frac{\delta E}{\delta \nu} \Big\rangle + \Big\langle \nu, \Big[\frac{\delta E}{\delta \nu}, \frac{\delta F}{\delta \nu}\Big] \Big\rangle.$$

REMARK 6.2. The Poisson bracket $\{E,F\}_{(G\circledS\mathfrak{g}_1^*)\circledS\mathfrak{g}_3^*}$ is independent of functions with respect to μ, that is, it does not involve $\delta E/\delta\mu$ and $\delta F/\delta\mu$. Its structure resembles the canonical Poisson bracket in Eq. (3.5) on $G \circledS \mathfrak{g}^*$. We recall that on $(G \circledS \mathfrak{g}^*) \circledS \mathfrak{g}^*$ there is another Poisson bracket given in Eq. (4.7) that involves $\delta E/\delta\mu$, $\delta F/\delta\mu$, $\delta E/\delta\nu$, and $\delta F/\delta\nu$. This latter comes from reduction of T^*TG by $\mathfrak{g}$.

The infinitesimal generator $X^{TT^*G}_{(0,0,\xi_3,0)}$ of the action in Eq. (6.3) corresponds to the element $\xi_3 \in \mathfrak{g}$ and is a right invariant vector field. Since the action is Hamiltonian, and the symplectic two-form is exact, we can derive the associated momentum map $\mathbf{J}^{\mathfrak{g}_2}_{TT^*G}$ from the equation

$$\langle \mathbf{J}^{\mathfrak{g}_2}_{TT^*G}(g,\mu,\xi,\nu), \xi_3 \rangle = \langle \theta_2, X^{TT^*G}_{(0,0,\xi_3,0)} \rangle = \langle \mu, \xi_3 \rangle,$$

where θ_2 is the potential one-form of Tulczyjew in Eq. (2.22) satisfying $d\theta_2 = \Omega_{TT^*G}$. We find that

$$\mathbf{J}^{\mathfrak{g}_2}_{TT^*G} \colon TT^*G \to \mathrm{Lie}^*(\mathfrak{g}_2) = \mathfrak{g}^* \colon (g,\mu,\xi,\nu) \to \mu \tag{6.5}$$

is the projection to the second entry in TT^*G. The preimage of an element $\mu \in \mathfrak{g}^*$ by $\mathbf{J}^{\mathfrak{g}_2}_{TT^*G}$ is the space $G \circledS (\mathfrak{g}_2 \circledS \mathfrak{g}_3^*)$. The following proposition describes the symplectic reduction of TT^*G with the action of $\mathfrak{g}_2$.

PROPOSITION 6.6. *The symplectic reduction of TT^*G under the action of $\mathfrak{g}_2$ given by Eq. (6.3) gives the reduced space*

$$(\mathbf{J}^{\mathfrak{g}_2}_{TT^*G})^{-1}(\mu) / \mathfrak{g}_2 \simeq G \circledS \mathfrak{g}_3^*$$

with the canonical symplectic two-form $\Omega_{G\circledS\mathfrak{g}_3^*}$ as in Eq. (2.6).

REMARK 6.3. Existence of the symplectic action of $\mathfrak{g}_2$ on TT^*G is directly related to the existence of a symplectic diffeomorphism

$$\bar{\sigma}_G \colon TT^*G \to T^*TG \colon (g,\mu,\xi,\nu) \to (g,\xi,\nu + \mathrm{ad}^*_\xi \mu, \mu)$$

in the Tulczyjew triplet described in [19].

6.1.3. *Reduction of TT^*G by $\mathfrak{g}^*$.* Induced from the group operation on TT^*G, there are two canonical actions of $\mathfrak{g}^*$ on TT^*G

$$\psi\colon \mathfrak{g}_1^* \times TT^*G \to TT^*G, \quad \phi\colon \mathfrak{g}_3^* \times TT^*G \to TT^*G$$

described by

$$\psi_\lambda(g,\mu,\xi,\nu) = (g,\mu+\lambda,\xi,\nu), \tag{6.6}$$

$$\phi_\lambda(g,\mu,\xi,\nu) = (g,\mu,\xi,\nu+\lambda). \tag{6.7}$$

PROPOSITION 6.7. *ψ is a symplectic action whereas ϕ is not.*

PROOF. Pushing forward of a vector field $X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ in the form of Eq. (2.19) by transformations ψ_λ and ϕ_λ results in right invariant vector fields

$$(\psi_\lambda)_* X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} = X^{TT^*G}_{(\xi_2,\nu_2-\mathrm{ad}^*_{\xi_2}\lambda,\xi_3,\nu_3-\mathrm{ad}^*_{\xi}\mathrm{ad}^*_{\xi_2}\lambda)},$$

$$(\phi_\lambda)_* X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} = X^{TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3-\mathrm{ad}^*_{\xi_2}\lambda)}.$$

If Ω_{TT^*G} is the symplectic two-form on TT^*G given in Eq. (2.20), direct calculations show that the identity

$$\psi_\lambda{}^*\Omega_{TT^*G}(X,Y)(g,\mu,\xi,\nu) = \Omega_{TT^*G}((\psi_\lambda)_*X,(\psi_\lambda)_*Y)(g,\mu+\lambda,\xi,\nu)$$

holds for all vector fields X and Y, and $\lambda \in \mathfrak{g}^*$, whereas

$$\phi_\lambda{}^*\Omega_{TT^*G}(X,Y)(g,\mu,\xi,\nu) = \Omega_{TT^*G}((\phi_\lambda)_*X,(\phi_\lambda)_*Y)(g,\mu,\xi,\nu+\lambda)$$

does not necessarily hold. Hence, ψ_λ is a symplectic action, but not ϕ_λ. □

PROPOSITION 6.8. *Poisson reduction of TT^*G under the action ψ of $\mathfrak{g}_1^*$ results in $G \circledS (\mathfrak{g}_2 \circledS \mathfrak{g}_3^*)$ endowed with the bracket*

$$\{E,F\}_{G\circledS(\mathfrak{g}_2\times\mathfrak{g}_3^*)}(g,\xi,\nu) = \left\langle T_e^*R_g\frac{\delta F}{\delta g},\frac{\delta E}{\delta\nu}\right\rangle - \left\langle T_e^*R_g\frac{\delta E}{\delta g},\frac{\delta F}{\delta\nu}\right\rangle + \left\langle \nu,\left[\frac{\delta E}{\delta\nu},\frac{\delta F}{\delta\nu}\right]\right\rangle.$$

REMARK 6.4. The Poisson bracket $\{E,F\}_{G\circledS(\mathfrak{g}_2\circledS\mathfrak{g}_3^*)}$ is independent of derivatives of functions with respect to ξ and it resembles the canonical Poisson bracket in Eq. (3.5) on $G\circledS\mathfrak{g}^*$. On the other hand, the space $G\circledS(\mathfrak{g}^*\times\mathfrak{g})$, which is isomorphic to $G\circledS(\mathfrak{g}_2\circledS\mathfrak{g}_3^*)$, has the Poisson bracket in Eq. (5.4) involving derivatives with respect to both ξ and ν. This latter is obtained from T^*TG via reduction by $\mathfrak{g}^*$.

The infinitesimal generator $X^{TT^*G}_{(0,\nu_2,0,0)}$ of the action are defined by $\nu_2 \in \mathrm{Lie}(\mathfrak{g}_1^*)$. We compute the associated momentum map from the equation

$$\langle \mathbf{J}^{\mathfrak{g}_1^*}_{TT^*G}(g,\mu,\xi,\nu),\nu_2\rangle = \langle \theta_1, X^{TT^*G}_{(0,\nu_2,0,0)}\rangle = -\langle \nu_2,\xi\rangle,$$

where θ_1 is the Tulczyjew potential one-form in Eq. (2.21). We find that

$$\mathbf{J}^{\mathfrak{g}_1^*}_{TT^*G}\colon TT^*G \to \mathrm{Lie}^*(\mathfrak{g}_1^*) \simeq \mathfrak{g}\colon (g,\mu,\xi,\nu) \to -\xi \tag{6.8}$$

is minus the projection to the third factor in TT^*G. The preimage of an element $\xi \in \mathfrak{g}$ is the space $G\circledS(\mathfrak{g}_1^*\times\mathfrak{g}_3^*)$.

PROPOSITION 6.9. *The symplectic reduction of TT^*G under the action of $\mathfrak{g}^*$ defined in Eq.* (6.3) *results in the reduced space*

$$(\mathbf{J}^{\mathfrak{g}_1^*}_{TT^*G})^{-1}(\xi)/\mathfrak{g}_1^* \simeq G\circledS(\mathfrak{g}_1^*\times\mathfrak{g}_3^*)/\mathfrak{g}_1^* \simeq G\circledS\mathfrak{g}_3^*$$

with the canonical symplectic two-form $\Omega_{G\circledS\mathfrak{g}_3^*}$ as given in Eq. (2.6).

REMARK 6.5. The existence of the symplectic action of $\mathfrak{g}^*$ on TT^*G can be traced back to the existence of the symplectic diffeomorphism

$$\Omega^\flat_{T^*G}\colon TT^*G\to T^*T^*G\colon (g,\mu,\xi,\nu)\to(g,\mu,\nu+\operatorname{ad}^*_\xi\mu,-\xi)$$

described in [19].

In the following proposition, we discuss the actions ψ and ϕ of $\mathfrak{g}^*$ on TT^*G in Eqs. (6.6) and (6.7) from a different point of view.

PROPOSITION 6.10. *The mappings*

$$\operatorname{Emb}_1\colon G\circledS\mathfrak{g}^*\hookrightarrow TT^*G\colon (g,\mu)\to(g,\mu,0,0)$$
$$\operatorname{Emb}_2\colon G\circledS\mathfrak{g}^*\hookrightarrow TT^*G\colon (g,\nu)\to(g,0,0,\nu)$$

define respectively Lagrangian and symplectic embeddings of $G\circledS\mathfrak{g}^*$ into TT^*G.

PROOF. The first embedding is Lagrangian because it is the zero section of the fibration $TT^*G\to G\circledS\mathfrak{g}_1^*$. The second one is symplectic because the pull-back of Ω_{TT^*G} to $G\circledS\mathfrak{g}^*$ by Emb_2 results in the symplectic two-form Ω_{T^*G} in Eq. (2.5). On the image of Emb_2, Hamilton's equations (6.1) reduce to trivialized Hamilton's equations (3.4) on $G\circledS\mathfrak{g}^*$. Consequently, the embedding $\mathfrak{g}_3^*\to TT^*G$ is a Poisson map. When $E=h(\nu)$, Hamilton's equations (6.1) reduce to the Lie–Poisson equations (3.8). □

6.1.4. *Reduction of TT^*G by $G\circledS\mathfrak{g}$.* The action

$$\vartheta\colon TT^*G\times(G\circledS\mathfrak{g}_2)\to TT^*G,$$

$$((g,\mu,\xi,\nu);(h,\eta))\mapsto\vartheta_{(h,\eta)}(g,\mu,\xi,\nu):=(gh,\mu,\xi+\operatorname{Ad}_g\eta,\nu-\operatorname{ad}^*_{\operatorname{Ad}_g\eta}\mu) \tag{6.9}$$

of $G\circledS\mathfrak{g}$ on TT^*G can be described as a composition

$$\vartheta_{(h,\eta)}=\vartheta_{(h,0)}\circ\vartheta_{(e,\operatorname{Ad}_g\eta)},$$

where $\vartheta_{(h,0)}$ and $\vartheta_{(e,\operatorname{Ad}_g\eta)}$ can be identified with the actions of G and $\mathfrak{g}$ on TT^*G, respectively. Since both of these are symplectic, the action ϑ of $G\circledS\mathfrak{g}$ on TT^*G is symplectic.

PROPOSITION 6.11. *The Poisson reduction of TT^*G under the action of $G\circledS\mathfrak{g}_2$ in Eq.* (6.9) *results in $\mathfrak{g}_1^*\times\mathfrak{g}_3^*$ endowed with the bracket*

$$\{E,F\}_{\mathfrak{g}_1^*\times\mathfrak{g}_3^*}(\mu,\nu)=\left\langle\nu,\left[\frac{\delta E}{\delta\nu},\frac{\delta F}{\delta\nu}\right]\right\rangle. \tag{6.10}$$

REMARK 6.6. Although the Poisson bracket (6.10) structurally resembles the Lie–Poisson bracket on $\mathfrak{g}_3^*$, it is not a Lie–Poisson bracket on $\mathfrak{g}_1^*\times\mathfrak{g}_3^*$ considered as a dual of the Lie algebra $\mathfrak{g}\circledS\mathfrak{g}$ of the group $G\circledS\mathfrak{g}$. We refer to the Poisson bracket in Eq. (4.8) for the Lie–Poisson structure on $\operatorname{Lie}^*(G\circledS\mathfrak{g})=\mathfrak{g}^*\times\mathfrak{g}^*$.

The right invariant vector field generating the action ϑ is associted to two tuples (ξ_2, ξ_3) in the Lie algebra of $G \circledS \mathfrak{g}_2$, and is given by

$$X^{TT^*G}_{(\xi_2,0,\xi_3,0)}(g,\mu,\xi,\nu) = (TR_g\xi_2, \operatorname{ad}^*_{\xi_2}\mu, \xi_3 + [\xi,\xi_2], \operatorname{ad}^*_{\xi_2}\nu).$$

The momentum map for this Hamiltonian action is defined by the equation

$$\langle \mathbf{J}^{G\circledS\mathfrak{g}_2}_{TT^*G}(g,\mu,\xi,\nu), (\xi_2,\xi_3)\rangle = \langle \theta_2, X^{TT^*G}_{(\xi_2,0,\xi_3,0)}\rangle = \langle \mu,\xi_3\rangle + \langle \nu + \operatorname{ad}^*_\xi \mu, \xi_2\rangle,$$

where θ_2, in Eq. (2.22), is the Tulczyjew potential one-form on TT^*G. We find

$$\mathbf{J}^{G\circledS\mathfrak{g}_2}_{TT^*G} \colon TT^*G \to \operatorname{Lie}^*(G\circledS\mathfrak{g}_2) = \mathfrak{g}^* \times \mathfrak{g}^* \colon (g,\mu,\xi,\nu) = (\nu + \operatorname{ad}^*_\xi \mu, \mu).$$

Note that we have the following relation

$$\mathbf{J}^{G\circledS\mathfrak{g}_2}_{TT^*G}(g,\mu,\xi,\nu) = (\mathbf{J}^{G}_{TT^*G}(g,\mu,\xi,\nu), \mathbf{J}^{\mathfrak{g}_2}_{TT^*G}(g,\mu,\xi,\nu))$$

for momentum maps in Eqs. (6.2) and (6.5) for the actions of G and $\mathfrak{g}_2$ on TT^*G. The preimage of a fixed element $(\lambda,\mu) \in \mathfrak{g}^* \times \mathfrak{g}^*$ is

$$(\mathbf{J}^{G\circledS\mathfrak{g}_2}_{TT^*G})^{-1}(\lambda,\mu) = \{(g,\mu,\xi,\nu) : \nu = \lambda - \operatorname{ad}^*_\xi \mu\},$$

which we may identify with the semidirect product $G\circledS\mathfrak{g}_2$. We recall the coadjoint action $\operatorname{Ad}^*_{(g,\xi)}$, in Eq. (4.11), of the group $G\circledS\mathfrak{g}_2$ on the dual $\mathfrak{g}^*\times\mathfrak{g}^*$ of its Lie algebra. The isotropy subgroup $(G\circledS\mathfrak{g}_2)_{(\lambda,\mu)}$ of this coadjoint action is

$$(G\circledS\mathfrak{g}_2)_{(\lambda,\mu)} = \{(g,\xi) \in G\circledS\mathfrak{g}_2 : \operatorname{Ad}^*_{(g,\xi)}(\lambda,\mu) = (\lambda,\mu)\}$$

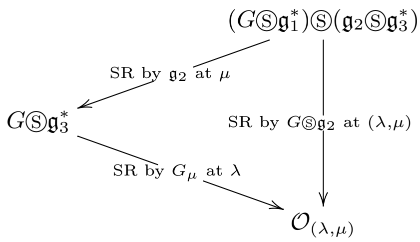
and acts on the preimage $(\mathbf{J}^{G\circledS\mathfrak{g}_2}_{TT^*G})^{-1}(\lambda,\mu)$. A generic quotient space

$$(\mathbf{J}^{G\circledS\mathfrak{g}_2}_{TT^*G})^{-1}(\lambda,\mu)\big/(G\circledS\mathfrak{g}_2)_{(\lambda,\mu)} \simeq G\circledS\mathfrak{g}_2\big/(G\circledS\mathfrak{g}_2)_{(\lambda,\mu)} \simeq \mathcal{O}_{(\lambda,\mu)}$$

is a coadjoint orbit in $\mathfrak{g}^*\times\mathfrak{g}^*$ through the point (λ,μ) under the coadjoint action $\operatorname{Ad}^*_{(g,\xi)}$ in Eq. (4.11).

Proposition 6.12. *The symplectic reduction of TT^*G under the action of $G\circledS\mathfrak{g}_2$ given in Eq. (6.9) results in the coadjoint orbit $\mathcal{O}_{(\lambda,\mu)}$ in $\mathfrak{g}^*\times\mathfrak{g}^*$ through the point (λ,μ) under the coadjoint action $\operatorname{Ad}^*_{(g,\xi)}$ in Eq. (4.11) as the total space and the symplectic two-form $\Omega_{\mathcal{O}_{(\lambda,\mu)}}$ in Eq. (4.13).*

It is also possible to obtain the symplectic space $\mathcal{O}_{(\lambda,\mu)}$ in two steps. Recall the symplectic reduction of TT^*G under the action of $\mathfrak{g}_2$ at $\mu\in\mathfrak{g}^*$ which results in $G\circledS\mathfrak{g}_3^*$ with the canonical symplectic two-form $\Omega_{G\circledS\mathfrak{g}_3^*}$. Then, consider the action of the isotropy subgroup G_μ on $G\circledS\mathfrak{g}_3^*$ and apply symplectic reduction which results in $(\mathcal{O}_{(\lambda,\mu)}, \Omega_{\mathcal{O}_{(\lambda,\mu)}})$. The diagram summarizing this two stage reduction of TT^*G follows



Reductions of TT^*G by $G\circledS\mathfrak{g}$.

6.1.5. *Reduction of TT^*G by $Gⓢ\mathfrak{g}^*$.* The action

$$\alpha\colon TT^*G\times(Gⓢ\mathfrak{g}_1^*)\to TT^*G\colon((g,\mu,\xi,\nu);(h,\lambda))\mapsto\alpha_{(h,\lambda)}(g,\mu,\xi,\nu)$$

of $Gⓢ\mathfrak{g}_1^*$ on TT^*G is given by

$$\alpha_{(h,\lambda)}(g,\mu,\xi,\nu)=(gh,\mu+\mathrm{Ad}_g^*\lambda,\xi,\nu). \tag{6.11}$$

As in the case of the action of $Gⓢ\mathfrak{g}_2$, it can also be described by composition of two actions

$$\alpha_{(h,\lambda)}=\alpha_{(h,0)}\circ\alpha_{(e,Ad_g^*\lambda)},$$

where $\alpha_{(h,0)}$ and $\alpha_{(e,Ad_g^*\lambda)}$ can be identified with the actions of G and $\mathfrak{g}_1^*$ on TT^*G, respectively. Since both of them are symplectic, α is also symplectic.

Proposition 6.13. *Poisson reduction of TT^*G under the action (6.11) of $Gⓢ\mathfrak{g}_1^*$ results in $\mathfrak{g}_2ⓢ\mathfrak{g}_3^*$ endowed with the bracket*

$$\{F,H\}_{\mathfrak{g}_2ⓢ\mathfrak{g}_3^*}(\xi,\nu)=\Big\langle\nu,\Big[\frac{\delta E}{\delta\nu},\frac{\delta F}{\delta\nu}\Big]\Big\rangle. \tag{6.12}$$

Remark 6.7. Regarding $\mathfrak{g}^*\times\mathfrak{g}$ as a dual of the Lie algebra $\mathfrak{g}ⓢ\mathfrak{g}^*$ of $Gⓢ\mathfrak{g}^*$, we obtained the Lie–Poisson bracket in Eq. (5.6). Although $\mathfrak{g}^*\times\mathfrak{g}$ and $\mathfrak{g}_2\times\mathfrak{g}_3^*$ are isomorphic as vector spaces, (5.6) is different from the Poisson bracket in Eq. (6.12) as manifestation of a group structure carried by adapted trivialization.

The infinitesimal generator of α is associated to the two tuple (ξ_2,ν_2) in the Lie algebra $\mathfrak{g}ⓢ\mathfrak{g}^*$ of $Gⓢ\mathfrak{g}_1^*$ and is of the form

$$X^{TT^*G}_{(\xi_2,\nu_2,0,0)}(g,\mu,\xi,\nu)=(TR_g\xi_2,\nu_2+\mathrm{ad}^*_{\xi_2}\mu,[\xi,\xi_2],\mathrm{ad}^*_{\xi_2}\nu-\mathrm{ad}^*_{\xi}\nu_2).$$

The momentum mapping $\mathbf{J}^{Gⓢ\mathfrak{g}_1^*}_{TT^*G}$ is defined by the equation

$$\langle\mathbf{J}^{Gⓢ\mathfrak{g}_1^*}_{TT^*G}(g,\mu,\xi,\nu),(\xi_2,\nu_2)\rangle=\langle\theta_1,X^{TT^*G}_{(\xi_2,\nu_2,0,0)}\rangle=-\langle\nu_2,\xi\rangle+\langle\nu+\mathrm{ad}^*_\xi\mu,\xi_2\rangle,$$

where θ_1 is the potential one-form given by Eq. (2.21). We obtain

$$\mathbf{J}^{Gⓢ\mathfrak{g}_1^*}_{TT^*G}\colon TT^*G\to\mathrm{Lie}^*(Gⓢ\mathfrak{g}_1^*)=\mathfrak{g}^*\times\mathfrak{g}\colon(g,\mu,\xi,\nu)\to(\nu+\mathrm{ad}^*_\xi\mu,-\xi),$$

which can be decomposed as

$$\mathbf{J}^{Gⓢ\mathfrak{g}_1^*}_{TT^*G}(g,\mu,\xi,\nu)=(\mathbf{J}^{G}_{TT^*G}(g,\mu,\xi,\nu),\mathbf{J}^{\mathfrak{g}_1^*}_{TT^*G}(g,\mu,\xi,\nu)),$$

where $\mathbf{J}^{G}_{TT^*G}$ and $\mathbf{J}^{\mathfrak{g}_1^*}_{TT^*G}$ are momentum mappings in Eqs. (6.2) and (6.8) for the actions of G and $\mathfrak{g}_1^*$ on TT^*G, respectively. The preimage of an element (λ,ξ) in $\mathfrak{g}^*\times\mathfrak{g}$ is

$$(\mathbf{J}^{Gⓢ\mathfrak{g}_1^*}_{TT^*G})^{-1}(\lambda,\xi)=\{(g,\mu,-\xi,\nu):\nu=\lambda+\mathrm{ad}^*_\xi\mu\},$$

which can be identified with the space $Gⓢ\mathfrak{g}_1^*$. The isotropy subgroup of coadjoint action of $Gⓢ\mathfrak{g}_2$ on $\mathfrak{g}^*\times\mathfrak{g}$ is

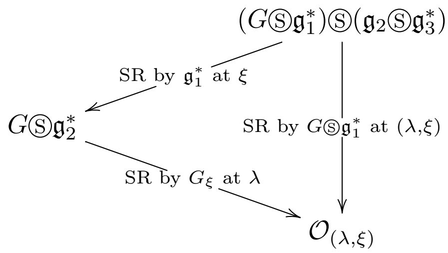
$$(Gⓢ\mathfrak{g}_1^*)_{(\lambda,\xi)}=\{(g,\mu)\in Gⓢ\mathfrak{g}_2:Ad^*_{(g,\mu)}(\lambda,\xi)=(\lambda,\xi)\},$$

where the coadjoint action is given by Eq. (5.10). The isotropy subgroup acts on the preimage of (λ,ξ) and results in the coadjoint orbit through the point $(\lambda,\xi)\in\mathfrak{g}^*\times\mathfrak{g}$

$$(\mathbf{J}^{Gⓢ\mathfrak{g}_1^*}_{TT^*G})^{-1}(\lambda,\xi)/(Gⓢ\mathfrak{g}_1^*)_{(\lambda,\xi)}\simeq Gⓢ\mathfrak{g}_1^*/(Gⓢ\mathfrak{g}_2)_{(\lambda,\xi)}\simeq\mathcal{O}_{(\lambda,\xi)}.$$

Proposition 6.14. *Symplectic reduction of TT^*G under the action of GⓈ$\mathfrak{g}_1^*$ given by Eq. (6.11) results in the coadjoint orbit $\mathcal{O}_{(\lambda,\xi)}$ and the symplectic two-form $\Omega_{\mathcal{O}_{(\lambda,\xi)}}$ in Eq. (5.11).*

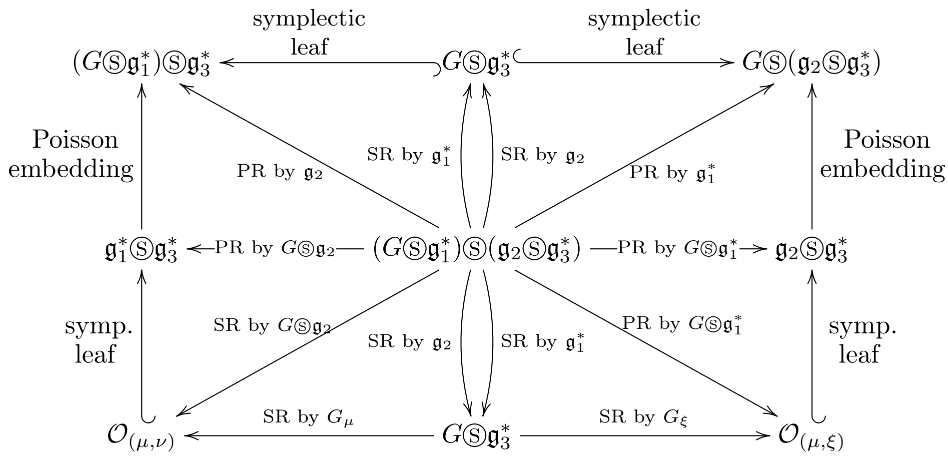
Similar to the reduction of TT^*G by GⓈ$\mathfrak{g}_2$, we may perform symplectic reduction of TT^*G with the action of GⓈ$\mathfrak{g}_1^*$ by two stages. Let us recall symplectic reduction of TT^*G with action of $\mathfrak{g}_1^*$ at $\xi \in \mathfrak{g}$ which results in GⓈ$\mathfrak{g}_3^*$ and the canonical symplectic two-form $\Omega_{G Ⓢ \mathfrak{g}_3^*}$. Then, consider the action of the isotropy subgroup G_ξ, defined in Eq. (5.12), on GⓈ$\mathfrak{g}_3^*$ and apply symplectic reduction. This gives $\mathcal{O}_{(\lambda,\xi)}$ and the symplectic two-form $\Omega_{\mathcal{O}_{(\lambda,\xi)}}$. The following diagram shows this two-stage reduction of TT^*G



Reduction of TT^*G by GⓈ$\mathfrak{g}_1^*$.

We summarize diagrammatically all possible reductions of Hamiltonian dynamics on the Tulczyjew symplectic space TT^*G.

(6.13)



Hamiltonian reductions of $TT^*Q = (G Ⓢ \mathfrak{g}_1^*) Ⓢ (\mathfrak{g}_2 Ⓢ \mathfrak{g}_3^*)$.

6.2. Lagrangian dynamics on TT^*G. As it is a tangent bundle, we can study Lagrangian dynamics on $TT^*G \simeq (G Ⓢ \mathfrak{g}_1^*) Ⓢ (\mathfrak{g}_2 Ⓢ \mathfrak{g}_3^*)$. We define variation of the base element $(g,\mu) \in G Ⓢ \mathfrak{g}_1^*$ by tangent lift of right translation of the Lie algebra element $(\eta,\lambda) \in \mathfrak{g} Ⓢ \mathfrak{g}^*$, that is,

$$\delta(g,\mu) = T_{(e,0)} R_{(g,\mu)}(\eta,\lambda) = (T_e R_g \eta, \lambda + \mathrm{ad}_\eta^* \mu).$$

To obtain the reduced variational principle $\delta(\xi,\nu)$ on the Lie algebra $\mathfrak{g}_2 \circledS \mathfrak{g}_3^*$ we compute

$$
\begin{aligned}
\delta(\xi,\nu) &= \frac{d}{dt}(\eta,\lambda) + [(\xi,\nu),(\eta,\lambda)]_{\mathfrak{g}\circledS\mathfrak{g}^*} \\
&= \frac{d}{dt}(\eta,\lambda) + ([\xi,\eta], \operatorname{ad}^*_\eta \nu - \operatorname{ad}^*_\xi \lambda) \\
&= (\dot\eta + [\xi,\eta], \dot\lambda + \operatorname{ad}^*_\eta \nu - \operatorname{ad}^*_\xi \lambda)
\end{aligned}
$$

for any $(\eta,\lambda) \in \mathfrak{g}\circledS\mathfrak{g}^*$. Assuming $\delta(\xi,\nu) = (\delta\xi,\delta\nu)$ and $\delta(g,\mu) = (\delta g,\delta\mu)$, we have the set of variations

$$
\delta g = T_e R_g \eta, \quad \delta\mu = \lambda + \operatorname{ad}^*_\eta \mu, \quad \delta\xi = \dot\eta + [\xi,\eta] \quad \delta\nu = \dot\lambda + \operatorname{ad}^*_\eta \nu - \operatorname{ad}^*_\xi \lambda \tag{6.14}
$$

for an arbitrary choice of $(\eta,\lambda) \in \mathfrak{g}\circledS\mathfrak{g}^*$. Note that these variations are the image of the right invariant vector field $X^{TT^*G}_{(\eta,\lambda,\dot\eta,\dot\lambda)}$ generated by $(\eta,\lambda,\dot\eta,\dot\lambda)$.

Proposition 6.15. *For a given Lagrangian E on TT^*G, extremals of the action integral are defined by the trivialized Euler-Lagrange equations*

$$
\begin{aligned}
\frac{d}{dt}\Big(\frac{\delta E}{\delta\xi}\Big) &= T_e^* R_g\Big(\frac{\delta E}{\delta g}\Big) - \operatorname{ad}^*_{\frac{\delta E}{\delta\mu}} \mu + \operatorname{ad}^*_\xi \Big(\frac{\delta E}{\delta\xi}\Big) - \operatorname{ad}^*_{\frac{\delta E}{\delta\nu}} \nu \\
\frac{d}{dt}\Big(\frac{\delta E}{\delta\nu}\Big) &= \frac{\delta E}{\delta\mu} - \operatorname{ad}_\xi \frac{\delta E}{\delta\nu}
\end{aligned} \tag{6.15}
$$

obtained by the variational principles in Eq. (6.14).

Proof. Let us begin with the observation that for any $(g,\xi) \in G\circledS\mathfrak{g}$, the variation $\delta(g,\xi) = (\delta g,\delta\xi)$ at $(\eta,\dot\eta) \in \mathfrak{g}\circledS\mathfrak{g}$ may be given by

$$
(\delta g,\delta\xi) = X^{TG}_{(\eta,\dot\eta)}(g,\xi) = (TR_g\eta, \dot\eta + [\eta,\xi]).
$$

Accordingly, given a Lagrangian $\mathfrak{L}\colon TG \cong G\circledS\mathfrak{g} \to \mathbb{R}$, the action integral

$$
\begin{aligned}
\delta\int_a^b \mathfrak{L}(g,\xi)dt &= \int_a^b \Big(\Big\langle \frac{\delta\mathfrak{L}}{\delta g}, \delta g\Big\rangle + \Big\langle \frac{\delta\mathfrak{L}}{\delta\xi}, \delta\xi\Big\rangle\Big)dt \\
&= \int_a^b \Big(\Big\langle \frac{\delta\mathfrak{L}}{\delta g}, TR_g\eta\Big\rangle + \Big\langle \frac{\delta\mathfrak{L}}{\delta\xi}, \dot\eta\Big\rangle + \Big\langle \frac{\delta\mathfrak{L}}{\delta\xi}, [\xi,\eta]\Big\rangle\Big)dt \\
&= \Big\langle \frac{\delta\mathfrak{L}}{\delta\xi}, \eta\Big\rangle\Big|_a^b + \int_a^b \Big(\Big\langle T^*R_g \frac{\delta\mathfrak{L}}{\delta g}, \eta\Big\rangle + \Big\langle -\frac{d}{dt}\frac{\delta\mathfrak{L}}{\delta\xi}, \eta\Big\rangle + \Big\langle \operatorname{ad}^*_\xi \frac{\delta\mathfrak{L}}{\delta\xi}, \eta\Big\rangle\Big)dt
\end{aligned}
$$

leads to the (trivialized) Euler–Lagrange equations

$$
\frac{d}{dt}\frac{\delta\mathfrak{L}}{\delta\xi} = T^*R_g \frac{\delta\mathfrak{L}}{\delta g} + \operatorname{ad}^*_\xi \frac{\delta\mathfrak{L}}{\delta\xi}.
$$

Accordingly, the (trivialized) Euler-Lagrange equations on TT^*G are given by

$$
\frac{d}{dt}\frac{\delta E}{\delta(\xi,\nu)} = T^*R_{(g,\mu)}\frac{\delta E}{\delta(g,\mu)} + \operatorname{ad}^*_{(\xi,\nu)}\frac{\delta E}{\delta(\xi,\nu)},
$$

where the first summand on the right hand side is

$$T^*R_{(g,\mu)}\frac{\delta E}{\delta(g,\mu)} = \Big(T^*R_g\frac{\delta E}{\delta g} - \operatorname{ad}^*_{\frac{\delta E}{\delta\mu}}\mu, \frac{\delta E}{\delta\mu}\Big),$$

while the second summand is

$$\operatorname{ad}^*_{(\xi,\nu)}\frac{\delta E}{\delta(\xi,\nu)} = \Big(\operatorname{ad}^*_\xi\frac{\delta E}{\delta\xi} - \operatorname{ad}^*_{\frac{\delta E}{\delta\nu}}\nu, \frac{\delta E}{\delta\nu} + \operatorname{ad}_\xi\frac{\delta E}{\delta\nu}\Big).$$

□

PROPOSITION 6.16. *Given a Lagrangian $E = E(g,\mu,\xi,\nu)$ on TT^*G, the quantity*

$$\Big\langle\frac{\delta E}{\delta\xi},\xi\Big\rangle + \Big\langle\frac{\delta E}{\delta\nu},\nu\Big\rangle - E$$

is constant.

PROOF. Let us begin with

$$\begin{aligned}\frac{dE}{dt} &= \Big\langle\frac{\partial E}{\partial g},\dot g\Big\rangle + \Big\langle\frac{\partial E}{\partial\mu},\dot\mu\Big\rangle + \Big\langle\frac{\partial E}{\partial\xi},\dot\xi\Big\rangle + \Big\langle\frac{\partial E}{\partial\nu},\dot\nu\Big\rangle\\ &= \Big\langle\frac{\partial E}{\partial g},TR_g\xi\Big\rangle + \Big\langle\frac{\partial E}{\partial\mu},\nu + \operatorname{ad}^*_\xi\mu\Big\rangle + \Big\langle\frac{\partial E}{\partial\xi},\dot\xi\Big\rangle + \Big\langle\frac{\partial E}{\partial\nu},\dot\nu\Big\rangle\\ &= \Big\langle T^*R_g\Big(\frac{\partial E}{\partial g}\Big) - \operatorname{ad}^*_{\frac{\partial E}{\partial\mu}}\mu,\xi\Big\rangle + \Big\langle\frac{\partial E}{\partial\mu},\nu\Big\rangle + \Big\langle\frac{\partial E}{\partial\xi},\dot\xi\Big\rangle + \Big\langle\frac{\partial E}{\partial\nu},\dot\nu\Big\rangle.\end{aligned}$$

Next, substituting the (trivialized) Euler–Lagrange equations (6.15), we obtain

$$\begin{aligned}\frac{dE}{dt} &= \Big\langle\frac{d}{dt}\Big(\frac{\partial E}{\partial\xi}\Big) - \operatorname{ad}^*_\xi\frac{\partial E}{\partial\xi} + \operatorname{ad}^*_{\frac{\partial E}{\partial\nu}}\nu,\xi\Big\rangle\\ &\quad + \Big\langle\nu,\frac{d}{dt}\Big(\frac{\partial E}{\partial\nu}\Big) - \operatorname{ad}_\xi\frac{\partial E}{\partial\nu}\Big\rangle + \Big\langle\frac{\partial E}{\partial\xi},\dot\xi\Big\rangle + \Big\langle\frac{\partial E}{\partial\nu},\dot\nu\Big\rangle\\ &= \Big\langle\frac{d}{dt}\Big(\frac{\partial E}{\partial\xi}\Big),\xi\Big\rangle + \Big\langle\frac{\partial E}{\partial\xi},\dot\xi\Big\rangle + \Big\langle\dot\nu,\frac{\partial E}{\partial\nu}\Big\rangle + \Big\langle\nu,\frac{d}{dt}\Big(\frac{\partial E}{\partial\nu}\Big)\Big\rangle\\ &\quad + \Big\langle\operatorname{ad}^*_{\frac{\partial E}{\partial\nu}}\nu,\xi\Big\rangle - \Big\langle\operatorname{ad}^*_\xi\frac{\partial E}{\partial\xi},\xi\Big\rangle - \Big\langle\nu,\operatorname{ad}_\xi\frac{\partial E}{\partial\nu}\Big\rangle\\ &= \frac{d}{dt}\Big(\Big\langle\frac{\partial E}{\partial\xi},\xi\Big\rangle + \Big\langle\nu,\frac{\partial E}{\partial\nu}\Big\rangle\Big),\end{aligned}$$

from which the result follows. □

6.2.1. *Reductions on TT^*G.* When the Lagrangian function E in the trivialized Euler-Lagrange equations (6.15) is independent of the group variable $g\in G$, we arrive at Euler–Lagrange equations (6.20) on $\mathfrak{g}_1^*Ⓢ(\mathfrak{g}_2Ⓢ\mathfrak{g}_3^*)$. In addition, if the Lagrangian E depends only on the fiber coordinates $E = E(\xi,\nu)$, we have the Euler–Poincaré equations (6.16).

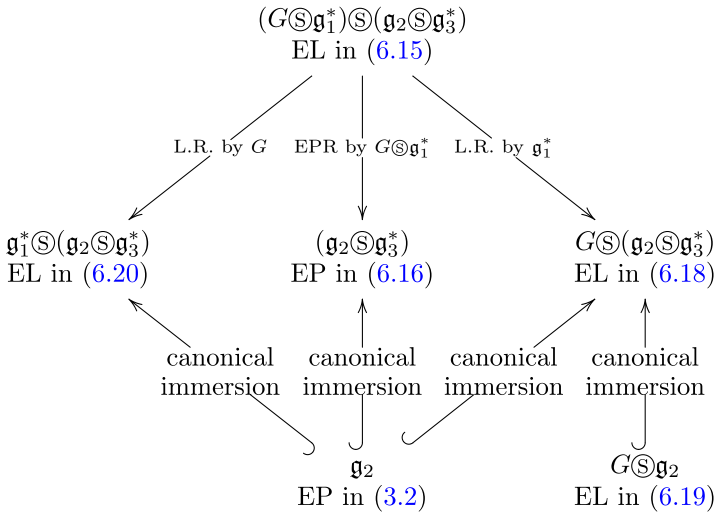
PROPOSITION 6.17. *The Euler–Poincaré equations on the Lie algebra $\mathfrak{g}_2Ⓢ\mathfrak{g}_3^*$ are*

$$\frac{d}{dt}\Big(\frac{\delta E}{\delta\xi}\Big) = \operatorname{ad}^*_\xi\Big(\frac{\delta E}{\delta\xi}\Big) - \operatorname{ad}^*_{\frac{\delta E}{\delta\nu}}\nu,\quad \frac{d}{dt}\Big(\frac{\delta E}{\delta\nu}\Big) = -\operatorname{ad}_\xi\frac{\delta E}{\delta\nu}. \tag{6.16}$$

If, moreover, $E = E(\xi)$, the Euler–Poincaré equations (3.2) on $\mathfrak{g}_2$ arise. This procedure is called reduction by stages [10, 28].

Alternatively, the Lagrangian function E in trivialized Euler–Lagrange equations (6.15) can be independent of $\mu \in \mathfrak{g}_1^*$, that is, E can be invariant under the action of $\mathfrak{g}_1^*$ on TT^*G. In this case, we have Euler–Lagrange equations (6.18) on $G \circledS (\mathfrak{g}_2 \circledS \mathfrak{g}_3^*)$. When $E = E(g, \xi)$, we have trivialized Euler–Lagrange equations (6.19) on $G \circledS \mathfrak{g}_2$. Referring to the notation in Remark 2.5, the following diagram summarizes this discussion.

(6.17)



Lagrangian reductions on TT^*G

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta E}{\delta \xi}\right) &= T_e^* R_g \left(\frac{\delta E}{\delta g}\right) + \operatorname{ad}_\xi^* \left(\frac{\delta E}{\delta \xi}\right) - \operatorname{ad}^*_{\frac{\delta E}{\delta \nu}} \nu \\ \frac{d}{dt}\left(\frac{\delta E}{\delta \nu}\right) &= -\operatorname{ad}_\xi \frac{\delta E}{\delta \nu} \end{aligned} \tag{6.18}$$

$$\frac{d}{dt}\left(\frac{\delta E}{\delta \xi}\right) = T_e^* R_g \left(\frac{\delta E}{\delta g}\right) + \operatorname{ad}_\xi^* \left(\frac{\delta E}{\delta \xi}\right) \tag{6.19}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta E}{\delta \xi}\right) &= -\operatorname{ad}^*_{\frac{\delta E}{\delta \mu}} \mu + \operatorname{ad}_\xi^* \left(\frac{\delta E}{\delta \xi}\right) - \operatorname{ad}^*_{\frac{\delta E}{\delta \nu}} \nu \\ \frac{d}{dt}\left(\frac{\delta E}{\delta \nu}\right) &= \frac{\delta E}{\delta \mu} - \operatorname{ad}_\xi \frac{\delta E}{\delta \nu} \end{aligned} \tag{6.20}$$

7. Summary, discussions and prospectives

We write Hamilton's equations on the cotangent bundles T^*TG and T^*T^*G. Symplectic and Poisson reductions of T^*TG are performed under actions of G, $\mathfrak{g}$ and $G \circledS \mathfrak{g}$ as shown in diagram (4.14). T^*T^*G is also reduced by actions of G, $\mathfrak{g}^*$ and $G \circledS \mathfrak{g}^*$ c.f. diagram (5.13).

On Tulczyjew's symplectic space $TT^*G = (G \circledS \mathfrak{g}_1^*) \circledS (\mathfrak{g}_2 \circledS \mathfrak{g}_3^*)$, we obtain both Hamilton's and Euler-Lagrange equations. Hamilton's equations are reduced by symplectic and Poisson actions of G, $\mathfrak{g}_1^*$, $\mathfrak{g}_2$, $G \circledS \mathfrak{g}_2$ and $G \circledS \mathfrak{g}_1^*$. These reductions are summarized in diagram (6.13). As it is a tangent bundle, Lagrangian reductions are performed with actions of G, $\mathfrak{g}_1^*$, $G \circledS \mathfrak{g}_1^*$ and $G \circledS (\mathfrak{g}_1^* \times \mathfrak{g}_2)$ and are shown in diagram (6.17).

Hamiltonian reductions of Tulczyjew's symplectic space TT^*G can be generalized to symplectic reduction of a tangent bundle of a symplectic manifold with lifted symplectic structure. This may be a first step towards reduction of special symplectic structures and reduction of Tulczyjew's triplet for the arbitrary configuration manifold $\mathcal{Q}$. In order to obtain this more general picture for trivialization and reduction of the Tulczyjew triplet, we plan to pursue a new project where the reduction is applied to Lagrangian dynamics on TQ and Hamiltonian dynamics on T^*Q for an arbitrary manifold Q. In this case, the reduced Lagrangian dynamics on the orbit space TQ/G is called Lagrange–Poincaré equations. If, particularly, $Q = G$ then the Lagrange–Poincaré equations turn out to be Euler–Poincaré equations on $\mathfrak{g}$. Similarly, the Hamiltonian dynamics on T^*Q/G is called Hamilton–Poincaré equations and, reduce to Lie–Poisson equations on $\mathfrak{g}^*$ for the case of $Q = G$. In the first paper [21] of that series, we have already presented the trivialization and reduction of the Tulczyjew triplet for an arbitrary manifold under the presence of an Ehresmann connection.

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TULCZYJEW-И ТРИПЛЕТИ ЛИЈЕВИХ ГРУПА III: ДИНАМИКА ВИШЕГ РЕДА И РЕДУКЦИЈЕ ИТЕРИРАНИХ РАСЛОЈЕЊА

Резиме. За дату Лијеву групу G разрађена је Хамилотнова динамика на T^*T^*G и T^*TG, као и динамика на Tulczyjew-овом симплектичком простору TT^*G која се може дефинисати како Лагранжијаном тако и Хамилтонијаном. Разматрају се тривијализације које су прилагођене груповној структури итерираних раслојења, што омогућује опис свих могућих редукција (Пуасонове, симплектичке или обе) динамика вишег реда у односу на дејство одговарајућих подгрупа.

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