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A TIME-DEPENDENT METRIC GRAPH WITH A FOURTH-ORDER OPERATOR ON THE EDGES

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Abstract. The metric graph model is suggested for description of elastic vibration in a network of rods under the assumption that the rod lengths vary in time. A single rod and star-like graph are considered. Influence of the length variation law on the vibration distribution is investigated. For high-frequency length variation one observes a fast transition to high-frequency amplitude distribution.

1. Introduction

The metric graph is a widely used low-dimensional model. To introduce the metric graph, one should define firstly its geometric structure, i.e. a set of edges (segments or curves) and a set of vertices (coupling and boundary points for edges). Then, one defines a space of functions at the edges satisfying some coupling conditions at the vertices and a differential operator acting in the space. A particular choice of the metric graph is related to the properties of the modelled physical system. Initiated in the 1930's as modelling of macromolecules, the approach became a powerful and, at the same time, simple tool for investigation of network-like systems. The largest class of such models, so-called quantum graphs, is presented in quantum theory [1,2]. In this case, the Schrödinger or Dirac operator is considered at the edges. As for other applications, e.g., in optics, hydrodynamics, elasticity, etc., various operators at the edges are studied [3–8]. Recently, great attention has been devoted to time-depending metric graphs, i.e. graphs having characteristics (e.g., edge lengths) varying in time [9–16]. This problem is in relation to time-dependent boundary conditions and time dependent potentials [17–21]. In the present paper we consider with varying edge lengths for the case of the fourth-order operator at the edges:

$$\frac{\partial^2}{\partial t^2}\Psi(x,t)=\frac{\partial^4}{\partial x^4}\Psi(x,t) \tag{1.1}$$

As for the scattering problem on a line or half-line, we can mention the works [22, 23]. In order to describe this system, we firstly solve the problem for graphs

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with constant edge lengths, and then proceed to the dynamic problem, where the lengths depend on time. The lengths of the edges of the graphs in these models depend on time according to a harmonic law:

$$L_0(t) = lL(t), \quad L(t) = a + b\cos(\omega t),$$

where $\omega = 2\pi T^{-1}$ is the oscillation frequency, T is the period of oscillation, and a, b are real parameters.

2. Segment graph

Let us consider the equations on a segment graph. It is the simplest graph which consists of two vertices and an edge between them. Firstly, we solve a problem with a constant length l of the edge. To obtain the eigenfunctions of the system we have to find the solutions for the following equation:

$$\frac{\partial^4}{\partial y^4}\phi(y) = k^4\phi(y), \tag{2.1}$$

where $\phi(y)$ is an eigenfunction satisfying boundary conditions:

$$\begin{cases} \phi(0) = \phi'(0) = 0 \\ \phi(l) = \phi'(l) = 0. \end{cases} \tag{2.2}$$

The solution for (2.1) has the form:

$$\phi(y) = C_1\sinh(ky) + C_2\cosh(ky) + C_3\sin(ky) + C_4\cos(ky).$$

Boundary conditions (2.2) give the following system for coefficients C_i:

$$\begin{cases} C_2 + C_4 = 0, \\ C_1 + C_3 = 0, \\ C_1\sinh(kl) + C_2\cosh(kl) + C_3\sin(kl) + C_4\cos(kl) = 0, \\ C_1\cosh(kl)) + C_2\sinh(kl) - C_1\cos(kl) + C_2\sin(kl) = 0. \end{cases} \tag{2.3}$$

The condition for the existence of non-trivial solutions for (2.3) (spectral equation) is as follows

$$\cosh(kl)\cos(kl) - 1 = 0. \tag{2.4}$$

Let k_n be the $n-$th positive root of (2.4). The eigenfunction corresponding to the eigenvalue k_n^4 has the form:

$$\phi_n(y) = C_1\Big[\sinh(k_ny) + \frac{\sinh(k_nl) - \sin(k_nl)}{\cos(k_nl) - \cosh(k_nl)}\cosh(k_ny) - \sin(k_ny) - \frac{\sinh(k_nl) - \sin(k_nl)}{\cos(k_nl) - \cosh(k_nl)}\cos(k_ny)\Big], \tag{2.5}$$

where the coefficient C_1 is determined from the normalization condition:

$$C_1 = [20k_nl + 2\sin(2k_nl) - 16\cos(k_nl)\sinh(k_nl) + \sinh(2k_nl)(\cos(2k_nl) + 2) - 16\sin(k_nl)\cosh(k_nl) + \sin(2k_nl)\cosh(2k_nl)]^{-1/2}.$$

The next step is to solve the problem of a graph with edges of variable length. In order to use the results of calculations from the stationary problem, it is necessary to make a change of variables.

$$\begin{cases} y = \frac{x}{L(t)}, \\ t_1 = t. \end{cases}$$

Below we use the same notation t for t_1.

The equation and the boundary conditions for the dynamical problem are as follows:

$$\frac{\partial^2}{\partial t^2}\Psi(x,t) = \frac{\partial^4}{\partial x^4}\Psi(x,t) \tag{2.6}$$

$$\begin{cases} \Psi(0,t) = \Psi'(0,t) = 0, \\ \Psi(L_0(t),t) = \Psi'(L_0(t),t) = 0. \end{cases} \tag{2.7}$$

After replacement of the variables in (2.6) and (2.7), we seek the wave function in the form of a Fourier series:

$$\Psi(y,t) = \sum_n C_n(t)\phi_n(t), \tag{2.8}$$

where ϕ_n is the $n-$th eigenfunction of the stationary problem (2.5). Below we use the notations $\frac{\partial^2 L(t)}{\partial t^2} = \ddot{L}$, $\frac{\partial L(t)}{\partial t} = \dot{L}$, $L(t) = L$ and replace $t_1 = t$. We substitute (2.8) in the equation and reduce it to the system of ordinary differential equations. One can find the details of the equation transformation in the Appendix.

As a result, we obtain the following system of differential equations for the coefficients of the Fourier series:

$$\begin{gathered} \frac{2\dot{L}^2 - \dot{L}L(t)}{L^2}\frac{\partial}{\partial y}\sum_n C_n(t)I_1 + \frac{\dot{L}^2}{L^2}\sum_n C_n(t)I_2 \\ 2\frac{L}{L}\sum_n \dot{C}_n(t)I_1 + \ddot{C}_m = \frac{k^4}{pL^4}\sum_n C_n(t). \end{gathered} \tag{2.9}$$

Here I_1, I_2 are matrices having the following entries:

$$I_1^{nm} = \int_0^l y\frac{\partial\phi^{(n)}(y)}{\partial y}\phi^{(m)}(y)dy, \quad I_2^{nm} = \int_0^l y^2\frac{\partial^2\phi^{(n)}(y)}{\partial y^2}\phi^{(m)}(y).$$

We solve system (2.9) using the ode45 function built into GNU Octave. Correspondingly, we present (2.9) as a system of differential equations of the first order by introducing a new variable:

$$D_n = \dot{C}_n.$$

Consequently, system (2.9) transforms to a system of $2N$ equations, where N is the number of eigenvalues taken into account (i.e. we cut series (A.2) at the $N-$th term). One obtains the following system in matrix form:

$$\begin{cases} D = \dot{C}, \\ \dot{C} = \frac{K^4}{L^4} * C + 2\frac{\dot{L}}{L} I_1 D - \frac{2\dot{L}^2 - \ddot{L}L(t)}{L^2} I_1 C - \frac{\dot{L}^2}{L^2} I_2 C. \end{cases}$$

Here $K, C, D, \dot{C}, \dot{D}$ are matrices of dimension $[N, 1]$ and I_1, I_2 are matrices of dimension $[N, N]$. For simulation, we take the following length variation law and the initial conditions

$$L(t) = a + b\cos(\omega t), \quad a = 1, \quad b = 0.1, \quad \omega = 50, \quad l = 1, \quad N = 10,$$

$$\Psi(y, 0) = 1 - \cos(2\pi y).$$

The result of simulation is shown in Figures 1 and 2.

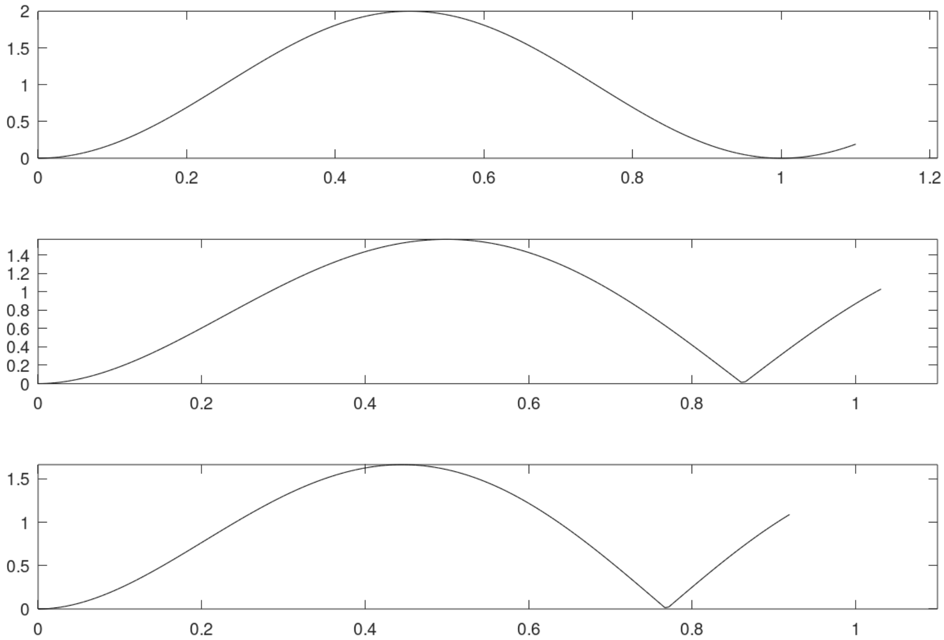

FIGURE 1. Modules $|\Psi|$ on the edge of the graph. From top to bottom: $t = 0$, $t = 0.2T$, $t = 0.4T$, where the period of oscillation of the length is $T = 2\pi/\omega$.

We took a smooth slowly varying initial condition and the length variation law of high frequency. Comparison of Figures 1 and 2 shows that, at first, the length variation does not lead to a change of the amplitude distribution. But later the distribution is determined mainly by the high-frequency variation of the segment length. There is a very fast transition from a smooth distribution of amplitude to a high-frequency one. The details of this transition are shown in Figure 3.

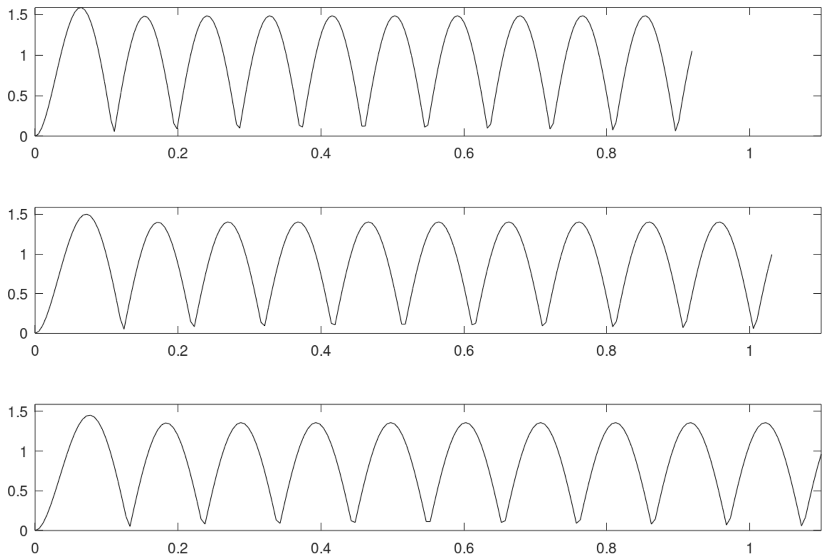

FIGURE 2. Modules $|\Psi|$ on the edge of the graph. From top to bottom: $t = 0.6T,\ t = 0.8T,\ t = T,$ where the period of oscillation of the length is $T = 2\pi/\omega$.

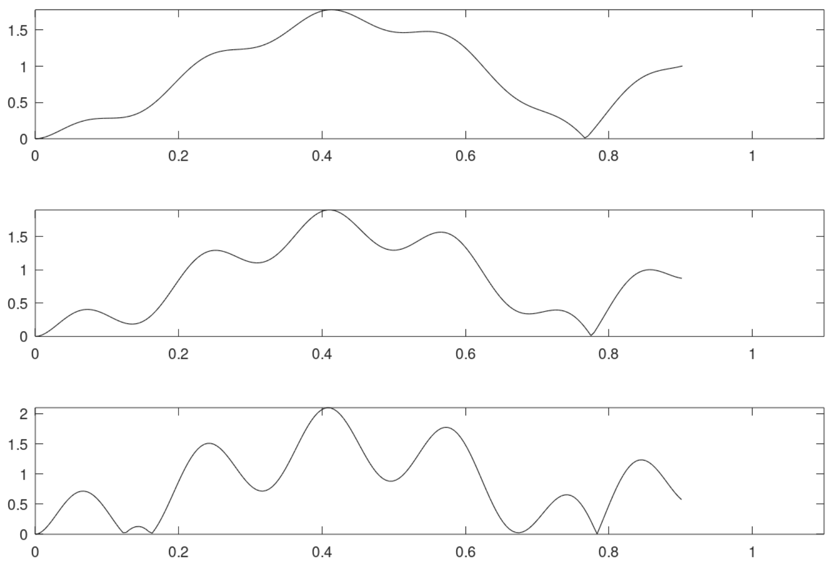

FIGURE 3. Modules $|\Psi|$ on the edge of the graph. From top to bottom: $t = 0.47T,\ t = 0.475T,\ t = 0.48T,$ where the period of oscillation of the length is $T = 2\pi/\omega$.

3. Star-like graph

Let us consider the corresponding star like graph with three edges of varying lengths $L_j(t)$, j $(j = 1, 2, 3)$:

$$L_j(t) = l_j L(t).$$

We will take a harmonic law for $L(t)$. The procedure is the same as for the segment. Firstly, we solve the problem for the edges of constant length. The operator acts as the fourth derivative at each edge. The operator domain consists of functions from the Sobolev space W_2^4 at each edge satisfying the boundary conditions (3.1). To obtain the eigenfunctions of the operator, we should find solutions $\phi_j(y)$ for equation (2.1) at each edge $(j = 1, 2, 3)$ and ensure satisfying the boundary conditions (3.1):

$$\begin{cases} \phi_1(0) = \phi_2(0) = \phi_3(0) = 0, \\ \phi_1(l_1) = \phi_2(l_2) = \phi_3(l_3), \\ \phi_1'(0) = \phi_2'(0) = \phi_3'(0) = 0, \\ \phi_1'(l_1) = \phi_2'(l_2) = \phi_3'(l_3) = 0, \\ \phi_1'''(l_1) + \phi_2'''(l_2) + \phi_3'''(l_3) = 0. \end{cases} \tag{3.1}$$

The solution for (2.1) at the $j-$th edge has the form:

$$\phi_j(y) = C_1^j \sinh(ky) + C_2^j \cosh(ky) + C_3^j \sin(ky) + C_4^j \cos(ky).$$

Correspondingly, (3.1) gives one

$$\phi_j(0) = C_2^j + C_4^j = 0 \Rightarrow C_2^j = -C_4^j,$$

$$\phi_j'(0) = k[C_1^j + C_3^j] = 0 \Rightarrow C_1^j = -C_3^j,$$

$$\phi_1'(l_1) = C_1^1(\cosh(kl_1) - \cos(kl_1)) + C_2^1(\sinh(kl_1) + \sin(kl_1)) = 0, \Rightarrow$$
$$C_2^1 = C_1^1 \frac{\cos(kl_1) - \cosh(kl_1)}{\sinh(kl_1) + \sin(kl_1)}.$$

For convenience, we put:

$$Q_j = \frac{\cos(kl_j) - \cosh(kl_j)}{\sinh(kl_j) + \sin(kl_j)}.$$

Conditions (3.1) give the following equation for eigenvalues:

$$\sum_{j=1}^{3} \frac{\cosh(kl_j)\sin(kl_j) + \cos(kl_j)\sinh(kl_j)}{\sinh(kl_j) + \sin(kl_j)} = 0.$$

The eigenfunction corresponding to the eigenvalue k_n at the j-th edge takes the form:

$$\phi_j^{(n)}(y) = C_1^{(n)}[\sinh(k_n y) + Q_j^{(n)} \cosh(k_n y) - \sin(k_n y) - Q_j^{(n)} \cos(k_n y)],$$

where $Q_j^{(n)}$ denotes Q_j corresponding to the eigenvalue k_n, and $C_1^{(n)}$ is the normalizing coefficient:

$$\sum_j \int_0^{l_j} (\phi_j^{(n)}(y))^2 dy = 1,$$

$$C_1^{(n)} = \sum_{j=1}^{3} \Big[\frac{-\sin(2k_n l_j) + \sinh(2k_n l_j)}{4k_n} + \frac{\cos(k_n l_j)\sinh(k_n l_j) - \sin(k_n l_j)\cosh(k_n l_j)}{k_n} + 2Q_j^{(n)} \frac{(\sin(k_n l_j) - \sinh(k_n l_j))^2}{2k_n} + (Q_j^{(n)})^2 \Big(\frac{4k_n l_j + \sin(2k_n l_j) + \sinh(2k_n l_j)}{4k_n} - \frac{\cos(k_n l_j)\sinh(k_n l_j) + \sin(k_n l_j)\cosh(k_n l_j)}{k_n} \Big) \Big].$$

Let us turn to the problem with a variable length of edges. We solve the problem analogously to the segment case. A difference appears in the integrals I_1 and I_2 due to the existence of three edges:

$$I_1 = \int_\Gamma y \frac{\partial \phi^{(n)}(y)}{\partial y} \phi^{(m)}(y) dy = \sum_{j=1}^{3} \int_0^{l_j} y \frac{\partial \phi_j^{(n)}(y)}{\partial y} \phi_j^{(m)}(y) dy,$$

$$I_2 = \int_\Gamma y^2 \frac{\partial^2 \phi^{(n)}(y)}{\partial y^2} \phi^{(m)}(y) dy = \sum_{j=1}^{3} \int_0^{l_j} y^2 \frac{\partial^2 \phi_j^{(n)}(y)}{\partial y^2} \phi_j^{(m)}(y) dy.$$

For simulation, we take the length variation law and the initial conditions as follows

$$L(t) = a + b\cos(\omega t)$$
$$a = 1, \quad b = 0.1, \quad \omega = 10$$
$$l_1 = 1, \quad l_2 = 2, \quad l_3 = 3, \quad N = 15$$
$$\Psi_1(y, 0) = 1 - \cos(2\pi y)$$
$$\Psi_j(y, 0) = 0, \qquad j = 2, 3.$$

Results of the simulation were shown in Figures 4 and 5.

We can see that a transition from a smooth amplitude distribution to a high-frequency one takes place for a very small value of time (at the moment of time 0.2T only a high-frequency distribution is observed. If one chooses different initial characteristics, it is possible to look into this transition in detail in Figures 6 and 7).

$$L(t) = a + b\cos(\omega t)$$
$$a = 1, \quad b = 0.5, \quad \omega = 30$$
$$l_1 = 1, \quad l_2 = 3, \quad l_3 = 5, \quad N = 15$$
$$\Psi_1(y, 0) = 1 - \cos(2\pi y)$$
$$\Psi_j(y, 0) = 0, \qquad j = 2, 3$$

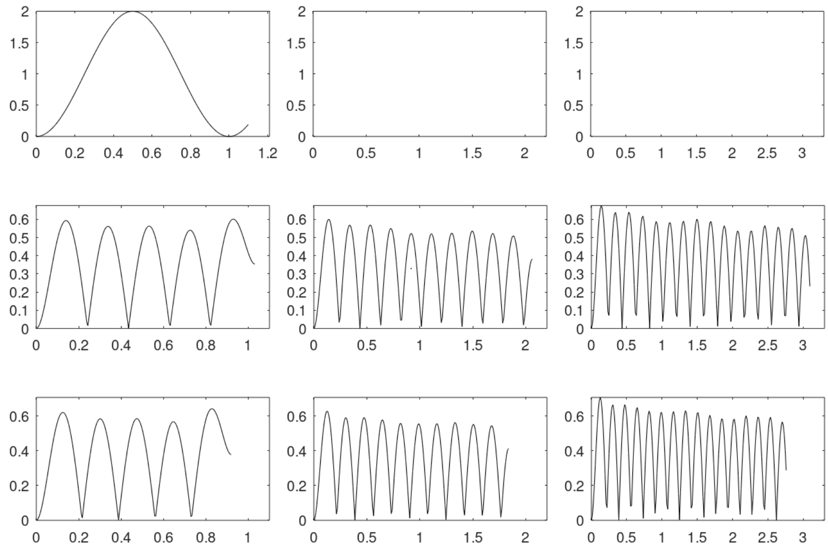

FIGURE 4. Modules $|\Psi_j|$ on the edge of the graph. From top to bottom: $t = 0$, $t = 0.2T$, $t = 0.4T$, where the period of oscillation of the length is $T = 2\pi/\omega$.

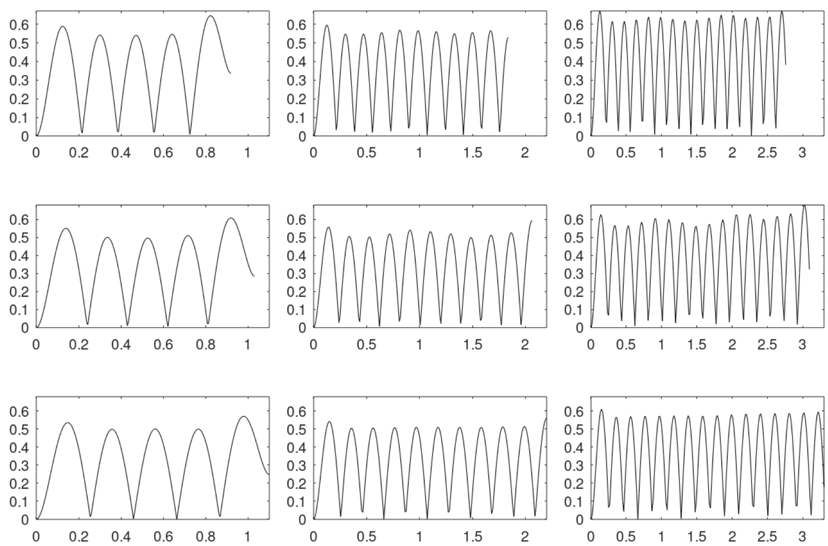

FIGURE 5. Modules $|\Psi_j|$ on the edge of the graph. From top to bottom: $t = 0.6$, $t = 0.8T$, $t = T$, where the period of oscillation of the length is $T = 2\pi/\omega$.

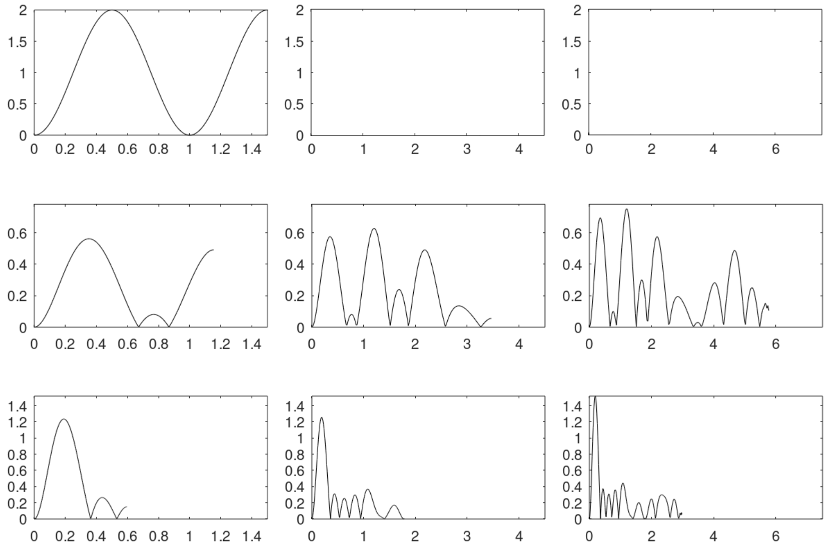

FIGURE 6. Modules $|\Psi_j|$ on the edge of the graph. From top to bottom: $t = 0$, $t = 0.2T$, $t = 0.4T$, where the period of oscillation of the length is $T = 2\pi/\omega$.

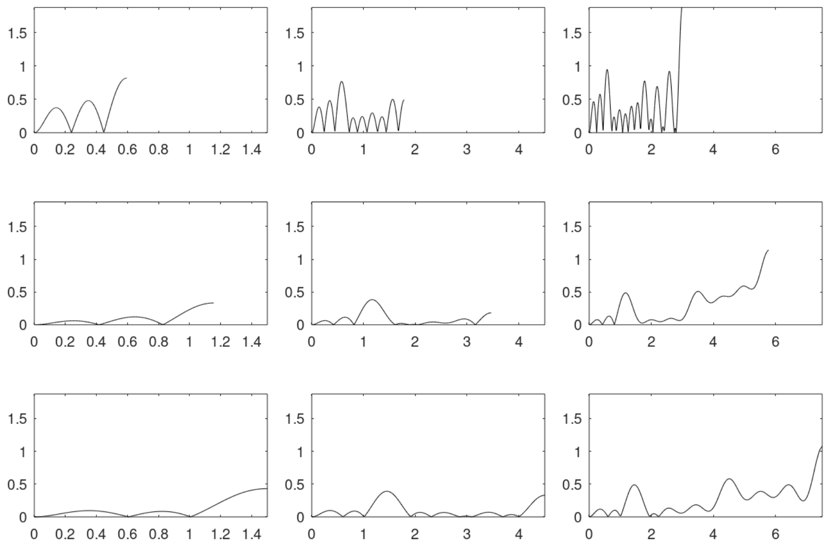

FIGURE 7. Modules $|\Psi_j|$ on the edge of the graph. From top to bottom: $t = 0.6$, $t = 0.8T$, $t = T$, where the period of oscillation of the length is $T = 2\pi/\omega$.

4. Conclusion

The metric graph model is suggested for description of elastic waves propagation in a rod and star-like system of rods. It is assumed that the rod lengths vary in time. The law of the length variation has a strong influence on the distribution of the vibration amplitude. If the frequency of the length vibration is essentially high, a fast transition to high-frequency distribution of the amplitude is observed. The model can be used for description of vibration of networks.

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Appendix

Let us find the expressions for the first spatial derivative in terms of new variables:

$$\frac{\partial}{\partial x}\Psi(x,t) = \frac{\partial}{\partial x}\Psi(y(x,t),t_1(x,t)) = \frac{\partial}{\partial y}\Psi(y,t_1)\frac{\partial y}{\partial x} + \frac{\partial}{\partial t_1}\Psi(y,t_1)\frac{\partial t_1}{\partial x}$$
$$= \frac{\partial}{\partial y}\Psi(y,t_1)\frac{1}{L(t)} + \frac{\partial}{\partial t_1}\Psi(y,t_1)\cdot 0 = \frac{1}{L(t)}\frac{\partial}{\partial y}\Psi(y,t_1).$$

Similarly, we find the second, third, and fourth derivatives:

$$\frac{\partial^2}{\partial x^2}\Psi(x,t) = \frac{\partial}{\partial x}\Big(\frac{1}{L(t)}\frac{\partial}{\partial y}\Psi(y,t_1)\Big) = \frac{1}{L(t)}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\Psi(y,t_1)$$
$$= \frac{1}{L(t)}\Big(\frac{\partial^2}{\partial y^2}\Psi(y,t_1)\frac{\partial y}{\partial x} + \frac{\partial^2}{\partial y\partial t_1}\Psi(y,t_1)\frac{\partial t_1}{\partial x}\Big) = \frac{1}{L^2(t)}\frac{\partial^2}{\partial y^2}\Psi(y,t_1),$$
$$\frac{\partial^3}{\partial x^3}\Psi(x,t) = \frac{1}{L^3(t)}\frac{\partial^3}{\partial y^3}\Psi(y,t_1),$$
$$\frac{\partial^4}{\partial x^4}\Psi(x,t) = \frac{1}{L^4(t)}\frac{\partial^4}{\partial y^4}\Psi(y,t_1). \tag{A.1}$$

We calculate the derivative of the variable

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\frac{x}{L(t)} = -\frac{1}{L^2(t)}\frac{\partial L(t)}{\partial t}x = -\frac{1}{L^2(t)}\frac{\partial L(t)}{\partial t}yL(t) = -\frac{1}{L(t)}\frac{\partial L(t)}{\partial t}y.$$

Then the first and the second time derivatives are as follows:

$$\frac{\partial}{\partial t}\Psi(x,t) = \frac{\partial}{\partial t}\Psi(y(x,t),t_1(x,t)) = \frac{\partial}{\partial y}\Psi(y,t_1)\frac{\partial y}{\partial t} + \frac{\partial}{\partial t_1}\Psi(y,t_1)\frac{\partial t}{\partial t_1}$$
$$= \frac{\partial}{\partial y}\Psi(y,t_1)\Big(-\frac{1}{L(t)}\frac{\partial L(t)}{\partial t}y\Big) + \frac{\partial}{\partial t_1}\Psi(y,t_1)\cdot 1$$
$$= -\frac{1}{L(t)}\frac{\partial L(t)}{\partial t}y\frac{\partial}{\partial y}\Psi(y,t_1) + \frac{\partial}{\partial t_1}\Psi(y,t_1),$$

$$
\begin{aligned}
\text{(A.2)} \quad \frac{\partial^2}{\partial t^2}\Psi(x,t) &= \frac{\partial}{\partial t}\Big(-\frac{1}{L(t)}\frac{\partial L(t)}{\partial t} y \frac{\partial}{\partial y}\Psi(y,t_1) + \frac{\partial}{\partial t_1}\Psi(y,t_1)\Big) \\
&= \frac{\partial}{\partial t}\Big(-\frac{1}{L(t)}\frac{\partial L(t)}{\partial t} y\Big) \frac{\partial}{\partial y}\Psi(y,t_1) - \frac{1}{L(t)}\frac{\partial L(t)}{\partial t} y \frac{\partial}{\partial t}\frac{\partial}{\partial y}\Psi(y,t_1) \\
&\quad + \frac{\partial}{\partial t}\frac{\partial}{\partial t_1}\Psi(y,t_1) = \frac{2\big(\frac{\partial L(t)}{\partial t}\big)^2 - \frac{\partial^2 L(t)}{\partial t^2}L(t)}{L^2(t)} y \frac{\partial}{\partial y}\Psi(y,t_1) \\
&\quad - \frac{\frac{\partial L(t)}{\partial t}}{L(t)} y \Big(\frac{\partial^2}{\partial y^2}\Psi(y,t_1)\frac{\partial y}{\partial t} + \frac{\partial^2}{\partial y \partial t_1}\Psi(y,t_1)\frac{\partial t_1}{\partial t}\Big) \\
&\quad + \frac{\partial^2}{\partial t_1 \partial y}\Psi(y,t_1)\frac{\partial y}{\partial t} + \frac{\partial^2}{\partial t_1^2}\Psi(y,t_1)\frac{\partial t_1}{\partial t} \\
&= \frac{2\big(\frac{\partial L(t)}{\partial t}\big)^2 - \frac{\partial^2 L(t)}{\partial t^2}L(t)}{L^2(t)} y \frac{\partial}{\partial y}\Psi(y,t_1) + \frac{\frac{\partial^2 L(t)}{\partial t^2}}{L^2(t)} y^2 \frac{\partial^2}{\partial y^2}\Psi(y,t_1) \\
&\quad - 2\frac{\frac{\partial L(t)}{\partial t}}{L(t)} y \frac{\partial^2}{\partial y \partial t_1}\Psi(y,t_1) + \frac{\partial^2}{\partial t_1^2}\Psi(y,t_1).
\end{aligned}
$$

We can substitute (2.8) in (A.2) and (A.1), and substitute them in (1.1):

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big(-\frac{L}{L} y\Big) \frac{\partial}{\partial y}\sum_n C_n(t)\phi_n(t) - \frac{L}{L} y \frac{\partial}{\partial t}\frac{\partial}{\partial y}\sum_n C_n(t)\phi_n(t) \\
&\quad + \frac{\partial}{\partial t}\frac{\partial}{\partial t_1}\sum_n C_n(t)\phi_n(t) = \frac{2\dot{L}^2 - \ddot{L}L(t)}{L^2} y \frac{\partial}{\partial y}\sum_n C_n(t)\phi_n(t) \\
&\quad - \frac{\dot{L}}{L} y \Big(\frac{\partial^2}{\partial y^2}\sum_n C_n(t)\phi_n(t)\frac{\partial y}{\partial t} + \frac{\partial^2}{\partial y \partial t_1}\sum_n C_n(t)\phi_n(t)\frac{\partial t_1}{\partial t}\Big) \\
&\quad + \frac{\partial^2}{\partial t_1 \partial y}\sum_n C_n(t)\phi_n(t)\frac{\partial y}{\partial t} + \frac{\partial^2}{\partial t_1^2}\sum_n C_n(t)\phi_n(t)\frac{\partial t_1}{\partial t} \\
&\quad = \frac{2\dot{L}^2 - \ddot{L}L}{L^2} y \frac{\partial}{\partial y}\sum_n C_n(t)\phi_n(t) + \frac{\ddot{L}}{L^2} y^2 \frac{\partial^2}{\partial y^2}\sum_n c_n(t)\phi_n(t) \\
&\quad - 2\frac{L}{L} y \frac{\partial^2}{\partial y \partial t_1}\sum_n C_n(t)\phi_n(t) + \frac{\partial^2}{\partial t_1^2}\sum_n C_n(t)\phi_n(t) \\
&\qquad = \frac{1}{L^4}\frac{\partial^4}{\partial y^4}\Psi(y,t_1).
\end{aligned}
$$

We multiply the resulting expression by $\phi^{(m)}$, integrate from 0 to l over dy and apply condition (2.1). Then, we come to the following equation

$$
\begin{aligned}
&\frac{2\dot{L}^2 - \ddot{L}L(t)}{L^2}\frac{\partial}{\partial y}\sum_n C_n(t)\int_0^l y \frac{\partial \phi^{(n)}(y)}{\partial y}\phi^{(m)}(y)dy \\
&\quad + \frac{\dot{L}^2}{L^2}\sum_n C_n(t)\int_0^l y^2 \frac{\partial^2 \phi^{(n)}(y)}{\partial y^2}\phi^{(m)}(y)dy
\end{aligned}
$$

$$-2\frac{L}{L}\sum_n \dot{c}_n(t)\int_0^l y\frac{\partial\phi^{(n)}(y)}{\partial y}\phi^{(m)}(y)dy+\ddot{C}_m$$
$$=\frac{k^4}{L^4}\sum_n C_n(t)\int_0^l \phi^{(n)}(y)\phi^{(m)}(y)dy.$$

Integrals in the obtained equation are independent of time; therefore, they can be denoted as I_1, I_2 and counted once:

$$I_1^{nm}=\int_0^l y\frac{\partial\phi^{(n)}(y)}{\partial y}\phi^{(m)}(y)dy,\quad I_2^{nm}=\int_0^l y^2\frac{\partial^2\phi^{(n)}(y)}{\partial y^2}\phi^{(m)}(y).$$

Eigenfunctions form an orthogonal and normalised system, i.e.

$$\int_0^l \phi^{(n)}(y)\phi^{(m)}(y)dy=\delta_{nm}.$$

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ВРЕМЕНСКИ-ЗАВИСНИ МЕТРИЧКИ ГРАФ СА ОПЕРАТОРОМ ЧЕТВРТОГ РЕДА НА ТЕМЕНИМА

Резиме. Предложен је модел метричког графа за опис еластичних вибрација у мрежи штапова под претпоставком да је дужина штапова променљива у времену. Разматрани су један случајеви графа са једном ивицом и звездастог графа. Истражује се утицај закона варијације дужине на расподелу вибрација. За варијацију дужине високе фреквенције примећује се брз прелаз на расподелу амплитуда високе фреквенције.

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